UDC 512.5

Xiaoqing Sun, Shangping Wang, Xiaoqin Shen, Jianghua Li
(School Sci., Xi'an Univ. Technology, China)

QUASI-UNIT REGULARITY AND QB-RINGS*

КВАЗІОДИНИЧНА РЕГУЛЯРНІСТЬ ТА QB-КІЛЬЦЯ

Some relations for quasiunit regular rings and QB-rings, as well as for pseudounit regular rings and QB_∞-rings, are obtained. In the first part of the paper, we prove that (an exchange ring R is a QB-ring) $\Leftrightarrow$ (whenever $x \in R$ is regular, there exists a quasiunit regular element $w \in R$ such that $x = xyx = xyw$ for some $y \in R$) $\Leftrightarrow$ (whenever $aR + bR = dR$ in R, there exists a quasiunit regular element $w \in R$ such that $a + bz = dw$ for some $z \in R$). Similarly, we also give necessary and sufficient conditions for QB_∞-rings in the second part of the paper.

Отримано деякі співвідношення для квазіодиничних регулярних кілець та QB-кілець, а також для псевдоодиничних регулярних кілець та QB_∞-кілець. У першій частині статті доведено, що (кільце R з властивістю заміни є QB-кільцем) $\Leftrightarrow$ (якщо $x \in R$ є регулярним, то існує квазіодиничний регулярний елемент $w \in R$ такий, що $x = xyx = xyw$ для деякого $y \in R$) $\Leftrightarrow$ (якщо $aR + bR = dR$ в R, то існує квазіодиничний регулярний елемент $w \in R$ такий, що $a + bz = dw$ для деякого $z \in R$). Аналогічним чином отримані необхідні та достатні умови для QB_∞-кілець наведено у другій частині статті.

1. Introduction. Let R be an associative ring with nonzero identity. Recall that a ring R is an exchange ring if for every right R-module A and any decomposition $A = M' \oplus N = \bigoplus_{i \in I} A_i$, where $M'_R \simeq R_R$ and the index set I is finite, there exist submodules $A'_i \subseteq A_i$ such that $A = M' \bigoplus (\bigoplus_{i \in I} A'_i)$ [8]. The class of exchange rings is large and includes all von Neumann regular rings, all π-regular rings and C^*-algebras of real rank zero [1] etc. The ring R is said to have stable range one provided that whenever $ax + b = 1$ in R, there exists $y \in R$ such that $a + by$ is a unit in R. An exchange ring R has stable range one if and only if whenever $x \in R$ is regular, there exists a unit-regular element $w \in R$ such that $x = xyx = xyw$ for some $y \in R$ if and only if whenever $aR + bR = dR$ in R, there exists a unit regular element $w \in R$ such that $a + bz = dw$ for some $z \in R$ [9]. Some necessary and sufficient conditions under which an exchange ring R has weakly stable range one are also proved.

Replacing invertibility with quasi-invertibility in stable range one Pere Ara discover a new class of rings, the QB-rings [2]. The ring R is a QB-ring provided whenever $aR + bR = R$ in R, there exists $y \in R$ such that $a + by$ is quasi-invertible in R. As well known, this definition is left-right symmetric. Replacing R_q^{-1} with R_∞^{-1} in the definition of QB-ring, we say that a ring is QB_∞-ring if whenever $aR + bR = R$ in R, there exists $y \in R$ such that $a + by \in R_\infty^{-1}$ [6].

In this paper, the definitions of quasi-unit regular and pseudo-unit regular are given. An element $x \in R$ is called quasi-unit regular (pseudo-unit regular) if there exists a quasi-invertible (pseudo-invertible) element $u \in R$ such that $x = xux$. The purpose of this article is to investigate the relations of quasi-unit regular and QB-rings, as well as pseudo-unit regular and QB_∞-rings. It is shown in Section 2 that an exchange ring R is a QB-ring if and only if whenever $x \in R$ is regular, there exists a quasi-unit regular element $w \in R$ such that $x = xyx = xyw$ for some $y \in R$ if and

*This paper is supported by National Nature Science Foundation of China (NSFC 61173192, 11101330) and Natural Science Foundation of Shaanxi Province (2011JQ1007) and Education Office Foundation of Shaanxi Province (2010JK728) and The Starting Research Fund from Xi'an University of Technology (108-211105).

only if for any regular $x \in R$ there exist a quasi-unit regular element $w \in R$ and an idempotent $e \in R$ such that $x = ew$ if and only if whenever $aR + bR = dR$ in R, there exists a quasi-unit regular element w such that $a + bz = dw$ for some $z \in R$. In Section 3, we extend these to QB_∞-ring. It is extended the results of Chen [7]. We prove that an exchange ring R is a QB_∞-ring if and only if whenever $x \in R$ is regular, there exists a pseudo-unit regular element $w \in R$ such that $x = xyx = xyw$ for some $y \in R$.

Throughout this paper, R denotes an associative ring with identity. We denote by R^{-1}, $E(R)$ the set of all units of R, the set of all idempotents in R, respectively. An element $x \in R$ is regular provided that $x = xyx$ for some $y \in R$, which is also commonly known as von Neumann regular.

