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EXTENSION OF HOLOMORPHIC MAPPINGS FOR FEW MOVING HYPERSURFACES*

ПРОДОВЖЕННЯ ГОЛОМОРФНИХ ВІДОБРАЖЕНЬ ДЛЯ ДЕКІЛЬКОХ ГІПЕРПОВЕРХОНЬ, ЩО РУХАЮТЬСЯ

We prove the big Picard theorem for holomorphic curves from a punctured disc into $\mathbf{P}^n(\mathbf{C})$ with $n+2$ hypersurfaces. We also prove a theorem on the extension of holomorphic mappings in several complex variables into a submanifold of $\mathbf{P}^n(\mathbf{C})$ with several moving hypersurfaces.

Доведено велику теорему Пікара для голоморфних кривих із проколотого круга в $\mathbf{P}^n(\mathbf{C})$ із $n+2$ гіперповерхнями. Також доведено теорему про продовження голоморфних відображень від декількох комплексних змінних у підбагатовид $\mathbf{P}^n(\mathbf{C})$ з декількома гіперповерхнями, що рухаються.

1. Introduction. Picard proved the following theorems for meromorphic functions in one complex variable.

Theorem A (Little Picard theorem). *Let $f(z)$ be a meromorphic function on the complex plane. If there exist three mutually distinct points w_1, w_2, and w_3 on the Riemann sphere such that $f(z)-w_i$, $i=1,2,3$, has no zero on the complex plane, then f is a constant.*

Theorem B (Big Picard theorem). *Let $f(z)$ be a meromorphic function on $\Delta^*=\{z\in\mathbf{C}\colon 1\leq |z|<+\infty\}$. If there exist three mutually distinct points w_1, w_2 and w_3 on the Riemann sphere such that $f(z)-w_i$, $i=1,2,3$, has no zero on Δ^*, then f does not have an essential singularity at ∞.*

In the case of higher dimension, H. Fujimoto [3] gave a Big Picard's theorem for holomorphic mappings from a complex manifold into $\mathbf{P}^n(\mathbf{C})$ as follows.

Theorem C (Theorem A [3]). *Let M be a complex manifold and let S be a regular thin analytic subset of M and let f be a holomorphic map of $M\setminus S$ into the n-dimensional complex projective space $\mathbf{P}^n(\mathbf{C})$. If f is of rank r somewhere and if $f(M-S)$ omits $2n-r+2$ hyperplanes in general position, then f can be extended to a holomorphic map of M into $\mathbf{P}^n(\mathbf{C})$, where the rank of f at a point $x\in M\setminus S$ means the rank of the Jacobian matrix of f at x.*

By using a criterion on normality and by applying little Picard theorems for holomorphic mappings, Z. H. Tu generalized the above theorems to the case of moving hyperplanes as follows.

Theorem D (Theorem 2.2 [11]). *Let S be an analytic subset of a domain D in $\mathbf{C}^n$ with codimension one, whose singularities are normal crossings. Let f be a holomorphic mapping from $D\setminus S$ into $\mathbf{P}^n(\mathbf{C})$. Let $a_1(z),\ldots,a_q(z)$ $(z\in D)$ be q $(q\geq 2n+1)$ moving hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in pointwise general position such that $f(z)$ intersects $a_j(z)$ on $D\setminus S$ with multiplicity at least m_j, $j=1,\ldots,q$, where $m_1,\ldots,m_q$ are positive integers and may be $+\infty$, with*

$$\sum_{j=1}^{q}\frac{1}{m_j}<\frac{q-(n+1)}{n}.$$

Then f extends to a holomorphic mapping from D into $\mathbf{P}^n(\mathbf{C})$.

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We would like to note that in Theorem D, the number of hyperplanes is assumed to be at least $2n+1$ and this assumption plays a very essential role in the proof. Then, the following question arises naturally: "*Are there any Big Picard's theorems which are analogous to Theorem C or Theorem D in the case where the moving hyperplanes are replaced by moving hypersurfaces and the number* q *is replaced by a smaller one* ?"

In the present paper we will give some positive answers for this question. First of all, let us recall some following.

Denote by $\mathcal{H}_D$ the ring of all holomorphic functions on a domain D in $\mathbf{C}^m$. Let Q be a homogeneous polynomial in $\mathcal{H}_D[x_0,\ldots,x_n]$ of degree $d\geq 1$. Denote by $Q(z)$ the homogeneous polynomial over $\mathbf{C}$ obtained by substituting a specific point $z\in D$ into the coefficients of Q. We also call a moving hypersurface in $\mathbf{P}^n(\mathbf{C})$ on D each homogeneous polynomial $Q\in\mathcal{H}_D[x_0,\ldots,x_n]$ such that the coefficients of Q have no common zero point.

Let $Q_1,\ldots,Q_q$ be q moving hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ on D.

Set

$$\mathcal{T}_d := \big\{(i_0,\ldots,i_n)\in N^{n+1}\ |i_0+i_1+\ldots+i_n=d\big\}.$$

Assume that

$$Q_j(z)=\sum_{I\in\mathcal{T}_{d_j}} a_{jI}(z)x^I,$$

where a_{jI} are holomorphic functions on D without common zeros, $x^I = x_0^{i_0}\ldots x_n^{i_n}$ for $x = (x_0,\ldots,x_n)$ and $I=(i_0,\ldots,i_n)\in\mathcal{T}_{d_j}$, $d_j=\deg(Q_j)$.

