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ON THE SUMMABILITY OF DOUBLE WALSH – FOURIER SERIES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

ПРО СУМОВНІСТЬ ПОДВІЙНИХ РЯДІВ УОЛША – ФУР'Є ФУНКЦІЙ ОБМЕЖЕНОЇ УЗАГАЛЬНЕНОЇ ВАРІАЦІЇ

The convergence of Cesàro means of negative order of double Walsh – Fourier series of functions of bounded generalized variation is investigated.

Досліджується збіжність середніх Чезаро від'ємного порядку від подвійних рядів Уолша – Фур'є функцій обмеженої узагальненої варіації.

1. Classes of functions of bounded generalized variation. In 1881 Jordan [14] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation etc., see [2, 15, 25, 23]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [13].

Let f be a real and measurable function of two variable of period 2π with respect to each variable. Given intervals $\Delta = (a,b)$, $J = (c,d)$ and points x, y from $I := [0,1)$ we denote

$$f(\Delta, y) := f(b,y) - f(a,y), \qquad f(x,J) = f(x,d) - f(x,c)$$

and

$$f(\Delta, J) := f(a,c) - f(a,d) - f(b,c) + f(b,d).$$

Let $E = \{\Delta_i\}$ be a collection of nonoverlapping intervals from I ordered in arbitrary way and let Ω be the set of all such collections E.

For the sequence of positive numbers $\Lambda^1 = \{\lambda_n^1\}_{n=1}^{\infty}$, $\Lambda^2 = \{\lambda_n^2\}_{n=1}^{\infty}$ and $I^2 := [0,1)^2$ we denote

$$\Lambda^1 V_1(f; I^2) = \sup_y \sup_{E\in\Omega} \sum_i \frac{|f(\Delta_i, y)|}{\lambda_i^1} \qquad (E = \{\Delta_i\}),$$

$$\Lambda^2 V_2(f; I^2) = \sup_x \sup_{F\in\Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j^2} \qquad (F = \{J_j\}),$$

$$\left(\Lambda^1 \Lambda^2\right) V_{1,2}(f; I^2) = \sup_{F,\, E\in\Omega} \sum_i \sum_j \frac{|f(\Delta_i, J_j)|}{\lambda_i^1 \lambda_j^2}.$$

Definition 1. *We say that the function f has bounded $\left(\Lambda^1, \Lambda^2\right)$-variation on I^2 and write $f \in \left(\Lambda^1, \Lambda^2\right) BV\left(I^2\right)$, if*

$$\left(\Lambda^1, \Lambda^2\right) V(f; I^2) := \Lambda^1 V_1(f; I^2) + \Lambda^2 V_2(f; I^2) + \left(\Lambda^1 \Lambda^2\right) V_{1,2}(f; I^2) < \infty.$$

We say that the function f has bounded partial Λ-variation and write $f \in P\Lambda BV\left(I^2\right)$ if

$$P\Lambda BV(f; I^2) := \Lambda V_1(f; I^2) + \Lambda V_2(f; I^2) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$) the class $P\Lambda BV$ coincide with the PBV class of bounded partial variation introduced by Goginava [7]. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{\Delta_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$1 < \lambda_1 \leq \lambda_2 \leq \ldots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{1}$$

We also suppose that $\sum_{n=1}^{\infty} (1/\lambda_n) = +\infty$.

In the case when $\lambda_n = n$, $n = 1, 2, \ldots$, we say *harmonic variation* instead of Λ-variation and write H instead of Λ (HBV, $PHBV$, $HV(f)$, etc.).

The notion of Λ-variation was introduced by Waterman [23] in one dimensional case, by Sahakian [20] in two dimensional case. The notion of bounded partial Λ-variation ($P\Lambda BV$) was introduced by Goginava and Sahakian [11].