2. Quasi-unit regular. Let us start by recalling the concept of quasi-invertibility. We say that elements x and y in a ring R are **centrally orthogonal** provided that $xRy = yRx = 0$, and we write $x \perp y$. An element u in an arbitrary ring R is said to be **quasi-invertible** if there exist elements a, b in R such that

$$(1 - ua) \perp (1 - bu). \tag{2.1}$$

The set of quasi-invertible elements in R will be denoted by R_q^{-1}. It is easily checked that $R^{-1}R_q^{-1} = R_q^{-1}$ and $R_q^{-1}R^{-1} = R_q^{-1}$.

If $u \in R_q^{-1}$, then we have the equation $(1 - ua)u(1 - bu) = 0$. Taking $v = a + b - aub$ this implies that $u = uvu$. By computation $1 - uv = (1 - ua)(1 - bu)$ and $1 - vu = (1 - au)(1 - ub)$, so that we have the relation $(1 - uv) \perp (1 - vu)$. We say in this situation that v is a **quasi-inverse** of u.

Definition 2.1. *Let R be a ring. An element $x \in R$ is **quasi-unit regular** if there exists a quasi-invertible element $u \in R$ such that $x = xux$. A ring R is quasi-unit regular if every element in R is quasi-unit regular.*

Lemma 2.1. *Let R be a ring and $x \in R$. Then the following are equivalent*:

(1) *x is quasi-unit regular*;

(2) *$x = xyx = xyu$, where $y, u \in R$ and $u \in R_q^{-1}$*;

(2′) *$x = xyx = uyx$, where $y, u \in R$ and $u \in R_q^{-1}$*;

(3) *$x = xyx = xyw$, where $y, w \in R$ and w is quasi-unit regular*;

(3′) *$x = xyx = wyx$, where $y, w \in R$ and w is quasi-unit regular.*

Proof. (1) $\Rightarrow$ (2). Since x is quasi-unit regular, there exists a quasi-invertible element $u \in R$ such that $x = xux$. Let $ux = e$ and $1 - xu = f$. Then $e,\ f \in E(R)$ and

$$euxu + uf = uxuxu + u(1 - xu) = u, \qquad e(uxu + uf) + (1 - e)uf = u.$$

Let $g = (1 - e)ufu_q^{-1}(1 - e)$ where u_q^{-1} is the quasi-inverse of u. Since $(1 - e)uf = (1 - e)u$, we have

$$g^2 = g,\ (1 - e)u = (1 - e)uu_q^{-1}(1 - e)u = g(1 - e)u = gu.$$

Therefore

$$u(x + fu_q^{-1}(1 - e))(1 - eufu_q^{-1}(1 - e))u = (ux + ufu_q^{-1}(1 - e))(1 - eufu_q^{-1}(1 - e))u =$$

$$= (e + ufu_q^{-1}(1 - e))(1 - eufu_q^{-1}(1 - e))u = (e(1 - eufu_q^{-1}(1 - e)) + ufu_q^{-1}(1 - e))u =$$

$$= (e + (1 - e)ufu_q^{-1}(1 - e))u = (e + g)u = u.$$

Let

$$v = (1 - eufu_q^{-1}(1-e))u = (1 + eufu_q^{-1}(1-e))^{-1}u, \qquad p = x + fu_q^{-1}(1-e).$$

Then $vpv = v$. Since $R^{-1}R_q^{-1} = R_q^{-1} = R_q^{-1}R^{-1}$, we have $v \in R_q^{-1}$.

Since $(1 - v_q^{-1}v)R(1 - vv_q^{-1}) = 0$, we have $(1 - v_q^{-1}v)p(1 - vv_q^{-1}) = 0$. Then $p = v_q^{-1} + 2p - v_q^{-1}vp - pvv_q^{-1} = v_q^{-1} + (1 - v_q^{-1}v)p + p(1 - vv_q^{-1})$. In view of Theorem 2.3 [2], we conclude that $p \in R_q^{-1}$. It is clear that

$$x = xux = xu(x + fu_q^{-1}(1-e)) = xup.$$

(2) $\Rightarrow$ (1). Suppose that $x = xyx = xyu$ where $u \in R_q^{-1}$. Let $z = yxy$. Then $x = xzx = xzu$ and $z = zxz$. Hence $z = z(x + (1 - xz)u)z$ where $x + (1 - xz)u = u \in R_q^{-1}$. That is, z is quasi-unit regular. It follows from (1) $\Rightarrow$ (2) that there exists a $p \in R_q^{-1}$ such that $z = zuz = zup$. Let $e = 1 - zx$ and $f = zu$. Then $e, f \in E(R)$ and

$$fpx(1-f) + e(1-f) = 1 - f, \qquad (1-f)e(1-f) = 1 - f.$$

Then

$$z + e(1-f)p = fp + e(1-f)p = (1 + fpx(1-f))^{-1}p \in R_q^{-1}.$$

It is clear that $x = x(z + e(1-f)p)x$ with $z + e(1-f)p \in R_q^{-1}$. Therefore, x is quasi-unit regular.