Denote by $\mathcal{R}\{Q_j\}$ the smallest field which contains $\mathbf{C}$ and all functions $\dfrac{a_{jI}}{a_{jJ}}$ with $a_{iJ}\not\equiv 0$. Sometime we write $\mathcal{R}$ for $\mathcal{R}\{Q_j\}$ if there is no confusion.

We say that moving hypersurfaces $\{Q_j\}_{j=1}^q$ in $\mathbf{P}^n(\mathbf{C})$ are located in general position (resp. in pointwise general position) on a subset $\Omega\subset D$ if there exists $z\in\Omega$ (resp. for all $z\in\Omega$) such that for any $1\leq j_0<\ldots<j_n\leq q$ the system of equations

$$Q_{j_i}(z)(w_0,\ldots,w_n)=0,\quad 0\leq i\leq n,$$

has only the trivial solution $w=(0,\ldots,0)$ in $\mathbf{C}^{n+1}$.

Let f be a meromorphic mapping of D into $\mathbf{P}^n(\mathbf{C})$ and let Q be a moving hypersurface of $\mathbf{P}^n(\mathbf{C})$ on D defined by $Q(z)=\sum_{I\in\mathcal{T}_d} a_{jI}(z)x^I$, where d is the degree of homogeneous polynomial Q. For $z_0\in D$, take a reduced representation $f=(f_0:\ldots:f_n)$ of f on a neighborhood U_{z_0} of z_0 and set $Q(f)(z)=Q(z)(f_0(z),\ldots,f_n(z))$ on U_{z_0}. We define $\mathrm{div}Q(f)(z)=\mathrm{div}(Q(f_0,\ldots,f_n))(z)$ if $Q(f)\not\equiv 0$ and $\mathrm{div}Q(f)(z)=\infty$ if $Q(f)\equiv 0$. Thus, $\mathrm{div}Q(f)$ is well-defined on D independently of the choice of reduced representations of f. If $\mathrm{div}Q(f)(z)\geq m_j$ for all $z\in D$, we say that f intersects Q on D with multiplicity at least m_j.

We set punctured discs on $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$ around ∞ by

$$\Delta^*=\{z\in\mathbf{C}\colon |z|\geq 1\},$$

$$\Delta^*(t)=\{z\in\mathbf{C}\colon |z|\geq t\},\quad 1\leq t\leq\infty,$$

and set $\Delta = \Delta^* \cup \{\infty\}$. For a moving hypersurface Q in $\mathbf{P}^n(\mathbf{C})$ on Δ^* defined by $Q(z) = \sum_{I\in\mathcal{T}_d} a_{jI}(z)x^I$, we say that Q is a moving hypersurface in $\mathbf{P}^n(\mathbf{C})$ on Δ if all coefficients a_{jI} are extendable over Δ.

Our first aim of this paper is to show a Big Picard's theorem for holomorphic curve from a punctured disc with only $n+2$ hypersurfaces. Namely, we will prove the following theorem.

Theorem 1. *Let f be a holomorphic curve from the punctured disc Δ^* into $\mathbf{P}^n(\mathbf{C})$, and let $Q_1, \ldots, Q_{n+2}$ be $n+2$ hypersurfaces in $\mathbf{P}^n(\mathbf{C})$ on Δ located in general position such that f is algebraically nondegenerate over $\mathcal{R}\{Q_i\}$. Assume that f intersects each Q_i on Δ^* with multiplicity at least m_i, where $m_1, \ldots, m_{n+2}$ are fixed positive integers and may be $+\infty$, with*

$$\sum_{i=1}^{n} \frac{1}{m_i} < \frac{1}{M},$$

where $M = (nd + [(n+1)^2(2^n-1)(d\epsilon)^{-1}]d)^n$. Then f extends at ∞ to a holomorphic curve $\tilde{f}$ from $\Delta = \Delta^* \cup \{\infty\}$ to $\mathbf{P}^n(\mathbf{C})$.

In the case of moving hypersurfaces and an arbitrary meromorphic mapping from a domain in $\mathbf{C}^m$ into a subvariety V of $\mathbf{P}^n(\mathbf{C})$, we shall prove the following, which is a generalization of the above result of H. Fujimoto.