Definition 2. *We say that the function f is continuous in $\left(\Lambda^1, \Lambda^2\right)$-variation on I^2 and write $f \in C\left(\Lambda^1, \Lambda^2\right) V\left(I^2\right)$, if*

$$\lim_{n\to\infty} \Lambda_n^1 V_1\left(f; I^2\right) = \lim_{n\to\infty} \Lambda_n^2 V_2\left(f; I^2\right) = 0$$

and

$$\lim_{n\to\infty} \left(\Lambda_n^1, \Lambda^2\right) V_{1,2}\left(f; I^2\right) = \lim_{n\to\infty} \left(\Lambda^1, \Lambda_n^2\right) V_{1,2}\left(f; I^2\right) = 0,$$

where $\Lambda_n^i := \left\{\lambda_k^i\right\}_{k=n}^{\infty} = \left\{\lambda_{k+n}^i\right\}_{k=0}^{\infty}$, $i = 1, 2$.

2. Walsh function. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$. The set of all integers by $\mathbf{Z}$ and the set of dyadic rational numbers in the unit interval $I := [0, 1)$ by $\mathbf{Q}$. In particular, each element of $\mathbf{Q}$ has the form $\dfrac{p}{2^n}$ for some $p, n \in \mathbf{N}$, $0 \leq p \leq 2^n$. By a dyadic interval in I we mean one of the form $I_N^l := [l2^{-N}, (l+1)\, 2^{-N})$ for some $l \in \mathbf{N}$, $0 \leq l < 2^N$. Given $N \in \mathbf{N}$ and $x \in I$, let $I_N(x)$ denote a dyadic interval of length 2^{-N} which contains the point x. Denote $I_N := [0, 2^{-N})$ and $\overline{I}_N := I \backslash I_N$. Set $(i, j) \leq (n, m)$ if $i \leq n$ and $j \leq m$.

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2), \\ -1, & \text{if } x \in [1/2, 1), \end{cases} \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1.$$

Let $w_0, w_1, \ldots$ represent the Walsh functions, i.e., $w_0(x) = 1$ and if $k = 2^{n_1} + \ldots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \ldots > n_s$ then

$$w_k(x) = r_{n_1}(x) \ldots r_{n_s}(x).$$

The Walsh – Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that [12, 21]

$$D_{2^n}(x) = \begin{cases} 2^n, \text{if } x \in [0, 2^{-n}), \\ 0, \ \text{ if } x \in [2^{-n}, 1), \end{cases} \tag{2}$$

and

$$D_{2^n+m}(x) = D_{2^n}(x) + w_{2^n}(x) D_m(x), \qquad 0 \le m < 2^n. \tag{3}$$

It is well known that [21]

$$D_n(t) = w_n(t) \sum_{j=0}^{\infty} n_j w_{2^j}(t) D_{2^j}(t), \tag{4}$$

where $n = \sum_{j=0}^{\infty} n_j 2^j$. Denote for $n \in \mathbf{P}$, $|n| := \max\{j \in \mathbf{N} \colon n_j \neq 0\}$, that is $2^{|n|} \le n < 2^{|n|+1}$.

Given $x \in I$, the expansion

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{5}$$

where each $x_k = 0$ or 1, will be called a dyadic expansion of x. If $x \in I \backslash \mathbf{Q}$, then (5) is uniquely determined. For the dyadic expansion $x \in \mathbf{Q}$ we choose the one for which $\lim_{k \to \infty} x_k = 0$.

The dyadic sum of $x, y \in I$ in terms of the dyadic expansion of x and y is defined by

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| \, 2^{-(k+1)}.$$

We say that $f(x, y)$ is continuous at (x, y) if

$$\lim_{h, \delta \to 0} f(x \dotplus h, y \dotplus \delta) = f(x, y). \tag{6}$$

Set

$$\omega(f; I_M(x) \times I_N(y)) := \sup_{(s,t) \in I_M \times I_N} |f(x \dotplus s, y \dotplus t) - f(x, y)|.$$

We consider the double system $\{w_n(x) \times w_m(y) \colon n, m \in \mathbf{N}\}$ on the unit square $I^2 = [0, 1) \times \times [0, 1)$.