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (2). Let $x = xyx = xyw$ where w is quasi-unit regular. It follows from (1) $\Rightarrow$ (2), we have $w = ep$ where $e^2 = e$ and $p \in R_q^{-1}$. It follows from the equation $xy + (1 - xy) = 1$ we have $xyw + (1 - xy)w = w$. Since $x = xyw$, we have $x + (1 - xy)w = w$. Then $xy + (1 - xy)wy = wy$. Hence $wy + (1 - xy)(1 - wy) = 1$. It follows that $ewy(1-e) + (1-xy)(1-wy)(1-e) = 1 - e$. Consequently,

$$e + (1-xy)(1-wy)(1-e) = 1 - ewy(1-e) = (1 + ewy(1-e))^{-1}$$

is invertible in R. Let

$$u = w + (1-xy)(1-wy)(1-e)p = (e + (1-xy)(1-wy)(1-e))p.$$

Since $R^{-1}R_q^{-1} = R_q^{-1}$ and $R_q^{-1}R^{-1} = R_q^{-1}$, we have $u \in R_q^{-1}$. It is easy to check that $x = xyx = xyw = xyu$ where $u \in R_q^{-1}$.

Similarly, we can prove equivalences of (1), (2′), (3′).

Lemma 2.1 is proved.

***Corollary* 2.1.** *Let R be a ring and $x \in R$ be regular. Then the following are equivalent*:

(1) *x is quasi-unit regular*;

(2) *there exist some idempotent $e \in R$ and some quasi-invertible element $u \in R$ such that $x = eu$*;

(2′) *there exist some idempotent $e \in R$ and some quasi-invertible element $u \in R$ such that $x = ue$*;

(3) *there exist some idempotent $e \in R$ and some quasi-unit regular element $w \in R$ such that $x = ew$*;

(3′) *there exist some idempotent $e \in R$ and some quasi-unit regular element $w \in R$ such that $x = we$.*

Proof. $(1) \Rightarrow (2)$. It follows from $(1) \Rightarrow (2)$ of Lemma 2.1.

$(2) \Rightarrow (3)$. It is obvious.

$(3) \Rightarrow (1)$. Assume $x = xyx = ew$, where $e \in R$ is an idempotent and w is quasi-unit regular. Let $w = wuw$ where u is a quasi-invertible in R. Since $xy + (1 - xy) = 1$, we have $ewy + (1 - xy) = 1$. It follows that

$$ewy(1 - e) + (1 - xy)(1 - e) = 1 - e.$$

Then

$$v := e + (1 - xy)(1 - e) = 1 - ewy(1 - e) = (1 + ewy(1 - e))^{-1}$$

is a unit in R. Let

$$p = x + (1 - xy)(1 - e)w = (e + (1 - xy)(1 - e))w = vw = vwuw = vw(uv^{-1})vw.$$

Since $R^{-1}R_q^{-1} = R_q^{-1}$ and $R_q^{-1}R^{-1} = R_q^{-1}$, we have $uv^{-1} \in R_q^{-1}$. Then q is quasi-unit regular. It is easy to check that $x = xyx = xy(x + (1 - xy)(1 - e)w) = xyp$. The result follows from Lemma 2.1.

Similarly, we can prove equivalences of (1), (2′), (3′).

Corollary 2.1 is proved.

By the result of Theorem 8.4 [2], an exchange ring R is a QB-ring if and only if every regular element in R is quasi-unit regular. It follows from Lemma 2.1, we immediately have the following characterizations of exchange QB-ring.

Theorem 2.1. *Let R be an exchange ring. Then the following are equivalent*:

(1) *R is a QB-ring*;

(2) *whenever $x \in R$ is regular, there exists a $u \in R_q^{-1}$ such that $x = xyx = xyu$ for some $y \in R$*;

(2′) *whenever $x \in R$ is regular, there exists a $u \in R_q^{-1}$ such that $x = xyx = uyx$ for some $y \in R$*;

(3) *whenever $x \in R$ is regular, there exists a quasi-unit regular element $w \in R$ such that $x = xyx = xyw$ for some $y \in R$*;

(3′) *whenever $x \in R$ is regular, there exists a quasi-unit regular element $w \in R$ such that $x = xyx = wyx$ for some $y \in R$.*

By Theorem 2.1, an exchange ring R is a QB-ring if and only if whenever $x = xyx \in R$, there exists a quasi-invertible element $u \in R$ such that $x = xyu$ if and only if whenever $x = xyx \in R$, there exists a quasi-invertible element $u \in R$ such that $x = uyx$. The following theorem gives a common quasi-invertible element $u \in R$ such that $x = xyu = uyx$.

Theorem 2.2. *Let R be an exchange ring. Then the following are equivalent*:

(1) *R is a QB-ring*;

(2) *whenever $x = xyx$, there exists a quasi-invertible element $u \in R$ such that $x = xyu = uyx$*;

(3) *whenever $x = xyx$, there exists a quasi-invertible element $u \in R$ such that $xyu = uyx$.*

