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A COMPANION OF DRAGOMIR'S GENERALIZATION OF OSTROWSKI'S INEQUALITY AND APPLICATIONS IN NUMERICAL INTEGRATION

АНАЛОГ УЗАГАЛЬНЕННЯ ДРАГОМІРА НЕРІВНОСТІ ОСТРОВСЬКОГО ТА ЗАСТОСУВАННЯ ДО ЧИСЕЛЬНОГО ІНТЕГРУВАННЯ

Some analogs of Dragomir's generalization of the Ostrowski integral inequality

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f(x)\right]-\int\limits_a^b f(t)dt\right|\leq$$

$$\leq\left[\frac{(b-a)^2}{4}\left(\lambda^2+(1-\lambda)^2\right)+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty$$

are established. Some sharp inequalities are proved. An application to a composite quadrature rule is provided.

Встановлено аналоги узагальнення Драгоміра інтегральної нерівності Островського

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f(x)\right]-\int\limits_a^b f(t)dt\right|\leq$$

$$\leq\left[\frac{(b-a)^2}{4}\left(\lambda^2+(1-\lambda)^2\right)+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty\,.$$

Отримано деякі точні нерівності. Наведено застосування до складеної квадратурної формули.

1. Introduction. In 1938, Ostrowski established a very interesting inequality for differentiable mappings with bounded derivatives, as follows:

Theorem 1. *Let $f\colon I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, the interior of the interval I, such that $f'\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'(x)|\leq M$. Then the following inequality:*

$$\left|f(x)-\frac{1}{b-a}\int\limits_a^b f(u)\,du\right|\leq M(b-a)\left[\frac{1}{4}+\frac{\left(x-\dfrac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

holds for all $x\in[a,b]$. The constant $\dfrac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

In [16], Dragomir, Cerone and Roumeliotis proved the following generalization of Ostrowski's inequality.

Theorem 2. *Let $f\colon[a,b]\to\mathbb{R}$ be a continuous on $[a,b]$, differentiable on (a,b) and whose derivative f' is bounded on (a,b). Denote $\|f'\|_\infty:=\sup_{t\in[a,b]}|f'(t)|<\infty$. Then*

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f(x)\right]-\int\limits_a^b f(t)dt\right|\leq$$

$$\leq\left[\frac{(b-a)^2}{4}\left(\lambda^2+(1-\lambda)^2\right)+\left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty \qquad (1.2)$$

for all $\lambda\in[0,1]$ *and* $a+\lambda\frac{b-a}{2}\leq x\leq b-\lambda\frac{b-a}{2}$.

Using (1.2), the authors obtained estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae. They also gave applications of the mentioned results in numerical integration and for special means. For recent results, generalizations and new inequalities of Hermite – Hadamard, Ostrowski and Simpson's type the reader may be refer to [1 – 20] and the references therein.

Motivated by [12], Dragomir in [14] has proved the following companion of the Ostrowski inequality:

Theorem 3. *Let* $f\colon[a,b]\to\mathbb{R}$ *be an absolutely continuous function on* $[a,b]$. *Then we have the inequalities*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int\limits_a^b f(t)\,dt\right|\leq$$

$$\leq\begin{cases}\left[\dfrac{1}{8}+2\left(\dfrac{x-\frac{3a+b}{4}}{b-a}\right)^2\right](b-a)\|f'\|_\infty, & f'\in L_\infty[a,b],\\[2ex] \dfrac{2^{1/q}}{(q+1)^{1/q}}\left[\left(\dfrac{x-a}{b-a}\right)^{q+1}+\left(\dfrac{\frac{a+b}{2}-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, & \\ \qquad p>1,\ \ \dfrac{1}{p}+\dfrac{1}{q}=1,\ \ \text{and}\ \ f'\in L_p[a,b], & \\[2ex] \left[\dfrac{1}{4}+\left|\dfrac{x-\frac{3a+b}{4}}{b-a}\right|\right]\|f'\|_{[a,b],1} & \end{cases} \qquad (1.3)$$

for all $x\in\left[a,\frac{a+b}{2}\right]$.

In [15], Dragomir established some inequalities for this companion for mappings of bounded variation.

Theorem 4. *Let* $f\colon[a,b]\to\mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$. *Then we have the inequalities*

$$\left|\frac{f(x)+f(a+b-x)}{2}-\frac{1}{b-a}\int\limits_a^b f(t)\,dt\right|\leq\left[\frac{1}{4}+\left|\frac{x-\frac{3a+b}{4}}{b-a}\right|\right]\cdot\bigvee_a^b(f) \qquad (1.4)$$

for any $x \in \left[a, \dfrac{a+b}{2}\right]$, *where* $\bigvee_a^b(f)$ *denotes the total variation of* f *on* $[a,b]$. *The constant* $1/4$ *is best possible.*

In [19], Liu introduced some companions of an Ostrowski type inequality for functions whose first derivative are absolutely continuous. In [9], Barnett, Dragomir and Gomma have proved some companions for the Ostrowski inequality and the generalized trapezoid inequality. Recently, Alomari [2] proved a companion inequality for differentiable mappings whose first derivatives are bounded.