If $f \in L^1(I^2)$, then

$$\hat{f}(n, m) = \int_{I^2} f(x, y) w_n(x) w_m(y) dx dy$$

is the (n, m)-th Walsh – Fourier coefficient of f.

The rectangular partial sums of double Fourier series with respect to the Walsh system are defined by

$$S_{M,N}f(x,y)=\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\hat{f}\left(m,n\right)w_m(x)w_n(y).$$

The Cesàro $(C;\alpha,\beta)$-means of double Walsh – Fourier series are defined as follows:

$$\sigma_{n,m}^{\alpha,\beta}f\left(x,y\right)=\frac{1}{A_{n-1}^{\alpha}A_{m-1}^{\beta}}\sum_{i=1}^{n}\sum_{j=1}^{m}A_{n-i}^{\alpha-1}A_{m-j}^{\beta-1}S_{i,j}f\left(x,y\right),$$

where

$$A_0^{\alpha}=1,\quad A_n^{\alpha}=\frac{(\alpha+1)\ldots(\alpha+n)}{n!},\quad \alpha\neq-1,-2,\ldots.$$

It is well-known that [27]

$$A_n^{\alpha}=\sum_{k=0}^{n}A_{n-k}^{\alpha-1},\tag{7}$$

$$A_n^{\alpha}\sim n^{\alpha}\tag{8}$$

and

$$\sigma_{n,m}^{\alpha,\beta}f\left(x,y\right)=\int\limits_{I^2}f\left(s,t\right)K_n^{\alpha}\left(x\dotplus s\right)K_m^{\beta}\left(y\dotplus t\right)dsdt,$$

where

$$K_n^{\alpha}\left(x\right):=\frac{1}{A_{n-1}^{\alpha}}\sum_{k=1}^{n}A_{n-k}^{\alpha-1}D_k\left(x\right).$$

Given a function $f\left(x,y\right)$, periodic in both variables with period 1, for $0\leq j<2^m$ and $0\leq i<$ $<2^n$ and integers $m,\,n\geq0$ we set

$$\Delta_j^m f\left(x,y\right)_1=f\left(x\dotplus 2j2^{-m-1},y\right)-f\left(x\dotplus(2j+1)\,2^{-m-1},y\right),$$

$$\Delta_i^n f\left(x,y\right)_2=f\left(x,y\dotplus 2i2^{-n-1}\right)-f\left(x,y\dotplus(2i+1)\,2^{-n-1}\right),$$

$$\Delta_{ji}^{mn}f\left(x,y\right)=\Delta_i^n\left(\Delta_j^m f\left(x,y\right)_1\right)_2=\Delta_j^m\left(\Delta_i^n f\left(x,y\right)_2\right)_1=$$

$$=f\left(x\dotplus 2j2^{-m-1},y\dotplus 2i2^{-n-1}\right)-f\left(x\dotplus(2j+1)\,2^{-m-1},y\dotplus 2i2^{-n-1}\right)-$$

$$-f\left(x\dotplus 2j2^{-m-1},y\dotplus(2i+1)\,2^{-n-1}\right)+$$

$$+f\left(x\dotplus(2j+1)\,2^{-m-1},y\dotplus(2i+1)\,2^{-n-1}\right).$$

3. Formulation of problems. The well known Dirichlet – Jordan theorem (see [27]) states that the Fourier series of a function $f(x),\ x\in T$ of bounded variation converges at every point x to the value $\left[f\left(x+0\right)+f\left(x-0\right)\right]/2$.

Hardy [13] generalized the Dirichlet – Jordan theorem to the double Fourier series. He proved that if function $f(x,y)$ has bounded variation in the sense of Hardy ($f \in BV$), then $S[f]$ converges at any point (x,y) to the value $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$.

Convergence of rectangular and spherical partial sums of d-dimensional trigonometric Fourier series of functions of bounded Λ-variation was investigated in details by Sahakian [20], Dyachenko [4, 5, 6], Bakhvalov [1], Sablin [19].