Theorem 2. *Let f be a holomorphic mapping of a domain $D \setminus S$ into X, where D is a domain in $\mathbf{C}^m$, S is an analytic subset of co-dimension one of D, whose singularities are only normal crossings, and X is an irreducible subvariety of $\mathbf{P}^n(\mathbf{C})$. Let $Q_0, \ldots, Q_{q-1}$ be q moving hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ on D located in pointwise subgeneral position with respect to X. Assume that f does not intersect each Q_i on $D \setminus S$ for all $1 \le i \le q-1$. If $q \ge 2\dim X + 1$. Then f extends to a holomorphic mapping $\tilde{f}$ from D into $\mathbf{P}^n(\mathbf{C})$.*

2. Notations. (a) We set punctured discs on $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ around ∞ by

$$\Delta^* = \{z \in \mathbf{C} \colon |z| \ge 1\},$$

$$\Delta^*(t) = \{z \in \mathbf{C} \colon |z| \ge t\}, \quad t \ge 1,$$

and set

$$\Gamma(r) = \{z \in \mathbf{C} \colon |z| = t\}, \quad t \ge 1.$$

In this paper, we always assume that functions on Δ^* and mappings from Δ^* are defined on a neighborhood of Δ^* in $\mathbf{C}$. Let ξ be a function on Δ^* satisfying that

(i) ξ is differentiable outside a discrete set of points,

(ii) ξ is locally written as a difference of two subharmonic functions.

Then by [5] (§1), we have

$$\int_1^t \frac{dt}{t} \int_{\Delta^*(t)} dd^c\xi = \frac{1}{4\pi} \int_{\Gamma(r)} \xi(re^{i\theta})d\theta - \frac{1}{4\pi} \int_{\Gamma(1)} \xi(re^{i\theta})d\theta - (\log r) \int_{\Gamma(1)} d^c\xi, \tag{2.1}$$

where $dd^c\xi$ is taken in the sense of current.

(b) A divisor E on Δ^* is given by a formal sum $E = \sum \mu_\nu p_\nu$, with $\{p_\nu\}$ is a locally finite family of distinct points in Δ^* and $\mu_\nu \in \mathbf{Z}$. We define the support of E by $\mathrm{Supp}\,(E) = \cup_{\nu \neq 0} p_\nu$. Let k be a positive integer or $+\infty$. We define the divisor $E^{(k)}$ by

$$E^{(k)} := \sum \min\{\mu_\nu, k\} p_\nu,$$

and define the *truncated counting function to level* k of E by

$$N^{(k)}(r, E) := \int\limits_1^r \frac{n^{(k)}(t, E)}{t} dt, \quad 1 < r < +\infty,$$

where

$$n^{(k)}(t, E) = \sum_{|z| \leq t} E^{(k)}(z).$$

We simply write $N(r, E)$ for $N^{(+\infty)}(r, E)$.

(c) Let $f\colon \Delta^* \to \mathbf{P}^n(\mathbf{C})$ be a holomorphic curve. For an arbitrary fixed homogeneous coordinates $(w_0 : \ldots : w_n)$ of $\mathbf{P}^n(\mathbf{C})$, it is easy to see that there exist a neighborhood U of Δ^* in $\mathbf{C}^m$ and a reduced representation $(f_0 : \ldots : f_n)$ on U of f, which means that $f_0, \ldots, f_n$ are holomorphic functions on U without common zeros. We set $\|f\| \colon = (|f_0|^2 + \ldots + |f_n|^2)^{\frac{1}{2}}$.

Denote by Ω the Fubibi – Study form of $\mathbf{P}^n(\mathbf{C})$. The *order function* or *characteristic function* of f with respect to Ω is defined by

$$T_f(r) := T_f(r; \Omega) = \int\limits_1^r \frac{dt}{t} \int\limits_{\Delta^*(t)} f^*\Omega, \quad r > 1. \tag{2.2}$$

Applying (2.1) to $\xi = \log \|f\|$, we obtain the following:

$$T_f(r) = \frac{1}{2\pi} \int\limits_{\Gamma(r)} \log \|f(re^{i\theta})\| d\theta - \frac{1}{2\pi} \int\limits_{\Gamma(1)} \log \|f(e^{i\theta})\| d\theta - (\log r) \int\limits_{\Gamma(1)} d^c \log \|f\|. \tag{2.3}$$

Let Q be a hypersurface in $\mathbf{P}^n(\mathbf{C})$ given by $Q(x) = \sum_{I \in \mathcal{T}_d} a_I x^I$, where the constants a_I are not all zeros and d is the degree of Q. We set $Q(f) = \sum_{i \in \mathcal{T}_d}^n a_I f^I$, where $f^I = f_0^{i_0} \ldots f_n^{i_n}$ for $I = (i_0, \ldots, i_n) \in \mathcal{T}_d$. Assume that $Q(f) \not\equiv 0$, we define the *proximity function* of f with respect to Q by

$$m_f(r, Q) = \frac{1}{2\pi} \int\limits_{\Gamma(r)} \log \frac{\|f\|^d}{|Q(f)|} d\theta - \frac{1}{2\pi} \int\limits_{\Gamma(1)} \log \frac{\|f\|^d}{|Q(f)|} d\theta.$$

Applying (2.1) to $\xi = \log |Q(f)|$, we have

$$N(r, \mathrm{div} Q(f)) = \frac{1}{2\pi} \int\limits_{\Gamma(r)} \log |Q(f)| d\theta - \frac{1}{2\pi} \int\limits_{\Gamma(1)} \log |Q(f)| \theta - (\log r) \int\limits_{\Gamma(1)} d^c \log |Q(f)|. \tag{2.4}$$