In this paper, we prove a companion of Dragomir's generalization of Ostrowski's inequality (1.2). Namely, inequalities for mappings of bounded variation and for absolutely continuous mappings whose first derivatives are belong to $L_\infty[a,b]$ and to $L_p[a,b]$ are established.

2. The case when f is of bounded variation.

Theorem 5. *Let* $f\colon [a,b] \to \mathbb{R}$ *be a mapping of bounded variation on* $[a,b]$. *Then for all* $\lambda \in [0,1]$ *and* $a + \lambda\dfrac{b-a}{2} \le x \le \dfrac{a+b}{2}$, *we have the inequality*

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int\limits_a^b f(t)dt\right| \le$$

$$\le \begin{cases} \max\left\{\lambda\dfrac{b-a}{2}, \left(x - \dfrac{(2-\lambda)a+\lambda b}{2}\right), \left(\dfrac{a+b}{2} - x\right)\right\} \cdot \bigvee_a^b(f), \\ \dfrac{b-a}{2}\max\left\{\bigvee_a^x(f), \bigvee_x^{a+b-x}(f), \bigvee_{a+b-x}^b(f)\right\}, \end{cases} \tag{2.1}$$

where $\bigvee_a^b(f)$ *denotes to the total variation of* f *over* $[a,b]$. *The constant* $\dfrac{1}{2}$ *in the second inequality is the best possible in the sense that it cannot be replaced by a smaller one.*

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\int\limits_a^x \left(t - \left(a + \lambda\frac{b-a}{2}\right)\right) df(t) = \left(x - a - \lambda\frac{b-a}{2}\right) f(x) + \lambda\frac{b-a}{2} f(a) - \int\limits_a^x f(t)dt,$$

$$\int\limits_x^{a+b-x} \left(t - \frac{a+b}{2}\right) df(t) = \left(\frac{a+b}{2} - x\right)(f(x) + f(a+b-x)) - \int\limits_x^{a+b-x} f(t)dt,$$

and

$$\int\limits_{a+b-x}^b \left(t - \left(b - \lambda\frac{b-a}{2}\right)\right) df(t) =$$

$$= \lambda\frac{b-a}{2} f(b) + \left(x - a - \lambda\frac{b-a}{2}\right) f(a+b-x) - \int\limits_{a+b-x}^b f(t)dt.$$

Adding the above equalities, we get

$$\int\limits_a^b K(x,t)\,f'(t)dt = (b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\,\frac{f(x)+f(a+b-x)}{2}\right] - \int\limits_a^b f(t)dt,$$

where

$$K(x,t) = \begin{cases} t - \left(a + \lambda\dfrac{b-a}{2}\right), & t\in[a,x], \\ t - \dfrac{a+b}{2}, & t\in(x, a+b-x], \\ t - \left(b - \lambda\dfrac{b-a}{2}\right), & t\in(a+b-x, b], \end{cases}$$

for all $\lambda\in[0,1]$ and $a+\lambda\dfrac{b-a}{2} \le x \le \dfrac{a+b}{2}$.

Now, we use the fact that for a continuous function $p\colon [c,d]\to\mathbb{R}$ and a function $\nu : [c,d]\to\mathbb{R}$ of bounded variation, one has the inequality

$$\left|\int\limits_c^d p(t)d\nu(t)\right| \le \sup_{t\in[c,d]} |p(t)| \bigvee_a^b(\nu). \tag{2.2}$$

Applying the inequality (2.2) for $p(t) = K(x,t)$, as above and $\nu(t) = f(t)$, $t\in[a,b]$, we get

$$\left|\int\limits_a^b K(x,t)\,df(t)\right| \le \left|\int\limits_a^x K(x,t)\,df(t)\right| + \left|\int\limits_x^{a+b-x} K(x,t)\,df(t)\right| + \left|\int\limits_{a+b-x}^b K(x,t)\,df(t)\right| \le$$

$$\le \sup_{t\in[a,x]} |K(x,t)|\cdot\bigvee_a^x(f) + \sup_{t\in[x,a+b-x]} |K(x,t)|\cdot\bigvee_x^{a+b-x}(f) +$$

$$+ \sup_{t\in[a+b-x,b]} |K(x,t)|\cdot\bigvee_{a+b-x}^b(f) =$$

$$= \max\left\{\lambda\frac{b-a}{2}, \left(x - \frac{(2-\lambda)\,a+\lambda b}{2}\right)\right\}\cdot\bigvee_a^x(f) + \left(\frac{a+b}{2} - x\right)\cdot\bigvee_x^{a+b-x}(f) +$$

$$+ \max\left\{\lambda\frac{b-a}{2}, \left(x - \frac{(2-\lambda)\,a+\lambda b}{2}\right)\right\}\cdot\bigvee_{a+b-x}^b(f) := M(x).$$