Proof. $(1) \Rightarrow (2)$. For any $x = xyx$ in R, we have $x = xzx$ and $z = zxz$ with $z = yxy$. By Theorem 8.4 [2], we have $z = zxz = zvz$ for some quasi-invertible element $v \in R$. Let

$$u = (1 - xz - vz)v(1 - zx - zv) = v - vzv + x.$$

Since $v \in R_q^{-1}$, there exist $a, b \in R$ such that $(1 - va) \perp (1 - bv)$. It is easily checked that $(1 - xz - vz)^2 = 1$ and $(1 - zx - zv)^2 = 1$. Then

$$(1-u(1-zx-zv)a(1-xz-vz))=(1-xz-vz)(1-va)(1-xz-vz),$$

$$(1-(1-zx-zv)b(1-xz-vz)u)=(1-zx-zv)(1-bv)(1-zx-zv).$$

Hence, $(1-u(1-zx-zv)a(1-xz-vz))\perp(1-(1-zx-zv)b(1-xz-vz)u)$. Therefore, u is quasi-invertible. It follows from

$$xzu=xzv-xzvzv+xzx=xzx=x,\qquad uzx=vzx-vzvzx+xzx=xzx=x$$

we obtain that $x=xyu=xzu=uzx=uyx$ with $u\in R_q^{-1}$.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). For any $x=xyx$ in R, there exists a quasi-invertible element $u\in R$ such that $xyu=uyx$. Define

$$\eta\colon\ xyR=xR\simeq yxR,\qquad r\in R,\quad \eta(xr)=yxr;$$

$$\alpha\colon\ (1-xy)R\to(1-yx)R,\qquad r\in R,\quad (1-xy)r\to(1-yx)u_q^{-1}(1-xy)r;$$

$$\beta\colon\ (1-yx)R\to(1-xy)R,\qquad r\in R,\quad (1-yx)r\to(1-xy)ur.$$

Since $(1-xy)u=u(1-yx)$, we easily check that α and β are right R-module homomorphisms. Define

$$\phi\colon\ R=xR\oplus(1-xy)R\to yxR\oplus(1-yx)R=R,$$

$$x_1\in xR,\qquad x_2\in(1-xy)R,\qquad \phi(x_1+x_2)=\eta(x_1)+\alpha(x_2);$$

$$\psi\colon\ R=yxR\oplus(1-yx)R\to xR\oplus(1-xy)R=R,$$

$$y_1\in yxR,\qquad y_2\in(1-yx)R,\qquad \psi(y_1+y_2)=\eta^{-1}(y_1)+\beta(y_2).$$

Then

$$(1-\psi\phi)(x_1+x_2)=x_2-(1-xy)u_q^{-1}ux_2=$$

$$=(1-xy)x_2-(1-xy)u_q^{-1}ux_2=(1-xy)(1-u_q^{-1}u)x_2$$

for any $x_1\in xR$, $x_2\in(1-xy)R$. On the other hand,

$$(1-\phi\psi)(y_1+y_2)=y_2-(1-yx)uu_q^{-1}y_2=(1-yx)(1-uu_q^{-1})y_2$$

for any $y_1\in yxR$, $y_2\in(1-yx)R$. Then we have ϕ is quasi-invertible such that $x=x\phi x$. Therefore R is a QB-ring.

Theorem 2.2 is proved.

Chen had shown that an exchange ring R is a QB-ring if and only if for any regular $x\in R$, there exist $e\in E(R)$ and $u\in R_q^{-1}$ such that $x=eu$ [5] (Theorem 5). Using Corollary 2.1, we have following corollary.

***Corollary* 2.2.** *Let R be an exchange ring. Then the following are equivalent*:

(1) *R is a QB-ring*;

(2) *whenever $x\in R$ is regular*, *there exists an idempotent $e\in R$ and a quasi-unit regular element $w\in R$ such that $x=ew$*;

(2$'$) *whenever $x\in R$ is regular*, *there exists an idempotent $e\in R$ and a quasi-unit regular element $w\in R$ such that $x=we$.*

Canfell showed that R has stable range one if and only if $aR + bR = R$ implies that there exists a unit $u \in R$ such that $a + by = du$ for some $y \in R$, by using the method of completion of diagrams [4] (Theorem 2.9). We generalize Canfell's result to QB-rings.

Proposition 2.1. *Let R be a ring. Then the following are equivalent*:

(1) *R is a QB-ring*;

(2) *whenever $aR + bR = R$, there exists some $z \in R$ such that $a + bz$ is quasi-invertible*;

(3) *whenever $aR + bR = dR$, there exists some quasi-invertible element $u \in R$ such that $a + bz = du$ for some $z \in R$.*

Proof. $(3) \Rightarrow (2) \Rightarrow (1)$ are obvious.

$(1) \Rightarrow (3)$. Let $aR + bR = dR$. Then $a, b \in dR$. Hence we may assume that $a = dr$ and $b = ds$ for some $r, s \in R$. Let $ax + by = d$. Equivalently we have $drx + dsy = d$. It follows that $dg = 0$ where $g = 1 - rx - sy$. Now from the fact that $rx + sy + g = 1$ we have there exists some $z' \in R$ such that $r + (sy + g)z' = u \in R_q^{-1}$. Hence

$$du = d(r + (sy + g)z') = a + byz' + dgz' = a + byz' = a + bz$$

where $z = yz'$.

Proposition 2.1 is proved.

In case R is an exchange ring. We even have the following more general result.

Theorem 2.3. *Let R be an exchange ring. Then the following are equivalent*:

(1) *R is a QB-ring*;

(2) *whenever $aR + bR = R$, there exists some quasi-unit regular element $w \in R$ such that $a + bz = w$ for some $z \in R$*;

(3) *whenever $aR + bR = dR$, there exists some quasi-unit regular element $w \in R$ such that $a + bz = dw$ for some $z \in R$.*

Proof. $(1) \Rightarrow (3)$. It follows from Proposition 2.1.