Combining (2.2) and (2.4), we have the **First Main Theorem** as follows:

$$dT_f(r) = N(r, \mathrm{div}Q(f)) + m_f(r, Q) + (\log r) \int\limits_{\Gamma(1)} d^c \log\left(\frac{\|f\|^d}{|Q(f)|}\right). \tag{2.5}$$

(d) For a meromorphic function φ on Δ^*, applying (2.1) to $\xi = \log|\varphi|$, we obtain

$$N(r, \mathrm{div}_0(\varphi)) + N(r, \mathrm{div}_\infty(\varphi)) =$$

$$= \frac{1}{2\pi} \int\limits_{\Gamma(r)} \log|\varphi| d\theta - \frac{1}{2\pi} \int\limits_{\Gamma(1)} \log|\varphi| d\theta - (\log r) \int\limits_{\Gamma(1)} d^c \log|\varphi|.$$

The proximity function $m(r, \varphi)$ is defined by

$$m(r, \varphi) = \frac{1}{2\pi} \int\limits_{\Gamma(r)} \log^+ |\varphi| d\theta,$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geqslant 0$. The Nevanlinna's characteristic function is defined by

$$T(r, \varphi) = N(r, \mathrm{div}_\infty(\varphi)) + m(r, \varphi).$$

We regard φ as a meromorphic mapping from $\mathbf{C}$ into $\mathbf{P}^1(\mathbf{C})$, there is a fact that

$$T_\varphi(r) = T(r, \varphi) + \mathrm{O}(\log r).$$

Theorem 3 (lemma on logarithmic derivative [5]). *Let φ be a nonzero meromorphic function on Δ^*. Then*

$$\Big\| \; m\left(r, \frac{\varphi'}{\varphi}\right) = O(\log^+ T_\varphi(r)) + C \log r, \tag{2.6}$$

where C is a positive constant which does not depend on φ.

As usual, by the notation "$\|$ P" we mean the assertion P holds for all $r \in (1, +\infty)$ excluding a finite Lebesgue measure subset E of $(1, +\infty)$.

3. Second main theorem for holomorphic curves from a punctured disc. Firstly, we prove a Second Main Theorem for holomorphic curves from the punctured disc Δ^* into $\mathbf{P}^n(\mathbf{C})$ for hypersurfaces with truncated multiplicities as follows.

Theorem 4. *Let f be an algebraically nondegenerate holomorphic curve from the punctured disc Δ^* into $\mathbf{P}^n(\mathbf{C})$ and let Q_i, $1 \le i \le q$, be q hypersurfaces of $\mathbf{P}^n(\mathbf{C})$ of degree d_i on Δ^* located in general position, $q \ge n+2$. Then for every $\epsilon > 0$, the following holds*

$$\| \; (q - n - 1 - \epsilon) T_f(r) \le \sum_{i=1}^{q} \frac{1}{d_i} N^{(M_0)}(r, \mathrm{div}(Q_i(f))) + O(\log r),$$

where $M_0 = (nd + [(n+1)^2(2^n - 1)(d\epsilon)^{-1}]d)^n$ with d is the least common multiple of the d_i's.

Proof. Take $(\omega_0 : \ldots : \omega_n)$ be a homogeneous coordinates of $\mathbf{P}^n(\mathbf{C})$ and take $(f_0 : \ldots : f_n)$ be a reduced representation of f on a neighborhood of Δ^* in $\mathbf{C}^m$. Replacing Q_j by Q_j^{d/d_j} if necessary, we may assume that $Q_1, \ldots, Q_q$ have the same degree of d.

Given $z \in \mathbf{C}^m$, there exists a renumbering $\{i_1, \ldots, i_q\}$ of the indices $\{1, \ldots, q\}$ such that

$$|Q_{i_1}(f)(z)| \leq |Q_{i_2}(f)(z)| \leq \ldots \leq |Q_{i_q}(f)(z)|.$$

We denote $\gamma := (Q_{i_1}, \ldots, Q_{i_n})$. Since $\{Q_j\}_{j=1}^q$ are in general position, by Hilbert's Nullstellensatz that for any integer k, $0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$X_k^{m_k} = \sum_{j=1}^{n+1} b_{jk} Q_{i_j}(X_0, \ldots, X_n),$$

where b_{jk}, $1 \leq j \leq n+1$, $0 \leq k \leq n$, are homogeneous forms with coefficients in $\mathbf{C}$ of degree $m_k - d$. So

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - d} \max\{|Q_{i_1}(f)(z)|, \ldots, |Q_{i_{n+1}}(f)(z)|\} =$$

$$= c_1 \|f(z)\|^{m_k - d} |Q_{i_{n+1}}(f)(z)|,$$

where c_1 is a positive constant which depends only on the coefficients of Q_i, $1 \leq i \leq q$. Therefore

$$\|f(z)\|^d \leq c_1 |Q_{i_{n+1}}(f)(z)|. \tag{3.1}$$

Fix big integer N, which will be chosen later, such that N divisible by d, denote by V_N the space of homogeneous polynomials in $\mathbf{C}[X_0, \ldots, X_n]$ of degree N. Arrange, by the lexicographic order, the n-tuples $(j) = (j_1, \ldots, j_n)$ of nonnegative integers such that $\sigma(j) := \sum_{k=1}^n j_k \leq \dfrac{N}{d}$. Define the spaces

$$W_{(j)} = \sum_{(e) \geq (j)} Q_{i_1}^{j_1} \ldots Q_{i_n}^{j_n} V_{N - d\sigma(e)}.$$

We put $\Delta_{(j)} := \dim \dfrac{W_{(j)}}{W_{(j')}}$, where (j') follows (j) in the ordering. From Lemma 3 [2], we have

$$\Delta_{(j)} = d^n, \tag{3.2}$$

provided $d\sigma(j) < N - nd$.