Now, observe that

$$M(x) \le \max\left\{\lambda\frac{b-a}{2}, \left(x - \frac{(2-\lambda)\,a+\lambda b}{2}\right),\ \left(\frac{a+b}{2} - x\right)\right\}\times$$

$$\times\left[\bigvee_a^x(f) + \bigvee_x^{a+b-x}(f) + \bigvee_{a+b-x}^b(f)\right] =$$

$$= \max\left\{\lambda\frac{b-a}{2}, \left(x - \frac{(2-\lambda)\,a + \lambda b}{2}\right), \left(\frac{a+b}{2} - x\right)\right\} \cdot \bigvee_a^b (f),$$

which proves the first inequality in (2.1). Also,

$$M(x) \le \max\left\{\bigvee_a^x (f), \bigvee_x^{a+b-x} (f), \bigvee_{a+b-x}^b (f)\right\} \times$$

$$\times \left[\lambda\frac{b-a}{2} + \left(x - a - \lambda\frac{b-a}{2}\right) + \left(\frac{a+b}{2} - x\right)\right] =$$

$$= \frac{b-a}{2} \max\left\{\bigvee_a^x (f), \bigvee_x^{a+b-x} (f), \bigvee_{a+b-x}^b (f)\right\}$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$, thus the second inequality in (2.1) is proved. To prove that the constant $\frac{1}{2}$ in the second inequality is sharp, assume that the second inequality holds with constant $C > 0$, i.e.,

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)dt\right| \le$$

$$\le C\,(b-a)\cdot\max\left\{\bigvee_a^x (f), \bigvee_x^{a+b-x} (f), \bigvee_{a+b-x}^b (f)\right\} \tag{2.3}$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$. Consider the mapping

$$f(t) = \begin{cases} 0, & t \in (a,b), \\ 1, & t = a, b, \end{cases}$$

then for $x = a$ and $\lambda = 0$, we have $\int_a^b f(t)dt = 0$, $\bigvee_a^b (f) = 2$, making of use (2.3), we get

$$(b-a) \le 2C\,(b-a),$$

which gives $\frac{1}{2} \le C$ and thus $\frac{1}{2}$ is the best possible, which completes the proof.

***Remark* 1.** In Theorem 5, choose $\lambda = 0$, then we get

$$\left|(b-a)\frac{f(x)+f(a+b-x)}{2} - \int_a^b f(t)dt\right| \le$$

$$\le (x-a)\cdot\bigvee_a^x (f) + \left(\frac{a+b}{2} - x\right)\cdot\bigvee_x^{a+b-x} (f) + (x-a)\cdot\bigvee_{a+b-x}^b (f) \le$$

$$\leq \max\left\{(x-a), \left(\frac{a+b}{2}-x\right)\right\} \cdot \bigvee_a^b (f) = \left[\frac{1}{4}(b-a) + \left|x - \frac{3a+b}{4}\right|\right] \cdot \bigvee_a^b (f),$$

which gives (1.4).

***Corollary* 1.** *Let f as in Theorem 5, then we have*

$$\left|(b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)dt\right| \leq$$

$$\leq \frac{b-a}{2}\left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right] \cdot \bigvee_a^b (f) \tag{2.4}$$

for all $\lambda \in [0,1]$. The 'first' constant $\frac{1}{2}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. In Theorem 5, choose $x = \dfrac{a+b}{2}$, we get

$$\left|(b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)dt\right| \leq$$

$$\leq \max\left\{\lambda \frac{b-a}{2}, (1-\lambda)\frac{b-a}{2}\right\} \cdot \bigvee_a^b (f) =$$

$$= \left[\frac{b-a}{2} \cdot \max\{\lambda, (1-\lambda)\}\right] \cdot \bigvee_a^b (f) = \frac{1}{2}\left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right](b-a) \cdot \bigvee_a^b (f)$$

which proves the inequality (2.4). To prove that the constant $\frac{1}{2}$ is sharp, assume that the inequality (2.4) holds with constant $C > 0$, i.e.,

$$\left|(b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right)\right] - \int_a^b f(t)dt\right| \leq$$

$$\leq C\left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right](b-a) \cdot \bigvee_a^b (f) \tag{2.5}$$

for all $\lambda \in [0,1]$. Consider the mapping

$$f(t) = \begin{cases} 0, & t \in [a,b] \setminus \left\{\dfrac{a+b}{2}\right\}, \\ 1, & t = \dfrac{a+b}{2}, \end{cases}$$

then we have $\int_a^b f(t)dt = 0$, $\bigvee_a^b (f) = 2$, and choose $\lambda = 0$, making of use (2.5), we get

$$b-a \leq 2C\,(b-a)\,,$$

which gives $\dfrac{1}{2} \leq C$ and thus $\dfrac{1}{2}$ is the best possible, which completes the proof.