$(3) \Rightarrow (2)$. It is obvious.

$(2) \Rightarrow (1)$. Let $x = xyx$ for some $y \in R$. Since $xy + (1 - xy) = 1$. By assumptions we have $x + (1 - xy)z = w$ is quasi-unit regular for some $z \in R$. Hence

$$x = xyx = xy(w - (1 - xy)z) = xyw.$$

The conclusion follows from Theorem 2.1.

Theorem 2.3 is proved.

Following a similar route above we give the following characterizations of QB-ring.

Theorem 2.4. *Let R be an exchange ring. Then the following are equivalent*:

(1) *R is a QB-ring*;

(2) *whenever $aR+bR = R$, there exists a quasi-unit regular element $w \in R$ such that $aw+by = 1$ for some $y \in R$*;

(3) *whenever $aR + bR = R$, there exist quasi-unit regular elements w_1, $w_2 \in R$ such that $aw_1 + bw_2 = 1$*;

(4) *whenever $a_1R + \ldots + a_mR = R$, there exist quasi-unit regular elements $w_1, \ldots, w_m \in R$ such that $aw_1 + \ldots + a_mw_m = 1$, where $m \geq 2$*;

(5) *whenever $aR+bR = dR$, there exists a quasi-unit regular element $w \in R$ such that $aw+by = = d$ for some $y \in R$*;

(6) *whenever* $aR + bR = dR$, *there exist quasi-unit regular elements* w_1, $w_2 \in R$ *such that* $aw_1 + bw_2 = d$;

(7) *whenever* $a_1R + \cdots + a_mR = dR$, *there exist quasi-unit regular elements* $w_1, \ldots, w_m \in R$ *such that* $aw_1 + \ldots + a_mw_m = d$, *where* $m \geq 2$.

Proof. (7) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (2) are obvious.

(1) $\Rightarrow$ (7). Assume that $a_1R + \ldots + a_mR = dR$. Then $a_i \in dR$, $i = 1, \ldots, m$. Let $a_i = dt_i$, $i = 1, \ldots, m$. Obviously we have $dt_1x_1 + \ldots + dt_mx_m = d$ for some $x_i \in R$, $i = 1, \ldots, m$. It follows that $dg = 0$, where $g = 1 - (dt_1x_1 + \ldots + dt_mx_m)$. Since $t_1x_1 + \ldots + t_mx_m + g = 1$ we obtain that $t_1R + \ldots + t_mR + gR = R$. Note that R is an exchange ring, so there exist idempotent $e_i \in R$, $i = 1, \ldots, m$, and idempotent $f \in R$, where e_i and f are orthogonal satisfying $e_1 + \ldots + e_m + f = 1$ such that $e_i = t_iy_i$, $i = 1, \ldots, m$, and $f = gz$ for some y_i, $z \in R$, $i = 1, \ldots, m$. Let $w_i = y_ie_i$, $i = 1, \ldots, m$. Then $t_iw_i = t_iy_ie_i = e_i$ and $w_it_iw_i = y_ie_ie_i = y_ie_i = w_i$. Since R is a QB-ring, we have w_i is quasi-unit regular by Theorem 8.4 [2]. It follows from $t_1w_1 + \ldots + t_mw_m + gz = e_1 + \ldots + e_m + f = 1$ that $aw_1 + \ldots + a_mw_m = d(t_1w_1 + \ldots + t_mw_m + gz) = d$.

(2) $\Rightarrow$ (1). Let $x = xyx$ for some $y \in R$. Since $yx + (1 - yx) = 1$, we have $yR + (1 - yx)R = R$. By assumptions there exists a quasi-unit regular element $w \in R$ such that $yw + (1 - yx)z = 1$ for some $z \in R$. Hence $x = xyx = x(yw + (1 - yx)z) = xyw$. It follows from Theorem 2.1 that R is a QB-ring.

Theorem 2.4 is proved.

The following proposition may be viewed as a supplement of Theorem 2.4 in case $m = 1$, which also generalizes Theorem 4 [5].

Proposition 2.2. *Let* R *be an exchange ring. Then the following are equivalent*:

(1) R *is a* QB*-ring*;

(2) *whenever* $aR = bR$, *there exists a quasi-invertible element* $u \in R$ *such that* $b = au$;

(3) *whenever* $aR = bR$, *there exists a quasi-unit regular element* $w \in R$ *such that* $b = aw$.

Proof. (1) $\Rightarrow$ (2). Given $aR = bR$, then $a = bx$ and $b = ay$ for $x, y \in R$. From $xy + (1 - xy) = 1$, we have $z \in R$ such that $x + (1 - xy)z = u \in R_q^{-1}$. It is easy to verify that $bxy = b$. Then $a = bx = b(x + (1 - xy)z) = bu$.

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (1). Let $x = xyx$ for some $y \in R$. Since $xR = xyR$, we can find a quasi-unit regular element $w \in R$ such that $x = xyw$. Then $x = xyx = xyw$. It follows from Theorem 2.1 that R is a QB-ring.

Proposition 2.2 is proved.