For the two-dimensional Walsh – Fourier series the convergence of partial sums of functions Harmonic bounded fluctuation and other bounded generalized variation were studied by Moricz [16, 17], Onnewer, Waterman [18], Waterman [24], Goginava [8, 9].

For the two-dimensional Walsh – Fourier series the summability by Cesáro method of negative order for functions of partial bounded variation investigated by the author.

Theorem G1 (Goginava [10]). *Let* $f \in C_w(I^2) \cap PBV$ *and* $\alpha + \beta < 1$, α, $\beta > 0$. *Then the double Walsh – Fourier series of the function* f *is uniformly* $(C; -\alpha, -\beta)$ *summable in the sense of Pringsheim.*

Theorem G2 (Goginava [10]). *Let* $\alpha + \beta \geq 1$, α, $\beta > 0$. *Then there exists a continuous function* $f_0 \in PBV$ *such that the Cesáro* $(C; -\alpha, -\beta)$ *means* $\sigma_{n,n}^{-\alpha,-\beta} f_0(0,0)$ *of the double Walsh – Fourier series of* f_0 *diverges.*

In this paper we consider the convergence of Cesáro means of negative order of double Walsh – Fourier series of functions from the classes $C\left(\left\{i^{1-\alpha}\right\}, \left\{i^{1-\beta}\right\}\right) V\left(I^2\right)$ (see Theorem 1) . We also consider the following problem: *Let* α, $\beta \in (0,1)$, $\alpha + \beta < 1$. Under what conditions on the sequence $\Lambda = \{\lambda_n\}$ the double Walsh – Fourier series of the function $f \in P\Lambda BV$ is $(C; -\alpha, -\beta)$ summable. The solution is given in Theorem 2 bellow.

4. Main results. The main results of this paper are presented in the following propositions:

Theorem 1. *Let* $f \in C\left(\left\{i^{1-\alpha}\right\}, \left\{i^{1-\beta}\right\}\right) V\left(I^2\right)$, $\alpha, \beta \in (0,1)$. *Then* $(C, -\alpha, -\beta)$*-means of double Walsh – Fourier series converges to* $f(x,y)$*, if* f *is continuous at* (x,y).

Theorem 2. *Let* $\Lambda = \{\lambda_n \colon n \geq 1\}$, $\alpha + \beta < 1$, α, $\beta > 0$, $\frac{\lambda_n}{n^{1-(\alpha+\beta)}} \downarrow 0$ *and* $f \in P\Lambda BV\left(I^2\right)$.

a) *If*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^{2-(\alpha+\beta)}} < \infty,$$

then $(C; -\alpha, -\beta)$*-means of double Walsh – Fourier series converges to* $f(x,y)$*, if* f *is continuous at* (x,y).

b) *If*

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{n^{2-(\alpha+\beta)}} = \infty,$$

then there exists a continuous function $f \in P\Lambda BV\left(I^2\right)$ *for which* $\sigma_{2^n,2^n}^{-\alpha,-\beta} f(0,0)$ *diverges.*

***Corollary* 1.** *Let* $\alpha, \beta \in (0,1)$, $\alpha + \beta < 1$.

a) *If* $f \in P\left\{\dfrac{n^{1-(\alpha+\beta)}}{\log^{1+\varepsilon}(n+1)}\right\}BV(I^2)$ *for some* $\varepsilon > 0$, *then the double Walsh – Fourier series of the function* f *is* $(C; -\alpha, -\beta)$ *summable to* $f(x,y)$, *if* f *is continuous at* (x,y).

b) *There exists a continuous function* $f \in P\left\{\dfrac{n^{1-(\alpha+\beta)}}{\log(n+1)}\right\}BV(I^2)$ *such that* $\sigma_{2^n,2^n}^{-\alpha,-\beta} f(0,0)$ *diverges.*

***Corollary* 2.** *Let* $\alpha,\ \beta \in (0,1)$, $\alpha+\beta < 1$ *and* $f \in PBV\left(I^2\right)$. *Then the double Walsh – Fourier series of the function* f *is* $(C; -\alpha, -\beta)$ *summable to* $f(x,y)$, *if* f *is continuous at* (x,y).

