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SPECTRAL PROBLEM FOR DISCONTINUOUS INTEGRO-DIFFERENTIAL OPERATOR

СПЕКТРАЛЬНА ЗАДАЧА ДЛЯ РОЗРИВНОГО ІНТЕГРО-ДИФЕРЕНЦІАЛЬНОГО ОПЕРАТОРА

A representation of solutions of a discontinuous integro-differential operator is obtained. The asymptotic behavior of the eigenvalues and eigenfunctions of this operator is described.

Отримано зображення розв'язків розривного інтегро-диференціального оператора. Описано асимптотичну поведінку власних чисел та власних функцій цього оператора.

1. Introduction. Let us consider the boundary-value problem L :

$$\ell y\colon = -y'' + q(x)y + \int\limits_0^x M(x-t)y(t)dt = \lambda y,\ x \in (0,d) \cup (d,\pi), \tag{1}$$

$$U(y)\colon = y(0) = 0, \tag{2}$$

$$V(y)\colon = y(\pi) = 0, \tag{3}$$

$$\begin{cases} y(d+0) = \alpha y(d-0), \\ y'(d+0) = \alpha^{-1} y'(d-0), \end{cases} \tag{4}$$

where $\alpha > 0,\ \alpha \neq 1,\ d \in (0,\pi)$ and $q(x)$ is a real valued function in $L_2(0,\pi)$, λ is the spectral parameter. Moreover, the functions $(\pi - x)M(x)$ and $\int_0^x M(t)dt$ are real valued functions in $L_2(0,\pi)$.

In [1] perturbation of a Sturm – Liouville operator by a Volterra integral operator is considered. The presence of an "aftereffect"in a mathematical model produces qualitative changes in the study of the inverse problem.

Boundary-value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disciplins ranging from engineering to the geosciences.

For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [4, 6]. Spectral information can be used to reconstruct the permittivity and conductivity profiles of a one-dimentional discontinuous medium [3, 5]. Further, it is known that spectral problems play an important role for investigating some nonlinear evolution equations of mathematical physics.

Consider the operator

$$T\colon =-\frac{d^2}{dx^2}+q(x)+\mathcal{M},$$

with the domain

$D(T)=\{y\colon y(x)$ and $y'(x)$ are absolutely continuous in $[0,d)\cup(d,\pi]$, $\ell y\in L_2(0,\pi)$, $y(0)=0$, $y(\pi)=0$, $y(d+0)=\alpha y(d-0)$, $y'(d+0)=\alpha^{-1}y'(d-0)\}$, such that $\mathcal{M}y=\int_0^x M(x-t)y(t)dt$, where $\mathcal{M}$ is an integral operator.

Let the function $\varphi(x,\lambda)$ be the solution of equation (1) satisfying the initial conditions $\varphi(0,\lambda)=0$, $\varphi'(0,\lambda)=1$ and the jump conditions (4).

It is shown in [1] that, the solution $\varphi(x,\lambda)$ has a representation as follows:

$$\varphi(x,\lambda)=\frac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\int\limits_0^x K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt,$$

where $K(x,t)$ is a continuous function and $K(x,0)=0$.

Firstly, let us try to get a similar representation for the solution $\varphi(x,\lambda)$ as follows:

$$\varphi(x,\lambda)=\begin{cases}\dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\displaystyle\int\limits_{-x}^x K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt, & x<d,\\ \alpha^+\dfrac{\sin\sqrt{\lambda}x}{\sqrt{\lambda}}+\alpha^-\dfrac{\sin\sqrt{\lambda}(2d-x)}{\sqrt{\lambda}}+\displaystyle\int\limits_{-x}^x K(x,t)\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}dt, & x>d.\end{cases} \tag{5}$$