***Corollary* 2.3.** *Let* R *be an exchange ring. Then the following are equivalent*:

(1) R *is a* QB*-ring*;

(2) *whenever* $\psi\colon aR \simeq bR$, *where* $a, b \in R$, *there exists a quasi-invertible element* $u \in R$ *such that* $\psi(a) = bu$;

(3) *whenever* $\psi\colon aR \simeq bR$, *where* $a, b \in R$, *there exists a quasi-unit regular element* $w \in R$ *such that* $\psi(a) = bw$.

Proof. (1) $\Rightarrow$ (2). If $\psi\colon aR \simeq bR$, then $b = \psi(ax)$ and $a = \psi^{-1}(by)$ for some $x, y \in R$. Then $b = \psi(ax) = \psi(\psi^{-1}(by)x) = by\psi(x)$. Since $y\psi(x) + (1 - y\psi(x)) = 1$ and R is a QB-ring, we have $y + (1 - y\psi(x))z = u \in R_q^{-1}$. Hence $\psi(a) = by = b(y + (1 - y\psi(x))z) = bu$.

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (1). It follows from Proposition 2.2.

Corollary 2.3 is proved.

The ideas of the following result come from Lemma 1.2 [3].

Proposition 2.3. *Let R be an exchange ring. Then the following are equivalent*:

(1) *R is a QB-ring*;

(2) *whenever $x = xyx$, there exists $a \in R$ such that $y - a$ is quasi-invertible and $1 - xa$ is invertible*;

(3) *whenever $x = xyx$, there exists $a \in R$ such that $x - a$ is quasi-unit regular and $1 - ya$ is invertible.*

Proof. (1) $\Rightarrow$ (2). Let $x = xyx$ for some $y \in R$. Since $yx + (1 - yx) = 1$ and R is a QB-ring, we have there exists some $z \in R$ such that $u\colon = y + (1 - yx)z$ is quasi-invertible. Let $a = -(1 - yx)z$. Then $y - a = u$. Moreover, since $x = xyx$, we have $1 - xa = 1 + x(1 - yx)z = 1$ is invertible.

(2) $\Rightarrow$ (3). Assume $x = xyx$. Let $z = yxy$. Obviously, $x = xzx$ and $z = zxz$. By assumption, there exists $a' \in R$ such that $u := x - a'$ is quasi-invertible and $1 - za'$ is invertible. Let $a = xya'$. Then

$$1 - ya = 1 - yxya' = 1 - za',\ x - a = xyx - xya' = xy(x - a') = eu,$$

where $e = xy$ is an idempotent and $u \in R_q^{-1}$. Hence $x - a$ is quasi-unit regular by Corollary 2.1.

(3) $\Rightarrow$ (1). For any $x = xyx$ in R, we have $x = xzx$ and $z = zxz$ with $z = yxy$. Then there exists $a' \in R$ such that $w := x - a'$ is quasi-unit regular and $u := 1 - za'$ is invertible. Hence

$$xyw = xy(x - a') = x - xya' = x - xyxya' = x - xza' = x(1 - za') = xu.$$

It follows that $x = xywu^{-1} = xyw'$ where $w' = wu^{-1}$. Assume that $w = wpw$, where p is the quasi-invertible in R. Then

$$w' = wu^{-1} = wpwu^{-1} = (wu^{-1})(up)(wu^{-1}) = w'(up)w',$$

where $up \in R^{-1}R_q^{-1} = R_q^{-1}$. Therefore, we have $x = xyx = xyw'$ with w' is quasi-unit regular. It follows from Theorem 2.1 that R is a QB-ring.

Proposition 2.3 is proved.

3. Pseudo-unit regular. Recall that two elements $x, y \in R$ are centrally orthogonal, denoted by $x \perp y$, if $xRy = 0 = yRx$. We say that two elements $x, y \in R$ are **pseudo-orthogonal**, denoted by $x \natural y$, if $RxRyR$ is nilpotent. Let $R_\infty^{-1} = \{u \in R \mid \exists\, a, b \in R \ \text{ such that } \ (1 - ua) \natural (1 - bu)\}$. It is also easily checked that $R^{-1}R_\infty^{-1} = R_\infty^{-1}$ and $R_\infty^{-1}R^{-1} = R_\infty^{-1}$.

A ring R is a QB_∞-ring provided that $aR + bR = R$ implies that there exists $y \in R$ such that $a + by \in R_\infty^{-1}$. Obviously, every QB-ring is a QB_∞-ring.

Definition 3.1. *Let R be a ring. An element $x \in R$ is* ***pseudo-unit regular*** *if there exists $u \in R_\infty^{-1}$ such that $x = xux$. A ring R is pseudo-unit regular if every element in R is pseudo-unit regular.*

Lemma 3.1. *Let R be a ring and $x \in R$. Then the following are equivalent*:

(1) *x is pseudo-unit regular*;

(2) *$x = xyx = xyu$, where $u, y \in R$ and $u \in R_\infty^{-1}$*;

(2′) *$x = xyx = uyx$, where $u, y \in R$ and $u \in R_\infty^{-1}$*;

(3) *$x = xyx = xyw$, where $w, y \in R$ and w is pseudo-unit regular*;