5. Auxiliary results.

Lemma 1. *Let* $\alpha \in (0,1)$ *and* $n := 2^{n_1} + 2^{n_2} + \ldots + 2^{n_r}$, $n_1 > n_2 > \ldots > n_r \geq 0$. *Then*

$$\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) = \sum_{l=1}^{r-1}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) D_{2^{n_l}}(x)\, A_{n^{(l-1)}-1}^{-\alpha} -$$

$$-\sum_{l=1}^{r}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x).$$

Proof. Set

$$n^{(k)} := 2^{n_{k+1}} + 2^{n_{k+2}} + \ldots + 2^{n_r}, \quad n_{k+1} > n_{k+2} > \ldots > n_r \geq 0, \quad n^{(0)} := n.$$

Then from (3) and (7) can write

$$\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) = \sum_{j=1}^{2^{n_1}} A_{n-j}^{-\alpha-1} D_j(x) + \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_{j+2^{n_1}}(x) =$$

$$= \sum_{j=1}^{2^{n_1}} A_{n-j}^{-\alpha-1} D_j(x) + D_{2^{n_1}}(x)\, A_{n^{(1)}-1}^{-\alpha} + w_{2^{n_1}}(x) \sum_{j=1}^{n^{(1)}} A_{n^{(1)}-j}^{-\alpha-1} D_j(x).$$

Iterating this equality gives

$$\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) =$$

$$= \sum_{l=1}^{r}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) \sum_{j=1}^{2^{n_l}} A_{n^{(l-1)}-j}^{-\alpha-1} D_j(x) + \sum_{l=1}^{r-1}\left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x)\right) A_{n^{(l)}-1}^{-\alpha} D_{2^{n_l}}(x). \qquad (9)$$

Since

$$D_{2^n-l}(x) = D_{2^n}(x) - w_{2^n-1}(x)\, D_l(x), \qquad l = 0, 1, \ldots, 2^n - 1,$$

we can write

$$\sum_{j=1}^{2^{n_l}} A_{n^{(l-1)}-j}^{-\alpha-1} D_j(x) = \sum_{j=1}^{2^{n_l}} A_{n^{(l)}+2^{n_l}-j}^{-\alpha-1} D_j(x) = \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_{2^{n_l}-j}(x) =$$

$$= D_{2^{n_l}}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} - w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x), \qquad l = 1, 2, ..., r. \tag{10}$$

Combining (9) and (10) we obtain

$$\sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) = \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) D_{2^{n_l}}(x) A_{n^{(l-1)}}^{-\alpha} -$$

$$- \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x) -$$

$$- \left(\prod_{k=1}^{r-1} w_{2^{n_k}}(x) \right) D_{2^{n_r}}(x) A_{n^{(r-1)}-1}^{-\alpha} = \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) D_{2^{n_l}}(x) A_{n^{(l-1)}-1}^{-\alpha} -$$

$$- \sum_{l=1}^{r} \left(\prod_{k=1}^{l-1} w_{2^{n_k}}(x) \right) w_{2^{n_l}-1}(x) \sum_{j=0}^{2^{n_l}-1} A_{n^{(l)}+j}^{-\alpha-1} D_j(x).$$

Lemma 1 is proved.