***Corollary* 2.** *In Corollary* 1, *if we choose*

(1) $\lambda = 0$, *then we get*

$$\left|(b-a)\,f\left(\frac{a+b}{2}\right) - \int\limits_a^b f(t)dt\right| \leq \frac{1}{2}\,(b-a)\cdot\bigvee_a^b (f)\,,$$

(2) $\lambda = \dfrac{1}{3}$, *then we get*

$$\left|\frac{b-a}{6}\left[f\,(a) + 4f\left(\frac{a+b}{2}\right) + f\,(b)\right] - \int\limits_a^b f(t)dt\right| \leq \frac{1}{3}\,(b-a)\cdot\bigvee_a^b (f)\,,$$

(3) $\lambda = \dfrac{1}{2}$, *then we get*

$$\left|\frac{b-a}{2}\left[\frac{f\,(a)+f\,(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \int\limits_a^b f(t)dt\right| \leq \frac{1}{4}\,(b-a)\cdot\bigvee_a^b (f)\,,$$

(4) $\lambda = 1$, *then we get*

$$\left|(b-a)\,\frac{f\,(a)+f\,(b)}{2} - \int\limits_a^b f(t)dt\right| \leq \frac{1}{2}\,(b-a)\cdot\bigvee_a^b (f)\,.$$

The constants $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}$ *and* $\dfrac{1}{2}$ *are the best possible.*

***Corollary* 3.** *In* (2.1), *choose* $\lambda = \dfrac{1}{4}$ *and* $x = \dfrac{2a+b}{3}$, *then we get the following* $3/8$-*Simpson's inequality*:

$$\left|\frac{b-a}{8}\left[f\,(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f\,(b)\right] - \int\limits_a^b f\,(t)\,dt\right| \leq$$

$$\leq \begin{cases} \dfrac{5b-a}{24}\cdot\bigvee_a^b (f)\,, \\ \dfrac{b-a}{2}\cdot\max\left\{\bigvee_a^{\frac{2a+b}{3}} (f)\,, \bigvee_{\frac{2a+b}{3}}^{\frac{a+2b}{3}} (f)\,, \bigvee_{\frac{a+2b}{3}}^b (f)\right\}. \end{cases} \tag{2.6}$$

3. The case when $f' \in L_\infty[a,b]$.

Theorem 6. *Let $f\colon I \subset \mathbb{R} \to \mathbb{R}$ be an absolutely continuous mapping on I°, the interior of the interval I, where $a, b \in I$ with $a < b$. If f' is bounded on $[a,b]$, i.e., $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$. Then the inequality*

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int\limits_a^b f(t)dt\right| \le$$

$$\le \left[\frac{(b-a)^2}{8}\left(2\lambda^2 + (1-\lambda)^2\right) + 2\left(x - \frac{(3-\lambda)a + (1+\lambda)b}{4}\right)^2\right]\left\|f'\right\|_\infty \tag{3.1}$$

holds, for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$.

Proof. Defining the mapping

$$K(x,t) = \begin{cases} t - \left(a + \lambda\frac{b-a}{2}\right), & t \in [a,x], \\ t - \frac{a+b}{2}, & t \in (x, a+b-x], \\ t - \left(b - \lambda\frac{b-a}{2}\right), & t \in (a+b-x, b], \end{cases} \tag{3.2}$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \le x \le \frac{a+b}{2}$.

Integrating by parts, we obtain

$$\int\limits_a^b K(x,t)\, f'(t)dt = (b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int\limits_a^b f(t)dt.$$

Since, f' is bounded, we can state that

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int\limits_a^b f(t)dt\right| \le$$

$$\le \int\limits_a^b |K(x,t)|\left|f'(t)\right| dt \le \left\|f'\right\|_\infty \int\limits_a^b |K(x,t)|\, dt.$$

Now, since

$$\int\limits_p^r |t-q|\, dt = \int\limits_p^q (q-t)\, dt + \int\limits_q^r (t-q)\, dt = \frac{(q-p)^2 + (r-q)^2}{2} =$$

$$= \frac{1}{4}(p-r)^2 + \left(q - \frac{r+p}{2}\right)^2 \tag{3.3}$$

for all r, p, q such that $p \leq q \leq r$. Then, we observe that

$$\int_a^x \left| t - \left(a + \lambda \frac{b-a}{2} \right) \right| dt = \frac{1}{4} (x-a)^2 + \left(\lambda \frac{b-a}{2} - \frac{x-a}{2} \right)^2,$$

$$\int_x^{a+b-x} \left| t - \frac{a+b}{2} \right| dt = \left(x - \frac{a+b}{2} \right)^2,$$

and

$$\int_{a+b-x}^{b} \left| t - \left(b - \lambda \frac{b-a}{2} \right) \right| dt = \frac{1}{4} (x-a)^2 + \left(\frac{x-a}{2} - \lambda \frac{b-a}{2} \right)^2.$$