It is clearly shown that the integral equation for the solution $\varphi(x,k)$ is of the following type:

for $x<d$,

$$\varphi(x,k)=\frac{\sin kx}{k}+\int\limits_0^x\frac{\sin k(x-t)}{k}q(t)\varphi(t,k)dt+$$

$$+\int\limits_0^x\int\limits_0^t\frac{\sin k(x-t)}{k}M(t-\tau)\varphi(\tau,k)d\tau dt,$$

and, for $x>d$,

$$\varphi(x,k)=\alpha^+\frac{\sin kx}{k}+\alpha^-\frac{\sin k(2d-x)}{k}+\frac{1}{k}\int\limits_0^d(\alpha^+\sin k(x-t)+$$

$$+\alpha^-\sin k(2d-x-t))q(t)\varphi(t,k)dt+$$

$$+\frac{1}{k}\int\limits_0^d\int\limits_0^t(\alpha^+\sin k(x-t)+\alpha^-\sin k(2d-x-t))M(t-\tau)\varphi(\tau,k)d\tau dt+$$

$$+\frac{1}{k}\int_{d}^{x}\sin k(x-t)q(t)\varphi(t,k)dt+\frac{1}{k}\int_{d}^{x}\int_{0}^{t}\sin k(x-t)M(t-\tau)\varphi(\tau,k)d\tau dt,$$

where $\alpha^{\pm}=\dfrac{1}{2}\left(\alpha+\dfrac{1}{\alpha}\right)$ and $\sqrt{\lambda}=k$.

In order to be solution of above equations of the function which has representation (5), the equality

$$\int_{-x}^{x}K(x,t)\frac{\sin kt}{k}dt=\frac{1}{k}\int_{0}^{d}(\alpha^{+}\sin k(x-t)+\alpha^{-}\sin k(2d-x-t))q(t)\frac{\sin kt}{k}dt+$$

$$+\frac{1}{k}\int_{0}^{d}(\alpha^{+}\sin k(x-t)+\alpha^{-}\sin k(2d-x-t))q(t)\left\{\int_{-t}^{t}K(t,\tau)\frac{\sin k\tau}{k}d\tau\right\}dt+$$

$$+\frac{1}{k}\int_{0}^{d}\int_{0}^{t}(\alpha^{+}\sin k(x-t)+\alpha^{-}\sin k(2d-x-t))M(t-\tau)\frac{\sin k\tau}{k}d\tau dt+$$

$$+\frac{1}{k}\int_{0}^{d}\int_{0}^{t}(\alpha^{+}\sin k(x-t)+\alpha^{-}\sin k(2d-x-t))M(t-\tau)\left\{\int_{-\tau}^{\tau}K(\tau,\xi)\frac{\sin k\xi}{k}d\xi\right\}d\tau dt+$$

$$+\frac{1}{k}\int_{d}^{x}\sin k(x-t)q(t)\left\{\alpha^{+}\frac{\sin kt}{k}+\alpha^{-}\frac{\sin k(2d-t)}{k}\right\}dt+$$

$$+\frac{1}{k}\int_{d}^{x}\sin k(x-t)q(t)\left\{\int_{-t}^{t}K(t,\tau)\frac{\sin k\tau}{k}d\tau\right\}dt+$$

$$+\frac{1}{k}\int_{d}^{x}\int_{0}^{t}\sin k(x-t)M(t-\tau)\left\{\alpha^{+}\frac{\sin k\tau}{k}+\alpha^{-}\frac{\sin k(2d-\tau)}{k}\right\}d\tau dt+$$

$$+\frac{1}{k}\int_{d}^{x}\int_{0}^{t}\sin k(x-t)M(t-\tau)\left\{\int_{-\tau}^{\tau}K(\tau,\xi)\frac{\sin k\xi}{k}d\xi\right\}d\tau dt$$

must be hold. For $d<x<2d$, $-x<t<x-2d<2d-x$, it is easy to get the following integral equation:

$$K(x,t)=\frac{\alpha^{+}}{2}\int_{(x-t)/2}^{x}q(s)ds+\frac{\alpha^{+}}{2}\int_{d-(x+t)/2}^{d}q(s)ds+\frac{\alpha^{+}}{2}\int_{0}^{x}\int_{t+s-x}^{t+x-s}M(s-\tau)d\tau ds+$$

