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ON THE COMPLEXITY OF THE IDEAL OF ABSOLUTE NULL SETS*

ПРО КОМПЛЕКСНІСТЬ ІДЕАЛУ АБСОЛЮТНИХ НУЛЬ-МНОЖИН

Answering a question of Banakh and Lyaskovska, we prove that for an arbitrary countable infinite amenable group G the ideal of sets having μ-measure zero for every Banach measure μ on G is an $F_{\sigma\delta}$ subset of $\{0,1\}^G$.

У відповідь на питання, поставлене Банахом і Лясківською, доведено, що для будь-якої зліченної аменабельної групи G ідеал множин, що мають нульову μ-міру для будь-якої міри Банаха μ на G, є $F_{\sigma\delta}$-підмножиною $\{0,1\}^G$.

1. Introduction. This note is related to a paper by T. Banakh and N. Lyaskovska [1]. Given an amenable group G, Banakh and Lyaskovska considered the ideal $\mathcal{N}$ of *absolute null* subsets of G, i.e., sets having μ-measure zero for every Banach measure μ on G (a finitely-additive, probability, left-invariant measure $\mu\colon \mathcal{P}(G)\to[0,1]$ defined on the family of all subsets of G; see [3]). Since each ideal on a countable infinite group G can be considered as a subspace of the Cantor set $\{0,1\}^G$ it makes sense to consider its descriptive properties. Banach and Lyaskovska asked ([1], Problem 4) whether the ideal of absolute null subsets of the group $\mathbb{Z}$ is co-analytic. In this note we prove (see Corollary 3.1) that for an arbitrary countable infinite amenable group G the ideal $\mathcal{N}$ is in fact $F_{\sigma\delta}$. This follows from a characterisation of absolute null subsets of an arbitrary amenable group (see Proposition 2.1) based on the notion of the intersection number of Kelly [2].

2. A characterisation of absolute null sets. Following Kelly [2] we define *the intersection number* $I(\mathcal{B})$ of a family $\mathcal{B}$ of subsets of a set X to be $\inf\{i(S)/n(S)\}$ where the infimum is taken over all finite sequences $S=(S_1,\ldots,S_n)$ of (not necessary distinct) elements of $\mathcal{B}$, $n=n(S)$ is the length of S and

$$i(S)=\sup\left\{\sum_{i=1}^{n}\chi_{S_i}(x)\colon\ x\in X\right\}.$$

Proposition 2.1. *Let G be an amenable group and $A\subseteq G$. Then the following are equivalent*:

(1) *A is absolute null.*

(2) *The intersection number of the family $\{gA\colon\ g\in G\}$ is zero.*

Proof. (1) $\Rightarrow$ (2). Assume that $I(\{gA\colon\ g\in G\})=\delta>0$. By a theorem of Kelly (see [2], Theorem 2), there is a finitely additive probability measure m defined on $\mathcal{P}(G)$ such that $m(gA)\geq\delta$ for each $g\in G$.

Let θ be a Banach measure on G. Following the proof of Invariant Extension Theorem (see [4], Theorem 10.8) define a function $\mu\colon\mathcal{P}(G)\to[0,1]$ by letting

$$\mu(B)=\int_G m(g^{-1}B)d\theta(g),\quad \text{for}\quad B\subseteq G.$$

It is easy to see that μ is a Banach measure on G. Moreover, we have

$$\mu(A)=\int_G m(g^{-1}A)d\theta(g)\geq\inf\{m(g^{-1}A)\colon\ g\in G\}\geq\delta>0,$$

which shows that $A\notin\mathcal{N}$.

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(2) $\Rightarrow$ (1). Let μ be an arbitrary Banach measure on G. Suppose that $\mu(A) = \epsilon > 0$. Then, since μ is left-invariant, we also have $\mu(gA) = \epsilon$ for every $g \in G$. Consequently, by [2] (Proposition 1), $I(\{gA\colon g \in G\}) \geq \epsilon > 0$.

Proposition 2.1 is proved.

3. The Borel complexity of the ideal $\mathcal{N}$. The following corollary of Proposition 2.1 gives an answer to a question of Banakh and Lyaskovska (see [1], Problem 4).

***Corollary* 3.1.** *Let G be an amenable group and $A \subseteq G$. Then the following are equivalent*:

(1) *A is absolute null.*

(2) $\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \exists \bar{g} \in G^{n+1}\ \forall S \subseteq \{1, \ldots, n+1\}\colon$

$$\frac{|S|}{n+1} > \frac{1}{k+1} \ \Rightarrow\ \bigcap_{i \in S} g_i A = \varnothing.$$

In particular, if G is countably infinity, then formula (2) *gives a $F_{\sigma\delta}$ definition of the ideal $\mathcal{N}$.*

Proof. It is easy to see that formula (2) simply states that $I(\{gA\colon g \in G\}) = 0$ so its equivalence with condition (1) was established in Proposition 2.1.

To prove the remaining part of the corollary, assume that G is countably infinity. Then it is enough to show that for fixed $n \in \mathbb{N}$, $\bar{g} \in G^{n+1}$ and $S \subseteq \{1, \ldots, n+1\}$ the family $\{A \subseteq G\colon \bigcap_{i \in S} g_i A = \varnothing\}$ is closed in $\mathcal{P}(G)$.

But this follows from the fact that for $A \subseteq G$ we have

$$\bigcap_{i \in S} g_i A = \varnothing \iff \forall g \in G\ \exists i \in S\colon\ g_i^{-1} g \notin A.$$

Corollary 3.1 is proved.

4. Some open problems. Let G be an arbitrary infinite group. Following a suggestion by Taras Banakh (personal communication) let us call a set $A \subseteq G$ *Kelly null* if the intersection number of the family $\{gA\colon g \in G\}$ is zero; denote by $\mathcal{K}$ the collection of all Kelly null subsets of G. In view of Proposition 2.1, $\mathcal{K}$ is an ideal of subsets of G *provided* the group G is amenable. On the other hand, Proposition 5.1 of [1] implies that if G has a free subgroup of rank 2, then $\mathcal{K}$ is not an ideal; in fact G is then the union of two Kelly null sets. In any case, however, $\mathcal{K}$ contains a (possibly proper) subfamily $\mathcal{A}_{\mathcal{K}} = \{A \subseteq G\colon \forall K \in \mathcal{K}\ K \cup A \in \mathcal{K}\}$ which already forms an ideal.

The remarks above lead to the following problems suggested by Banakh.

Problem 1. Characterise groups G for which $\mathcal{K}$ is an ideal.

Problem 2. Characterise groups G which are finite unions of elements of $\mathcal{K}$.

Problem 3. Given a countably infinite group G find a combinatorial description of elements of the ideal $\mathcal{A}_{\mathcal{K}}$. What is its descriptive complexity? In particular, is it Borel?

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