Set $M := \dim V_N$. We now chose a suitable basis as follows: We start with the last nonzero $W_{(j)}^\gamma$, pick any basic of it. Then we continue inductively as follows, for $(j') > (j)$ such that $d\sigma(j), d\sigma(j') \leq$ $\leq N$. Assume that we have chosen a basic of $W_{(j')}^\gamma$, we pick representatives in $W_{(j)}^\gamma$ of the basic of $W_{(j)}^\gamma / W_{(j')}^\gamma$ which are the form $Q_{i_1}^{j_1} \ldots Q_{i_n}^{j_n} q$, where $q \in V_{N - d\sigma(j)}$. We extend the previously constructed basic in $W_{(j')}^\gamma$ by adding these representations, then we have a basic of $W_{(j)}^\gamma$. If $W_{(j)}^\gamma =$ $= V_N$ then we stop the process and we obtain a basic of V_N.

Now we estimate $\log \prod_{t=1}^M |\psi_t^\gamma(f)(z)|$. With ψ be an element of the basic constructed with respect to $W_{(j)}^\gamma / W_{(j')}^\gamma$, $\psi = Q_{i_1}^{j_1} \ldots Q_{i_n}^{j_n} q$, $q \in V_{N - d\sigma(j)}$. Then we have

$$|\psi(f)(z)| \leq C_2\|f(z)\|^{N-d\sigma(j)}|Q_{i_1}(f)(z)|^{j_1}\dots|Q_{i_n}(f)(z)|^{j_n},$$

then

$$\prod_{t=1}^{M}|\psi_t^{\gamma}(f)(z)| \leq C_3\|f(z)\|^{\sum_{(j)}\Delta_{(j)}^{\gamma}(N-d\sigma(j)}\prod_{k=1}^{n}|Q_{i_k}(f)(z)|^{\sum_{(j)}\Delta_{(j)}^{\gamma}j_k}, \qquad (3.3)$$

where C_2, C_3 are constants which depend only on N and the coefficients of $\{Q_i\}_{i=1}^q$ $\Big($the sum and product are taken over all n-tuples (i), such that $\sigma(i) \leq \dfrac{N}{d}\Big)$.

We fix $\phi_1,\dots,\phi_M$, a basic of V_N, $\psi_t^{\gamma}(f) = L_t^{\gamma}(F)$, where L_t^{γ} are linear forms and

$$F = (\phi_1(f):\dots:\phi_M(f)).$$

We set

$$b_k^{\gamma} = \sum_{(j)}\Delta_{(j)}^{\gamma}j_k, \quad 1 \leq j \leq n,$$

$$a^{\gamma} = \sum_{(j)}\Delta_{(j)}^{\gamma}(N-d\sigma(j)),$$

where the sums are taken over all n-tuples (j) such that $\sigma(j) \leq N/d$. We note that

$$a^{\gamma} + \sum_{k=1}^{n} db_k^{\gamma} = NM.$$

From (3.3) we have that

$$\log\prod_{t=1}^{M}|L_t^{\gamma}(F)(z)| \leq \log\left(\prod_{j=1}^{n}|Q_{i_j}(f)(z)|^{b_j^{\gamma}}\right) + \log\|f(z)\|^{a^{\gamma}} + C_4,$$

where C_4 is a constant which depends only on N and the coefficients of $\{Q_i\}_{i=1}^q$.

We set $b = \min_{k,\gamma} b_k^{\gamma}$. Because f is algebraically non degenerate over $\mathbf{C}$,

$$F = (\phi_1(f):\dots:\phi_M(f))$$

is linearly non degenerate over $\mathbf{C}$, then we have

$$W(\phi_i(f)) = \det\left(\frac{\partial^i(\phi_j(f))}{\partial z^i}\right)_{1\leq i,j\leq M} \not\equiv 0.$$

We also have

$$\log\frac{\|f(z)\|^{(q-n)db}|W(\phi_i(f))(z)|}{(\prod_{j=1}^{q}|Q_j(f)(z)|^b)\|f(z)\|^{(NM-ndb)}} \leq \log\frac{W(\phi_i(f))(z)|}{(\prod_{k=1}^{n}|Q_{i_k}(f)(z)|^b)\|f(z)\|^{(NM-ndb)}} \leq$$

$$\leq \log\frac{W(\phi_i(f))(z)|C_5}{(\prod_{k=1}^{n}|Q_{i_k}(f)(z)|^{b_k^{\gamma}})\|f(z)\|^{(NM-d\sum_{k=1}^{n}b_k^{\gamma})}} \leq$$

$$\leq \log \frac{W(\phi_i(f))(z)|C_3C_5}{\prod_{i=1}^{M}|\psi_i^{\gamma}(f)(z)|} \leq \log \frac{W(\psi_i^{\gamma}(f))(z)|C_6}{\prod_{i=1}^{M}|\psi_i^{\gamma}(f)(z)|}, \tag{3.4}$$

where C_5, C_6 are constants, which are depend only on N and $\{Q_i\}_{i=1}^{q}$.