$$+\frac{\alpha^-}{2}\int\limits_0^d\int\limits_{t+s+x-2d}^{t+2d-x-s}M(s-\tau)d\tau ds-\frac{\alpha^-}{2}\int\limits_d^x\int\limits_{t+s+2d-x}^{t-s+2d+x}M(s-\tau)d\tau ds+$$

$$+\frac{\alpha^+}{2}\int\limits_0^d q(s)\int\limits_{t+s+x-2d}^{t+2d-x-s}K(s,\tau)d\tau ds+\frac{\alpha^-}{2}\int\limits_0^d q(s)\int\limits_{t-x+s}^{t+x-s}K(s,\tau)d\tau ds+$$

$$+\frac{\alpha^+}{2}\int\limits_0^d\int\limits_0^s\int\limits_{t+s-x}^{t+x-s}M(s-\tau)K(\tau,\xi)d\xi d\tau ds+\frac{\alpha^-}{2}\int\limits_0^d\int\limits_0^s\int\limits_{t+s+x-2d}^{t+2d-x-s}M(s-\tau)K(\tau,\xi)d\xi d\tau ds+$$

$$+\frac{1}{2}\int\limits_d^x q(s)\int\limits_{t-x+s}^{t+x-s}K(s,\tau)d\tau ds+\frac{1}{2}\int\limits_0^d\int\limits_0^s\int\limits_{t-x+s}^{t+x-s}M(s-\tau)K(\tau,\xi)d\xi d\tau ds.$$

The same integral equations can be obtained for (i) $2d < x, -x < t < 2d - x$; (ii) $d < x < 2d$, $x - 2d < t < 2d - x$; (iii) $2d < x$, $2d - x < t < x - 2d$; (iv) $2d < x$, $2d - x < t < x$ and (v) $d < x < 2d$, $x - 2d < t < x$.

For solving this integral equation:

put

$$K_0(x,t)=\frac{\alpha^+}{2}\int\limits_{(x-t)/2}^{x}q(s)ds+\frac{\alpha^+}{2}\int\limits_{d-(x+t)/2}^{d}q(s)ds+\frac{\alpha^+}{2}\int\limits_0^x\int\limits_{t+s-x}^{t+x-s}M(s-\tau)d\tau ds+$$

$$+\frac{\alpha^-}{2}\int\limits_0^d\int\limits_{t+s+x-2d}^{t+2d-x-s}M(s-\tau)d\tau ds-\frac{\alpha^-}{2}\int\limits_d^x\int\limits_{t+s+2d-x}^{t-s+2d+x}M(s-\tau)d\tau ds$$

and

$$K_{n+1}(x,t)=\frac{\alpha^+}{2}\int\limits_0^d q(s)\int\limits_{t+s+x-2d}^{t+2d-x-s}K_n(s,\tau)d\tau ds+\frac{\alpha^-}{2}\int\limits_0^d q(s)\int\limits_{t-x+s}^{t+x-s}K_n(s,\tau)d\tau ds+$$

$$+\frac{\alpha^+}{2}\int\limits_0^d\int\limits_0^s\int\limits_{t+s-x}^{t+x-s}M(s-\tau)K_n(\tau,\xi)d\xi d\tau ds+\frac{\alpha^-}{2}\int\limits_0^d\int\limits_0^s\int\limits_{t+s+x-2d}^{t+2d-x-s}M(s-\tau)K_n(\tau,\xi)d\xi d\tau d+$$

$$+\frac{1}{2}\int\limits_d^x q(s)\int\limits_{t-x+s}^{t+x-s}K_n(s,\tau)d\tau ds+\frac{1}{2}\int\limits_0^d\int\limits_0^s\int\limits_{t-x+s}^{t+x-s}M(s-\tau)K_n(\tau,\xi)d\xi d\tau ds.$$

It is shown by the successive approximation method, the following theorem is true.