Lemma 2. *Let* $\alpha \in (0, 1)$. *Then*

$$\left| K_n^{-\alpha}(x) \right| \le \frac{c(\alpha)}{A_{n-1}^{-\alpha}} \sum_{l=0}^{|n|} 2^{-l\alpha} D_{2^l}(x).$$

Proof. From Lemma 1 we can write

$$\left| \sum_{j=1}^{n} A_{n-j}^{-\alpha-1} D_j(x) \right| \le \sum_{l=1}^{r} D_{2^{n_l}}(x) A_{n^{(l-1)}}^{-\alpha} +$$

$$+ \sum_{k=1}^{r} \sum_{j=1}^{2^{n_k}-1} \left| A_{n^{(k)}+j}^{-\alpha-1} \right| |D_j(x)| := B_1 + B_2. \tag{11}$$

From (8) we have

$$B_1 \le c(\alpha) \sum_{l=0}^{|n|} 2^{-l\alpha} D_{2^l}(x). \tag{12}$$

For B_2 we can write

$$B_2=\sum_{k=1}^{r}\sum_{m=1}^{n_k}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(k)}+j}^{-\alpha-1}\right|\left|D_j\left(x\right)\right| \quad =$$

$$=\sum_{k=1}^{r}\sum_{m=1}^{n_{k+1}}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(k)}+j}^{-\alpha-1}\right|\left|D_j\left(x\right)\right|+\sum_{k=1}^{r}\sum_{m=n_{k+1}+1}^{n_k}\sum_{j=2^{m-1}}^{2^m-1}\left|A_{n^{(k)}+j}^{-\alpha-1}\right|\left|D_j\left(x\right)\right|.$$

From (4) and (8) we have

$$B_2\le c\left(\alpha\right)\left\{\sum_{k=1}^{r}2^{n_{k+1}(-\alpha-1)}\sum_{m=1}^{n_{k+1}}2^m\sum_{l=0}^{m}D_{2^l}\left(x\right)+\sum_{k=1}^{r}\sum_{m=n_{k+1}+1}^{n_k}2^{m(-\alpha-1)}2^m\sum_{l=0}^{m}D_{2^l}\left(x\right)\right\}\le$$

$$\le c\left(\alpha\right)\sum_{k=1}^{n_1}2^{-\alpha k}\sum_{l=0}^{k}D_{2^l}\left(x\right)\ \le\ c\left(\alpha\right)\sum_{l=0}^{n_1}2^{-\alpha l}D_{2^l}\left(x\right). \qquad (13)$$

Combining (11) – (13) we complete the proof of Lemma 2.

***Corollary* 3.** *Let* $\alpha\in(0,1)$. *Then*

$$\left|K_n^{-\alpha}\left(x\right)\right|\le c\min\left\{\frac{1}{A_{n-1}^{-\alpha}}\frac{1}{x^{1-\alpha}},n\right\}.$$

Theorem B (Bakhvalov). *Let* $\Lambda^i:=\left\{\lambda_n^i\colon n\ge1\right\}$ *and* $\Gamma^i=\left\{\gamma_n^i\colon n\ge1\right\}$ *such that* $\gamma_n^i=o\left(\lambda_n^i\right)$, $i=1,2$. *Then*

$$\left(\Gamma^1,\Gamma^2\right)BV\left(I^2\right)\subset C\left(\Lambda^1,\Lambda^2\right)V\left(I^2\right).$$

Theorem 3. *Let* $\Lambda=\left\{\lambda_n\colon n\ge1\right\}$, $\alpha+\beta<1$, $\alpha,\,\beta>0$. *If*

$$\frac{\lambda_n}{n^{1-(\alpha+\beta)}}\downarrow0 \quad \textit{and} \quad \sum_{n=1}^{\infty}\frac{\lambda_n}{n^{2-(\alpha+\beta)}}<\infty,$$

then there exists a sequence $\Gamma^i=\left\{\gamma_n^i\colon n\ge1\right\}$, $i=1,2$, *such that* $\gamma_n^1=o\left(n^{1-\alpha}\right)$, $\gamma_n^2=o\left(n^{1-\beta}\right)$ *and* $P\Lambda BV\left(I^2\right)\subset\left(\Gamma^1,\Gamma^2\right)BV\left(I^2\right)$.