(3′) *$x = xyx = wyx$, where $w, y \in R$ and w is pseudo-unit regular.*

Proof. (1) ⇒ (2). Since x is pseudo-unit regular, there exists $u \in R_\infty^{-1}$ such that $x = xux$. Let $ux = e$ and $1 - xu = f$. Then $e^2 = uxux = ux = e$ and $f^2 = (1 - xu)(1 - xu) = 1 - xu = f$. Hence $euxu + uf = uxuxu + u(1 - xu) = u$ and $e(uxu + uf) + (1 - e)uf = u$. Since $u \in R_\infty^{-1}$, there exists $v \in R$ such that $(1 - uv)\natural(1 - vu)$ and $(R(u - uvu)R)^m = 0 = (R(v - vuv)R)^m$ for some $m \in N$ by Lemma 2.1 [6]. Let $g = (1 - e)ufv(1 - e)$. Since $(1 - e)uf = (1 - e)u$, we see that

$$(1 - e)ufv(1 - e))u = (1 - e)uvu - (1 - e)uveu = (1 - e)uvu - (1 - e)uvuxu =$$

$$= (1 - e)uvu + (1 - e)(u - uvu)xu - (1 - e)uxu =$$

$$= (1 - e)uvu - (1 - e)(u - uvu) + (1 - e)(u - uvu)xu.$$

As a result, $(1 - e)u \equiv (1 - e)ufv(1 - e)u \equiv gu \pmod{R(u - uvu)R}$. Similarly, we have

$$g^2 \equiv (1 - e)ufv(1 - e)ufv(1 - e) \equiv (1 - e)ufv(1 - e) \equiv g \pmod{R(u - uvu)R}.$$

Then

$$u(x + fv(1 - e))(1 - eufv(1 - e))u = (ux + ufv(1 - e))(1 - eufv(1 - e))u =$$

$$= (e + ufv(1 - e))(1 - eufv(1 - e))u = (e(1 - eufv(1 - e)) + ufv(1 - e))u =$$

$$= (e + (1 - e)ufv(1 - e))u = (e + g)u \equiv u (\operatorname{mod} R(u - uvu)R).$$

Let $p = x + fv(1 - e)$ and $q = (1 - eufv(1 - e))u = (1 + eufv(1 - e))^{-1}u$. Then $qpq = q$. Since $R^{-1}R_\infty^{-1} = R_\infty^{-1}$ and $R_\infty^{-1}R^{-1} = R_\infty^{-1}$, we have $q \in R_q^{-1}$. Hence $\bar{q}\bar{p}\bar{q} = \bar{q}$ in $R/R(u - uvu)R$. Since $\bar{q} \in (R/R(u - uvu)R)_\infty^{-1}$, there exist $\bar{a}, \bar{b} \in R/R(u - uvu)R$ such that $(\bar{1} - \bar{q}\bar{a})\natural(\bar{1} - \bar{b}\bar{q})$. It follows from $(\bar{1} - \bar{q}\bar{p}) = (\bar{1} - \bar{q}\bar{p})(\bar{1} - \bar{q}\bar{a})$ and $(\bar{1} - \bar{p}\bar{q}) = (\bar{1} - \bar{b}\bar{q})(\bar{1} - \bar{b}\bar{p})$ that $(\bar{1} - \bar{q}\bar{p})\natural(\bar{1} - \bar{p}\bar{q})$. Then $\bar{p} \in (R/R(u - uvu)R)_\infty^{-1}$. By Lemma 2.5 [6], $p \in R_\infty^{-1}$. Hence $x = xux = xu(x + fu_q^{-1}(1 - e)) = = xup$.

(2) ⇒ (1). Suppose that $x = xyx = xyu$ where $u \in R_\infty^{-1}$. Let $z = yxy$. Then $x = xzx = xzu$ and $z = zxz$. Hence $z = z(x + (1 - xz)u)z$ where $x + (1 - xz)u = u \in R_\infty^{-1}$. z is pseudo-unit regular. It follows from (1) ⇒ (2) that there exists a $p \in R_\infty^{-1}$ such that $z = zuz = zup$. Let $e = 1 - zx$ and $f = zu$. Then $e^2 = e$ and $f^2 = f$. It is easily checked that

$$fpx(1 - f) + e(1 - f) = 1 - f \quad \text{and} \quad (1 - f)e(1 - f) = 1 - f.$$

Then

$$z + e(1 - f)p = fp + e(1 - f)p = (1 + fwx(1 - f))^{-1}p \in R_\infty^{-1}.$$

It is clear that $x = x(z + e(1 - f)p)x$ with $z + e(1 - f)p \in R_\infty^{-1}$. Therefore, x is pseudo-unit regular.

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (2). Let $x = xyx = xyw$ where w is quasi-unit regular. It follows from $(1) \Rightarrow (2)$, we have $w = ep$ where $e^2 = e$ and $p \in R_\infty^{-1}$. It follows from the equation $xy + (1 - xy) = 1$ we obtain $xyw + (1 - xy)w = w$. Since $x = xyw$, we have $x + (1 - xy)w = w$. Then $xy + (1 - xy)wy = wy$. Hence $wy + (1 - xy)(1 - wy) = 1$. It follows that $ewy(1 - e) + (1 - xy)(1 - wy)(1 - e) = 1 - e$. Consequently,

$$e + (1 - xy)(1 - wy)(1 - e) = 1 - ewy(1 - e) = (1 + ewy(1 - e))^{-1}$$

is invertible in R. Let

$$u = w + (1 - xy)(1 - wy)(1 - e)p = (e + (1 - xy)(1 - wy)(1 - e))p.$$

Since $R^{-1}R_\infty^{-1} = R_\infty^{-1}$ and $R_\infty^{-1}R^{-1} = R_\infty^{-1}$, we have $u \in R_\infty^{-1}$. It is easy to check that $x = xyx = xyw = xyu$ where $u \in R_\infty^{-1}$.