Theorem 1. *For the solution of* (1) *which satisfies the initial conditions* $\varphi(0) = 0$ $\varphi'(0) = 1$ *and the jump condition* (4) *has the form,*

$$\varphi(x,k) = \begin{cases} \dfrac{\sin kx}{k} + \displaystyle\int\limits_{-x}^{x} K(x,t)\dfrac{\sin kt}{k}dt, & x < d, \\ \alpha^+\dfrac{\sin kx}{k} + \alpha^-\dfrac{\sin k(2d-x)}{k} + \displaystyle\int\limits_{-x}^{x} K(x,t)\dfrac{\sin kt}{k}dt, & x > d, \end{cases}$$

and also

$$\int\limits_{-x}^{x} |K(x,t)|\, dt \le e^{C\sigma(x)} - 1,$$

where

$$\sigma(x) = \int\limits_{0}^{x} (x-t)\left[|q(t)| + \int\limits_{0}^{t} |M(t-\tau)|\, d\tau\right] dt$$

and $C = \alpha^+ + |\alpha^-| + 1.$

Denote $\Delta(\lambda) = \varphi(\pi,\lambda)$.The eigenvalues $\{\lambda_n\}_{n\ge 1}$ of the boundary-value problem L coincide with the zeros of the function $\Delta(\lambda)$.

Theorem 2. *The eigenvalues* λ_n *and eigenfunctions* $\varphi(x,k_n)$ *of the problem* (1)–(4) *satisfy the following asymptotic estimates for sufficiently large* n :

$$\sqrt{\lambda_n} = k_n = k_n^0 + o\left(\frac{1}{k_n^0}\right), \tag{6}$$

$$\varphi(x,k_n) = \frac{\sin k_n^0 x}{k_n^0} + o\left(\frac{1}{k_n^0}\right), \quad x < d, \tag{7}$$

$$\varphi(x,k_n) = \alpha^+\frac{\sin k_n^0 x}{k_n^0} + \alpha^-\frac{\sin k_n^0(2d-x)}{k_n^0} + o\left(\frac{1}{k_n^0}\right), \quad x > d, \tag{8}$$

where k_n^0 *are the roots of* $\Delta_0(k)\colon = \alpha^+\dfrac{\sin k\pi}{k} + \alpha^-\dfrac{\sin k(\pi-2d)}{k}$ *and* $k_n^0 = n + h_n$, $h_n \in \ell_\infty$.

Proof. From (5)

$$\Delta(k) = \alpha^+\frac{\sin k\pi}{k} + \alpha^-\frac{\sin k(2d-\pi)}{k} + \int\limits_{-\pi}^{\pi} K(\pi,t)\frac{\sin kt}{k}dt.$$

Denote $\Gamma_n\colon = \{\lambda\colon \lambda = k^2,\ |k| = k_n^0 + \delta\}$, $n = 0, 1, \ldots$. Since $\Delta(k) - \Delta_0(k) = o\left(\dfrac{e^{|\operatorname{Im} k\pi|}}{|k|}\right)$ and $|\Delta_0(k)| \ge C\dfrac{e^{|\operatorname{Im} k\pi|}}{|k|}$ for all $\lambda \in \Gamma_n$, we establish by the Rouche's theorem that $k_n =$

$= k_n^0 + \varepsilon_n$, where $\varepsilon_n = o\,(1)$. Moreover, $\varepsilon_n = o\left(\dfrac{1}{k_n^0}\right)$ is obtained from the equality $\Delta(k_n) = $ $= \left(\Delta_0'(k_n^0) + o(1)\right)\varepsilon_n + o\left(\dfrac{1}{k_n^0}\right) = 0$. This completes the proof of (6).

From (5) and (6), one can easily prove that the asymptotic formulae (7) and (8) are true.

Theorem 2 is proved.

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