From (3.4), for all $z \in \mathbf{C} \setminus I(f)$, which are not zero of $Q_i(f)$, $1 \leq i \leq q$, we have

$$\log \frac{\|f(z)\|^{(q-n)db}|W^{\alpha}(\phi_i^{\gamma}(f))(z)|}{\left(\prod_{j=1}^{q}|Q_j(f)(z)|^b\right)\|f(z)\|^{(NM-ndb)}} \leq \sum_{\gamma} \log^+ \left(\frac{W^{\alpha}(\psi_i^{\gamma}(f))(z)|C_6}{\prod_{i=1}^{M}|\psi_i^{\gamma}(f)(z)|}\right).$$

Integrating both sides of the above inequality over $\Gamma(r)$, we obtain

$$\left\| \left(q - \frac{NM}{db}\right) T_f(r) \leq \right.$$

$$\leq \sum_{j=1}^{q} \frac{1}{d} N(r, \mathrm{div} Q_j(f)) - \frac{1}{db} N(r, \mathrm{div}(W^{\alpha}(\phi_i(f)))) + O(\log^+(T_f(r))) + C \log r, \tag{3.5}$$

where C is a positive constant (may be depend on f and Q_i).

We now have some estimates. First,

$$M = \binom{N+n}{n} = \frac{(N+1)\ldots(N+n)}{1\ldots n}.$$

Second, since the number of nonnegative integer p-tuples with summation $\leq T$ is equal to the number of nonnegative integer $(p+1)$-tuples with summation exactly equal $T \in \mathbf{Z}$, which is $\begin{pmatrix} T+m \\ m \end{pmatrix}$, since the sum below is independent of k, we have that

$$b_k^{\gamma} = \sum_{\sigma(j) le N/d} \Delta_{(j)} j_k \geq \sum_{\sigma(j) \leq N/d-n} \Delta_{(j)} j_k =$$

$$= \sum_{\sigma(j) \leq N/d-n} d^n j_k = \frac{d^n}{n+1} \sum_{\sigma(j) \leq N/d-n} \sum_{k=1}^{n+1} j_k =$$

$$= \frac{d^n}{n+1} \sum_{\sigma(j) \leq N/d-n} \frac{N}{d} = \frac{d^n N}{(n+1)d} \binom{N/d}{n} = \frac{d^n N(N/d-1)\ldots(N/d-n)}{1\ldots(n+1)d}.$$

This implies that

$$\frac{NM}{db} \leq (n+1) \frac{(N+1)\cdots(N+n)}{(N-d)\cdots(N-nd)} \leq$$

$$\leq (n+1) \prod_{k=1}^{n} \frac{n+k}{N-(n+1)d+kd} \leq (n+1) \left(\frac{N+1}{N-nd}\right)^n.$$

We chose

$$N \geq nd + \left[\frac{n+1/d}{(1+\epsilon/(n+1))^{1/n}-1}\right] d.$$

Then N divisible by d and one gets

$$N \geq nd + \left[(n+1)^2(2^n-1)(d\epsilon)^{-1}\right] d \qquad \text{and} \qquad \left(q - \frac{NM}{db}\right) \geq (q-n-1-\epsilon).$$

Thus, from (3.5) we obtain

$$\|(q-n-1-\epsilon)T_f(r) \leq$$

$$\leq \sum_{j=1}^{q} \frac{1}{d} N(r, \mathrm{div} Q_j(f)) - \frac{1}{db} N(r, \mathrm{div}(W^{\alpha}(\phi_i(f)))) + O(\log^+(T_f(r))). \tag{3.6}$$

We now estimate $\left(\sum_{j=1}^{q} \mathrm{div}(Q_j(f)) - \frac{1}{b}\mathrm{div}(W(\phi_i(f)))\right)$. Fix $z \in \mathbf{C}^m$, we may assume that

$$\mathrm{div}(Q_{j_1}(f))(z) \geq \ldots \geq \mathrm{div}(Q_{j_k}(f))(z) > 0 = \mathrm{div}(Q_{j_{k+1}}(f))(z) = \ldots = \mathrm{div}(Q_{j_q}(f))(z),$$

where $0 \leq k \leq n$ (k may be zero). Put $\gamma = (Q_{j_1}, \ldots, Q_{j_n})$, then we have

$$\mathrm{div}(W(\phi_i(f)))(z) = \mathrm{div}(W(\psi_i^{\gamma}(f)))(z) \geq \sum_{t=1}^{M} \max\{\mathrm{div}(\psi_t^{\gamma}(f))(z) - M, 0\}.$$