Then, we have

$$\int_a^b |K(x,t)|\, dt = \frac{(x-a)^2 + ((x-a) - \lambda (b-a))^2}{2} + \left(x - \frac{a+b}{2} \right)^2 =$$

$$= \underbrace{\frac{1}{4}\lambda^2 (b-a)^2 + \left(x - \frac{(2-\lambda) a + \lambda b}{2} \right)^2}_{\text{by (3.3)}} + \left(x - \frac{a+b}{2} \right)^2 =$$

$$= \frac{\lambda^2}{4} (b-a)^2 + \underbrace{\frac{(1-\lambda)^2}{8} (b-a)^2 + 2 \left(x - \frac{(3-\lambda) a + (1+\lambda) b}{4} \right)^2}_{\text{by (3.3)}} =$$

$$= \frac{(b-a)^2}{8} \left(2\lambda^2 + (1-\lambda)^2 \right) + 2 \left(x - \frac{(3-\lambda) a + (1+\lambda) b}{4} \right)^2,$$

which gives that

$$\left| (b-a) \left[\lambda \frac{f(a) + f(b)}{2} + (1-\lambda) \frac{f(x) + f(a+b-x)}{2} \right] - \int_a^b f(t)dt \right| \leq$$

$$\leq \left[\frac{(b-a)^2}{8} \left(2\lambda^2 + (1-\lambda)^2 \right) + 2 \left(x - \frac{(3-\lambda) a + (1+\lambda) b}{4} \right)^2 \right] \|f'\|_\infty$$

for all $\lambda \in [0,1]$ and $a + \lambda \dfrac{b-a}{2} \leq x \leq \dfrac{a+b}{2}$, which gives the required result.

***Remark* 2.** In (3.1), choose $\lambda = 0$, then we have

$$\left| (b-a) \frac{f(x) + f(a+b-x)}{2} - \int_a^b f(t)dt \right| \leq$$

$$\leq \left[\frac{(b-a)^2}{8} + 2\left(x - \frac{3a+b}{4}\right)^2\right] \left\|f'\right\|_\infty,$$

which is equivalent to the first inequality in (1.3), and if we choose $x = \dfrac{3a+b}{4}$, then we have

$$\left|\frac{b-a}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \int\limits_a^b f(t)dt\right| \leq \frac{(b-a)^2}{8}\left\|f'\right\|_\infty.$$

***Corollary* 4.** *Let f as in Theorem* 6, *then we get*

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right)\right] - \int\limits_a^b f(t)dt\right| \leq$$

$$\leq \left(\lambda^2 + (1-\lambda)^2\right)\frac{(b-a)^2}{4}\left\|f'\right\|_\infty. \tag{3.4}$$

The constant $\dfrac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Proof. In the proof of Theorem 6, choose $x = \dfrac{a+b}{2}$ we get the required result. To show that $1/4$ is the best possible (3.4). Assume (3.4) holds with constant $C > 0$, i.e.,

$$\left|(b-a)\left[\lambda\frac{f(a)+f(b)}{2} + (1-\lambda) f\left(\frac{a+b}{2}\right)\right] - \int\limits_a^b f(t)dt\right| \leq$$

$$\leq C\left(\lambda^2 + (1-\lambda)^2\right)(b-a)^2 \cdot \left\|f'\right\|_\infty \tag{3.5}$$

for all $\lambda \in [0,1]$. Consider the function $f(t) = \left|t - \dfrac{a+b}{2}\right|$, $t \in [a,b]$, then $\displaystyle\int\limits_a^b f(t)dt = \frac{(b-a)^2}{4}$ and $\left\|f'\right\|_\infty = 1$. Using (3.5) with $\lambda = 1$, we get $\dfrac{1}{4} \leq C$, which shows that $1/4$ is the best possible, which completes the proof.

***Corollary* 5.** *In Corollary* 4, *if we choose*

(1) $\lambda = 0$, *then we get*

$$\left|(b-a) f\left(\frac{a+b}{2}\right) - \int\limits_a^b f(t)dt\right| \leq \frac{1}{4}(b-a)^2\left\|f'\right\|_\infty,$$

(2) $\lambda = \dfrac{1}{3}$, *then we get*

$$\left|\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \int\limits_a^b f(t)dt\right| \leq \frac{5}{36}(b-a)^2\left\|f'\right\|_\infty,$$

(3) $\lambda = \dfrac{1}{2}$, *then we get*

$$\left| \frac{b-a}{2} \left[\frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \int_a^b f(t)dt \right| \leq \frac{1}{8} (b-a)^2 \left\| f' \right\|_\infty,$$

(4) $\lambda = 1$, *then we get*

$$\left| (b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(t)dt \right| \leq \frac{1}{4} (b-a)^2 \left\| f' \right\|_\infty.$$

The constants $\frac{1}{4}$, $\frac{5}{36}$, $\frac{1}{8}$ *and* $\frac{1}{4}$ *are the best possible.*