Similarly, we can prove equivalences of (1), $(2')$, $(3')$.

Lemme 3.1 is proved.

***Corollary* 3.1.** *Let R be a ring and $x \in R$ be regular. Then the following are equivalent*:

(1) *x is pseudo-unit regular*;

(2) *there exist some idempotent $e \in R$ and some $u \in R_\infty^{-1}$ such that $x = eu$*;

(2′) *there exist some idempotent $e \in R$ and some $u \in R_\infty^{-1}$ such that $x = ue$*;

(3) *there exist some idempotent $e \in R$ and some pseudo-unit regular element $w \in R$ such that $x = ew$*;

(3′) *there exist some idempotent $e \in R$ and some pseudo-unit regular element $w \in R$ such that $x = we$.*

Proof. (1) $\Rightarrow$ (2). It follows from (1) $\Rightarrow$ (2) of Lemma 3.1.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). Assume $x = xyx = ew$, where $e \in R$ is an idempotent and w is pseudo-unit regular. Let $w = wuw$ where $u \in R_\infty^{-1}$. Since $xy + (1 - xy) = 1$, we have $ewy + (1 - xy) = 1$. It follows that $ewy(1 - e) + (1 - xy)(1 - e) = 1 - e$. Then

$$v := e + (1 - xy)(1 - e) = 1 - ewy(1 - e) = (1 + ewy(1 - e))^{-1}$$

is a unit in R. Let

$$p = x + (1 - xy)(1 - e)w = (e + (1 - xy)(1 - e))w = vw = vwuw = vw(uv^{-1})vw.$$

Since $R^{-1}R_\infty^{-1} = R_\infty^{-1}$ and $R_\infty^{-1}R^{-1} = R_\infty^{-1}$, we have $uv^{-1} \in R_\infty^{-1}$. Then q is pseudo-unit regular. It is easy to check that $x = xyx = xy(x + (1 - xy)(1 - e)w) = xyp$. The result follows from Lemma 3.1.

Similarly, we can prove equivalences of (1), (2′), (3′).

Corollary 3.1 is proved.

By the result of Theorem 2.1 [7], an exchange ring R is a QB_∞-ring if and only if every regular element in R is pseudo-unit regular. It follows from Lemma 3.1 and Corollary 3.1, we immediately have the following characterizations of exchange QB_∞-ring.

Theorem 3.1. *Let R be an exchange ring. Then the following are equivalent*:

(1) *R is a QB_∞-ring*;

(2) *whenever $x \in R$ is regular*, *there exists a $u \in R_\infty^{-1}$ such that $x = xyx = xyu$ for some $y \in R$*;

(2′) *whenever $x \in R$ is regular, there exists a $u \in R_\infty^{-1}$ such that $x = xyx = uyx$ for some $y \in R$*;

(3) *whenever $x \in R$ is regular, there exists a pseudo-unit regular element $w \in R$ such that $x = xyx = xyw$ for some $y \in R$*;

(3′) *whenever $x \in R$ is regular, there exists a pseudo-unit regular element $w \in R$ such that $x = xyx = wyx$ for some $y \in R$.*

By Lemma 3.1 and Theorem 3.1, the proof of Theorems 2.2, 2.3 and 2.4, Propositions 2.1, 2.2 and 2.3 could be similarly extended to QB_∞-ring.

1. *Ara P., Goodeal K. R., O'Meara K. C., Pardo E.* Separative cancellation for projective modules over exchange rings // Isr. J. Math. – 1998. – **105**. – P. 105 – 137.
2. *Ara P., Pedersen G. K., Pereva F.* An infinite analogue of rings with stable range one // J. Algebra. – 2000. – **230**. – P. 608 – 655.
3. *Camps R., Menal P.* Power-cancellation for artinian modulea // Communs Algebra. – 1991. – **19**. – P. 2081 – 2095.
4. *Canfell M. J.* Completions of diagrams by automorphism and Bass'first stable range condition // J. Algebra. – 1995. – **176**. – P. 480 – 513.
5. *Chen H.* On exchange QB-rings // Communs Algebra. – 2003. – **31**. – P. 831 – 841.
6. *Chen H.* On QB_∞-rings // Communs Algebra. – 2006. – **34**. – P. 2057 – 2068.
7. *Chen H.* On exchange QB_∞-rings // Alg. colloq. – 2007. – **14**. – P. 613 – 623.
8. *Warfield R. B.* (*Jr.*) Exchange rings and decompositions of modules // Math. Ann. – 1972. – **199**. – P. 31 – 36.
9. *Wei J.* Unit-regularity and stable range conditions // Communs Algebra. – 2005. – **33**. – P. 1937 – 1946.

Received 20.07.11,
after revision — 10.02.12