For $\psi = Q_{j_1}^{i_1} \ldots Q_{j_n}^{i_n} q \in \{\psi_t^{\gamma}\}_{t=1}^{M}$, we have

$$\psi(f)(z) = Q_{j_1}^{i_1}(f)(z) \ldots Q_{j_n}^{i_n}(f)(z).q(f)(z).$$

Hence

$$\max\{\mathrm{div}(\psi(f))(z) - M, 0\} \geq \sum_{k=1}^{n} \max\{\mathrm{div}(Q_{j_k}^{i_k}(f))(z) - M, 0\} \geq$$

$$\geq \sum_{k=1}^{n} i_k \max\{\mathrm{div}(Q_{j_k}(f))(z) - M, 0\}.$$

This implies that

$$\sum_{t=1}^{M} \max\{\mathrm{div}(\psi_t(f))(z) - M, 0\} \geq \sum_{(i)} \Delta_{(i)}^{\gamma} \sum_{k=1}^{n} i_k \max\{\mathrm{div}(Q_{j_k}(f))(z) - M, 0\} =$$

$$= \sum_{k=1}^{n} b_k^{\gamma} \max\{\mathrm{div}(Q_{j_k}(f))(z) - M, 0\} \geq$$

$$\geq \sum_{k=1}^{n} b \max\{\mathrm{div}(Q_{j_k}(f))(z) - M, 0\}.$$

Hence

$$\sum_{j=1}^{q} \mathrm{div} Q_j(f)(z) - \frac{1}{b} \mathrm{div} W^{\alpha}(\phi_j(f))(z) \leq$$

$$\leq \sum_{j=1}^{q} (\mathrm{div} Q_j(f)(z) - \max\{\mathrm{div} Q_j(f)(z) - M, 0\}) =$$

$$= \sum_{j=1}^{q} \mathrm{div} Q_j(f)^{[M]}(z). \tag{3.7}$$

From (3.7) we obtain

$$\sum_{j=1}^{q} N(r, \mathrm{div} Q_j(f)) - \frac{1}{b} N(r, \mathrm{div}(W^{\alpha}(\phi_j(f)))) \leq \sum_{j=1}^{q} N^{(M)}(r, Q_j(f)). \tag{3.8}$$

Combining (3.6) and (3.8), we have

$$\Big\|(q - n - 1 - \epsilon) T_f(r) \leq \sum_{j=1}^{q} \frac{1}{d} N^{(M)}(r, Q_j(f)) + O(\log^+(T_f(r))).$$

One can be estimated that

$$M \leq \binom{N+n}{N} \leq N^n \leq \big(nd + [(n+1)^2(2^n - 1)(d\epsilon)^{-1}]d\big)^n \leq M_0.$$

Theorem 4 is proved.

4. Proof of Theorem 1. Let $f\colon \Delta^* \to V$ be a holomorphic curve into a complex projective algebraic variety V. We know the following characterization of a removable singularity (see [5]).

Lemma 1. *Let $f\colon \Delta^* \to V$ be as above and let $T_f(r)$ be a characteristic function with respect an ample line bundle over V. Then f extends at ∞ to a holomorphic curve $\tilde{f}$ from $\Delta = \Delta^* \cup \{\infty\}$ into V if and only if*

$$\liminf_{r\to\infty} T_f(r)/(\log r) < \infty.$$

***Proof of Theorem* 1.** For $0 < \epsilon < 1 - \sum_{i=1}^{n+2} \dfrac{M}{m_i d_i}$, it follows from Theorem 4 and the assumption that

$$\|(1-\epsilon) T_f(r) \leq \sum_{j=1}^{n+2} \frac{1}{d_i} N^{(M)}(r, \mathrm{div}(Q_i(f))) + O(\log T_f(r)) + O(\log r) \leq$$

$$\leq \sum_{j=1}^{n+2} \frac{M}{m_i d_i} N(r, \mathrm{div} Q_i(f)) + O(\log T_f(r)) + O(\log r) \leq$$

$$\leq \left(\sum_{j=1}^{n+2} \frac{M}{m_i d_i}\right) T_f(r) + O(\log T_f(r)) + O(\log r).$$

This implies that

$$\|T_f(r) = O(\log T_f(r)) + O(\log r).$$

Therefore

$$\liminf_{r\to+\infty} T_f(r)/(\log r) < +\infty.$$

By Lemma 1 we have the required extension of f.

Theorem 1 is proved.

5. Proof of Theorem 2. In order to prove Theorem 2, we need some following.

Definition 1 (Definition 3.1 [11]). *Let Ω be a hyperbolic domain and let M be a complete complex Hermitian manifold with metric ds^2_M. A holomorphic mapping $f(z)$ from Ω into M is said to be a normal holomorphic mapping from Ω into M if and only if there exists a positive constant C such that for all $z \in \Omega$ and all $\xi \in T_z(\Omega)$,*

$$ds^2_M(f(z), df(z)(\xi)) \le CK_\Omega(z, \xi),$$

where $df(z)$ is the mapping from $T_z(\Omega)$ into $T_{f(z)}(M)$ induced by f and K_Ω denotes the infinitesimal Kobayashi metric on Ω.