***Corollary* 6.** *In* (3.1), *choose* $\lambda = \dfrac{1}{4}$ *and* $x = \dfrac{2a+b}{3}$, *then we get the following* $3/8$*-Simpson's inequality*

$$\left| \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \int_a^b f(t)\,dt \right| \leq$$

$$\leq \frac{25}{288} (b-a)^2 \cdot \left\| f' \right\|_\infty. \tag{3.6}$$

4. The case when $f' \in L_p[a,b]$.

Theorem 7. *Let* $f\colon I \subset \mathbb{R} \to \mathbb{R}$ *be an absolutely continuous mapping on* I°, *the interior of the interval* I, *where* $a, b \in I$ *with* $a < b$. *If* f' *is belong to* $L_p[a,b]$, $p > 1$. *Then we have the following inequality*:

$$\left| (b-a) \left[\lambda \frac{f(a) + f(b)}{2} + (1-\lambda) \frac{f(x) + f(a+b-x)}{2} \right] - \int_a^b f(t)dt \right| \leq$$

$$\leq \left(\frac{2}{q+1}\right)^{1/q} \left\| f' \right\|_p \left[\left(\lambda \frac{b-a}{2}\right)^{q+1} + \left(\frac{a+b}{2} - x\right)^{q+1} + \left(x - \frac{(2-\lambda)a + \lambda b}{2}\right)^{q+1} \right]^{1/q} \tag{4.1}$$

for all $\lambda \in [0,1]$, $a + \lambda \dfrac{b-a}{2} \leq x \leq \dfrac{a+b}{2}$, *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $p > 1$.

Proof. Using Hölder inequality, we have

$$\left| \frac{b-a}{2} \left[\lambda (f(a) + f(b)) + (1-\lambda)(f(x) + f(a+b-x)) \right] - \int_a^b f(t)dt \right| \leq$$

$$\leq \left(\int_a^b |K(x,t)|^q \, dt \right)^{1/q} \left(\int_a^b \left| f'(t) \right|^p dt \right)^{1/p} =$$

$$= \|f'\|_p \left[\int\limits_a^x \left| t - \left(a + \lambda \frac{b-a}{2} \right) \right|^q dt + \int\limits_x^{a+b-x} \left| t - \frac{a+b}{2} \right|^q dt + \int\limits_{a+b-x}^b \left| t - \left(b - \lambda \frac{b-a}{2} \right) \right|^q dt \right] =$$

$$= \left(\frac{2}{q+1} \right)^{1/q} \|f'\|_p \left[\left(\lambda \frac{b-a}{2} \right)^{q+1} + \left(\frac{a+b}{2} - x \right)^{q+1} + \left(x - \frac{(2-\lambda)\, a + \lambda b}{2} \right)^{q+1} \right]^{1/q}$$

for all $\lambda \in [0,1]$, $a + \lambda \dfrac{b-a}{2} \le x \le \dfrac{a+b}{2}$, and $\dfrac{1}{p} + \frac{1}{q} = 1$, $p > 1$.

***Remark* 3.** In Theorem 7, choose $\lambda = 0$, then we have

$$\left| (b-a) \frac{f(x) + f(a+b-x)}{2} - \int\limits_a^b f(t)dt \right| \le$$

$$\le \left(\frac{2}{q+1} \right)^{1/q} \|f'\|_p \left[\left(\frac{a+b}{2} - x \right)^{q+1} + (x-a)^{q+1} \right]^{1/q},$$

which is equivalent to the second inequality in (1.3), and if $x = \dfrac{3a+b}{4}$, then we have

$$\left| \frac{b-a}{2} \left[f\left(\frac{3a+b}{4} \right) + f\left(\frac{a+3b}{4} \right) \right] - \int\limits_a^b f(t)dt \right| \le \frac{(b-a)^{(q+1)/q}}{4\,(q+1)^{1/q}} \|f'\|_p .$$

***Corollary* 7.** *In Theorem* (7), *choose* $x = \dfrac{a+b}{2}$, *we get*

$$\left| (b-a) \left[\lambda \frac{f(a) + f(b)}{2} + (1-\lambda)\, f\left(\frac{a+b}{2} \right) \right] - \int\limits_a^b f(t)dt \right| \le$$

$$\le \frac{1}{2} \left(\frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{q+1} \right)^{1/q} (b-a)^{(q+1)/q} \|f'\|_p . \tag{4.2}$$

The constant $\dfrac{1}{2}$ *is the best possible in the sense that it cannot be replaced by a smaller one.*

Proof. In the proof of Theorem 7, choose $x = \dfrac{a+b}{2}$ we get the required result. To show that $1/2$ is the best possible (4.2). Assume (4.2) holds with constant $C > 0$, i.e.,

$$\left| (b-a) \left[\lambda \frac{f(a) + f(b)}{2} + (1-\lambda)\, f\left(\frac{a+b}{2} \right) \right] - \int\limits_a^b f(t)dt \right| \le$$

$$\le C \left(\frac{\lambda^{q+1} + (1-\lambda)^{q+1}}{q+1} \right)^{1/q} (b-a)^{\frac{q+1}{q}} \|f'\|_p \tag{4.3}$$

for all $\lambda \in [0,1]$. Consider the function $f(t) = \left| t - \frac{a+b}{2} \right|$, $t \in [a,b]$, then $\int_a^b f(t)dt = \frac{(b-a)^2}{4}$ and $\|f'\|_p = (b-a)^{1/p}$. Using (4.3) with $\lambda = 0$, we get

$$\frac{(b-a)^2}{4} \le C \frac{1}{(q+1)^{1/q}} (b-a)^{(q+1)/q} (b-a)^{1/p},$$

which gives

$$\frac{1}{4} \le \frac{C}{(q+1)^{1/q}}$$

for any $q > 1$. Letting $q \to 1^+$, we deduce that $C \ge \frac{1}{2}$, and the sharpness of the constant in (4.2) is proved, which completes the proof.

***Corollary* 8.** *In Corollary* 7, *if we choose*

(1) $\lambda = 0$, *then we get*

$$\left| (b-a) f\left(\frac{a+b}{2}\right) - \int\limits_a^b f(t)dt \right| \le \frac{(b-a)^{(q+1)/q}}{2(q+1)^{1/q}} \|f'\|_p,$$

(2) $\lambda = \frac{1}{3}$, *then we get*

$$\left| \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \int\limits_a^b f(t)dt \right| \le$$

$$\le \frac{1}{6} \left(\frac{1+2^{q+1}}{3(q+1)} \right)^{1/q} (b-a)^{(q+1)/q} \|f'\|_p,$$

(3) $\lambda = \frac{1}{2}$, *then we get*

$$\left| \frac{b-a}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right) \right] - \int\limits_a^b f(t)dt \right| \le \frac{(b-a)^{(q+1)/q}}{4(q+1)^{1/q}} \|f'\|_p,$$

(4) $\lambda = 1$, *then we get*

$$\left| b - a\frac{f(a)+f(b)}{2} - \int\limits_a^b f(t)dt \right| \le \frac{(b-a)^{(q+1)/q}}{2(q+1)^{1/q}} \|f'\|_p.$$

The constants $\frac{1}{2(q+1)^{1/q}}$, $\frac{1}{6}\left(\frac{1+2^{q+1}}{3(q+1)}\right)^{1/q}$, $\frac{1}{4(q+1)^{1/q}}$ *and* $\frac{1}{2(q+1)^{1/q}}$ *are the best possible.*

***Corollary* 9.** *In* (4.1)*, choose* $\lambda = \dfrac{1}{4}$ *and* $x = \dfrac{2a+b}{3}$, *then we get the following* $3/8$*-Simpson's inequality*:

$$\left| \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \int\limits_a^b f(t)\,dt \right| \leq$$

$$\leq \left(\frac{2}{q+1}\right)^{1/q} \left[\left(\frac{1}{8}\right)^{q+1} + \left(\frac{1}{6}\right)^{q+1} + \left(\frac{5}{24}\right)^{q+1} \right]^{1/q} (b-a)^{(q+1)/q} \left\|f'\right\|_p . \qquad (4.4)$$

5. A composite quadrature formula. Let $I_n \colon a = x_0 < x_1 < \ldots < x_n = b$ be a division of the interval $[a,b]$ and $h_i = x_{i+1} - x_i$, $i = 0, 1, 2, \ldots, n-1$.

Consider the general quadrature formula

$$Q_n(I_n, f) := \sum_{i=0}^{n-1} \frac{h_i}{2} \Big[\lambda \left(f(x_i) + f(x_{i+1}) \right) + (1-\lambda) \left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i) \right) \Big] \qquad (5.1)$$

for all $\lambda \in [0,1]$ and $x_i + \lambda \dfrac{x_{i+1} - x_i}{2} \leq \alpha_i \leq \dfrac{x_i + x_{i+1}}{2}$.

The following result holds.

Theorem 8. *Let* f *as in Theorem* 5, *then we have*

$$\int\limits_a^b f(t)dt = Q_n(I_n, f) + R_n(I_n, f),$$

where $Q_n(I_n, f)$ *is defined by formula* (5.1), *and the remainder satisfies the estimates*

$$|R_n(I_n, f)| \leq \begin{cases} \displaystyle\sum_{i=0}^{n-1} \max\left\{ \lambda \frac{h_i}{2}, \left(\alpha_i - \frac{(2-\lambda)x_i + \lambda x_{i+1}}{2} \right), \left(\frac{x_i + x_{i+1}}{2} - \alpha_i \right) \right\} \cdot \bigvee_{x_i}^{x_{i+1}}(f), \\ \displaystyle\sum_{i=0}^{n-1} \frac{h_i}{2} \cdot \max\left\{ \bigvee_{x_i}^{\alpha_i}(f), \bigvee_{\alpha_i}^{x_i + x_{i+1} - \alpha_i}(f), \bigvee_{x_i + x_{i+1} - \alpha_i}^{x_{i+1}}(f) \right\} \end{cases}$$

for all $\lambda \in [0,1]$ *and* $x_i + \lambda \dfrac{x_{i+1} - x_i}{2} \leq \alpha_i \leq \dfrac{x_i + x_{i+1}}{2}$.