Lemma 2 (see [11]). *Let f be a holomorphic mapping from a bounded domain Ω in $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ such that for every sequence of holomorphic mappings $\varphi_k(z)$ from the unit disc U in $\mathbf{C}$ into Ω, the sequence $\{f \circ \varphi_k(z)\}_{k=1}^{\infty}$ from U into $\mathbf{P}^n(\mathbf{C})$ is a normal family on U. Then f is a normal holomorphic mapping from Ω into $\mathbf{P}^n(\mathbf{C})$.*

Theorem 5 (Theorem 3.1 [1], Theorem 2.5 [10]). *Let Ω be a domain in $\mathbf{C}^m$. Let M be a compact complex Hermitian space. Let $\mathcal{F} \subset \mathrm{Hol}(\Omega, M)$. Then the family $\mathcal{F}$ is not normal if and only if there exist sequences $\{pj\} \in \Omega$ with $\{pj\} \to p_0$, $(f_j) \subset \mathcal{F}$, $\{\rho_j\} \subset \mathbf{R}$ with $\rho_j > 0$ and $\{\rho_j\} \to 0$ such that*

$$g_j(\xi) := f_j(p_j + \rho_j \xi)$$

converges uniformly on compact subsets of $\mathbf{C}^m$ to a non-constant holomorphic map $g\colon \mathbf{C}^m \to M$.

***Proof of Theorem* 2.** For $z_0 \in S$, we take a relative compact subdomain Ω containing z_0 of D. It suffices to prove that f extends over $\Omega \setminus S$ to a holomorphic mapping.

Firstly, we shall prove that f is normal on $\Omega \setminus S$. Indeed, suppose that f is not normal on $\Omega \setminus S$, then there exists a sequence of holomorphic mappings $\{\varphi_i\colon U \to \Omega \setminus S\}_{j=1}^{\infty}$ such that $\{f \circ \varphi_j\}$ is not normal, where U denotes the unit disc in $\mathbf{C}$. By Lemma 2, we may assume that there exist sequences $\{p_j\} \in U$, $\{r_j\} \in \mathbf{R}$ with $r_j > 0$ and $r_j \searrow 0$, $p_j \to p_0 \in U$ such that $g_j(\xi) := f \circ \varphi_j(p_j + r_j\xi)$ converges uniformly on compact subsets of $\mathbf{C}$ to a non-constant holomorphic mapping g of $\mathbf{C}$ into $\mathbf{P}^n(\mathbf{C})$. Because $\Omega \setminus S$ is bounded, $\{\varphi_j\}$ is a normal family of holomorphic mappings. Hence, there exists a sub-sequence (again denoted by $\{\varphi_j\}$) of $\{\varphi_j\}$ which converges uniformly on compact subsets of U to a holomorphic mapping $\varphi\colon U \to \overline{\Omega}$. Then $\lim_{j\to\infty} \varphi_j(p_j + r_j\xi) = \varphi(p_0) \in \overline{\Omega}$. Since $f(z)$ does not intersect $Q_i(z)$, then g does not intersect $Q_i(\varphi(p_0))$ or $g(\mathbf{C})$ is included in $Q_i(\varphi(p_0))$ for all $0 \le i \le q-1$ by Hurwitz's theorem. Hence, there exists a subset I of $\{1, \ldots, q\}$ such that $g(\mathbf{C}) \subset (\cap_{i\in I} Q_i(\varphi(p_0)) \setminus \cup_{i\notin I} Q_i(\varphi(p_0))) \cap X$. By Corollary 1.4 [7], we have that the set $\cap_{i\in I} Q_i(\varphi(p_0)) \setminus \cup_{i\notin I} Q_i(\varphi(p_0))$ is hyperbolic imbedded into X. Then g must be constant. This is a contradiction. Hence, f is normal.

By the assumption of Theorem 2, $S \cap \Omega$ is an analytic subset of domain Ω with codimension one, whose singularities are normal crossings. Then f extends to a holomorphic mapping from Ω into $\mathbf{P}^n(\mathbf{C})$ by Theorem 2.3 in Joseph and Kwack [4].

Theorem 2 is proved.

Remark. Let f be a holomorphic mapping of a domain $D \setminus S$ into X, where D is a domain in $\mathbf{C}^m$, S is an analytic subset of co-dimension at least two of D and X is an irreducible subvariety of $\mathbf{P}^n(\mathbf{C})$. Let Q be a moving hypersurface of $\mathbf{P}^n(\mathbf{C})$ on D. Assume that f does not intersect Q on D, then f extends to a holomorphic mapping of D into X.

Indeed, by Corollary 3.3.44 [6], f extends to a meromorphic mapping of D into X (denoted again by f). It suffices to show that f is holomorphic on D.

Suppose that f is not holomorphic on D. We denote by I the indeterminancy locus of f which is a non empty analytic subset of codimension two of D.

It is easy to see that $I \subset \operatorname{Supp}(\operatorname{div} Q(f))$. Then $\operatorname{Supp}(\operatorname{div} Q(f))$ is a non empty analytic subset of codimension one of D. Therefore $\operatorname{Supp}(\operatorname{div} Q(f)) \cap (D \setminus S) \neq \varnothing$. This contradicts to the assumption that f does not intersect Q on $D \setminus S$. Hence f is holomorphic on D.

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