Proof. Applying inequality (2.1) on the intervals $[x_i, x_{i+1}]$, we may state that

$$R_i(I_i, f) = \int\limits_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h_i}{2} \Big[\lambda \left(f(x_i) + f(x_{i+1}) \right) + (1-\lambda) \left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i) \right) \Big].$$

Summing the above inequality over i from 0 to $n-1$, we get

$$R_n(I_n, f) =$$

$$= \sum_{i=0}^{n-1} \int\limits_{x_i}^{x_{i+1}} f(t)dt - \sum_{i=0}^{n-1} \frac{h_i}{2} \left[\lambda \left(f(x_i) + f(x_{i+1}) \right) + (1-\lambda) \left(f(\alpha_i) + f(x_i + x_{i+1} - \alpha_i) \right) \right] =$$

$$= \int\limits_a^b f(t)dt - \sum_{i=0}^{n-1} \frac{h_i}{2}\Big[\lambda\left(f\left(x_i\right)+f\left(x_{i+1}\right)\right)+(1-\lambda)\left(f\left(\alpha_i\right)+f\left(x_i+x_{i+1}-\alpha_i\right)\right)\Big],$$

which follows form (2.1), that

$$|R_n\left(I_n,f\right)| =$$

$$= \left|\int\limits_a^b f(t)dt - \sum_{i=0}^{n-1} \frac{h_i}{2}\Big[\lambda\left(f\left(x_i\right)+f\left(x_{i+1}\right)\right)+(1-\lambda)\left(f\left(\alpha_i\right)+f\left(x_i+x_{i+1}-\alpha_i\right)\right)\Big]\right| \le$$

$$\le \begin{cases} \displaystyle\sum_{i=0}^{n-1} \max\left\{\lambda\frac{h_i}{2}, \left(\alpha_i - \frac{(2-\lambda)\,x_i+\lambda x_{i+1}}{2}\right), \left(\frac{x_i+x_{i+1}}{2}-\alpha_i\right)\right\}\cdot \bigvee_{x_i}^{x_{i+1}}(f), \\ \displaystyle\sum_{i=0}^{n-1} \frac{h_i}{2}\cdot\max\left\{\bigvee_{x_i}^{\alpha_i}(f), \bigvee_{\alpha_i}^{x_i+x_{i+1}-\alpha_i}(f), \bigvee_{x_i+x_{i+1}-\alpha_i}^{x_{i+1}}(f)\right\}, \end{cases}$$

which completes the proof.

Theorem 9. *Let f as in Theorem* 6, *then we have*

$$\int\limits_a^b f(t)dt = Q_n\left(I_n,f\right)+R_n\left(I_n,f\right),$$

where $Q_n\left(I_n,f\right)$ is defined by formula (5.1), *and the remainder satisfies the estimates*

$$|R_n\left(I_n,f\right)| \le$$

$$\le \left\|f'\right\|_\infty \sum_{i=0}^{n-1}\left[\frac{h_i^2}{8}\left(2\lambda^2+(1-\lambda)^2\right)+2\left(\alpha_i-\frac{(3-\lambda)\,x_i+(1+\lambda)\,x_{i+1}}{4}\right)^2\right]$$

for all $\lambda\in[0,1]$ and $x_i+\lambda\dfrac{x_{i+1}-x_i}{2}\le\alpha_i\le\dfrac{x_i+x_{i+1}}{2}$.

Proof. The proof is similar to that of Theorem 7, using Theorem 6. We shall omit the details.

Theorem 10. *Let f as in Theorem* 7, *then we have*

$$\int\limits_a^b f(t)dt = Q_n\left(I_n,f\right)+R_n\left(I_n,f\right),$$

where $Q_n\left(I_n,f\right)$ is defined by formula (5.1), *and the remainder satisfies the estimates*

$$|R_n\left(I_n,f\right)| \le \left(\frac{2}{q+1}\right)^{1/q}\left\|f'\right\|_p \times$$

$$\times\sum_{i=0}^{n-1}\left[\left(\lambda\frac{h_i}{2}\right)^{q+1}+\left(\frac{x_i+x_{i+1}}{2}-\alpha_i\right)^{q+1}+\left(\alpha_i-\frac{(2-\lambda)\,x_i+\lambda x_{i+1}}{2}\right)^{q+1}\right]^{1/q}$$

for all $\lambda\in[0,1]$ and $x_i+\lambda\dfrac{x_{i+1}-x_i}{2}\le\alpha_i\le\dfrac{x_i+x_{i+1}}{2}$.

Proof. The proof is similar to that of Theorem 8, using Theorem 8. We shall omit the details.

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