Proof. By definition it is enough to prove that there exists a sequence $\Gamma^i=\left\{\gamma_n^i\colon n\ge1\right\}$, $i=1,2$, with $\gamma_n^1=o\left(n^{1-\alpha}\right)$, $\gamma_n^2=o\left(n^{1-\beta}\right)$ such that for any $f\in P\Lambda BV\left(I^2\right)$

$$\Gamma^1V_1\left(f;I^2\right)+\Gamma^2V_2\left(f;I^2\right)+\left(\Gamma^1,\Gamma^2\right)V_{1,2}\left(f;I^2\right)<\infty.$$

Let the sequence $\{A_n\colon n\ge1\}$ be such that

$$A_n\uparrow\infty,\quad \frac{\lambda_nA_n}{n^{1-(\alpha+\beta)}}\downarrow0,\quad \sum_{n=1}^{\infty}\frac{\lambda_nA_n^2}{n^{2-(\alpha+\beta)}}<\infty.$$

We set

$$\Gamma^1:=\left\{\gamma_n^1:=\frac{n^{1-\alpha}}{A_n}\colon n\ge1\right\},\qquad \Gamma^2:=\left\{\gamma_n^2:=\frac{n^{1-\beta}}{A_n}\colon n\ge1\right\}.$$

We can write

$$\sum_{i,j} \frac{|f(\Delta_i, J_j)|}{\gamma_i^1 \gamma_j^2} = \sum_{i \le j} \frac{|f(\Delta_i, J_j)|}{\gamma_i^1 \gamma_j^2} + \sum_{i > j} \frac{|f(\Delta_i, J_j)|}{\gamma_i^1 \gamma_j^2} := F_1 + F_2. \tag{14}$$

From the condition of the Theorem 3 we have

$$F_1 \le \sum_{i=1}^{\infty} \frac{1}{\gamma_i^1} \sum_{j=i}^{\infty} \frac{|f(\Delta_i, J_j)|}{\gamma_j^2} = \sum_{i=1}^{\infty} \frac{A_i}{i^{1-\alpha}} \sum_{j=i}^{\infty} \frac{|f(\Delta_i, J_j)|}{j^{1-\beta}} A_j \le$$

$$\le 2 \sum_{i=1}^{\infty} \frac{A_i}{i^{1-\alpha}} \sup_x \sum_{j=i}^{\infty} \frac{|f(x, J_j)|}{j^{1-\beta}} A_j = 2 \sum_{i=1}^{\infty} \frac{A_i}{i^{1-\alpha}} \sup_x \sum_{j=i}^{\infty} \frac{|f(x, J_j)|}{\lambda_j} \frac{\lambda_j A_j}{j^{1-\beta}} \le$$

$$\le 2\Lambda V_2\left(f; I^2\right) \sum_{i=1}^{\infty} \frac{\lambda_i A_i^2}{i^{2-(\alpha+\beta)}} < \infty. \tag{15}$$

Analogously, we can prove that

$$F_2 < \infty. \tag{16}$$

Combining (14) – (16) we complete the proof of Theorem 3.

Theorem DF (Daly, Fridli [3]). *Let* n, $N \in \mathbf{N}$ *and* $1 < q \le 2$. *Then for any real numbers* c_k, $1 \le k \le 2^n$, *we have*

$$\int_{2^{-N}}^{1} \left| \sum_{k=1}^{2^n} c_k D_k(x) \right| dx \le c 2^{N(1-1/q)} \left(\sum_{k=1}^{2^n} |c_k|^q \right)^{1/q}.$$

6. Proofs of main results. ***Proof of Theorem* 1.** It is easy to show that

$$\sigma_{n,m}^{-\alpha,-\beta} f(x,y) - f(x,y) =$$

$$= \frac{1}{A_{n-1}^{-\alpha}} \frac{1}{A_{m-1}^{-\beta}} \int_{I^2} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{-\alpha-1} A_{m-j}^{-\beta-1} D_i(s) D_j(t) \Delta f(x,y,s,t)\, ds dt =$$

$$= \left(\int_{I_{N-1} \times I_{M-1}} + \int_{\overline{I}_{N-1} \times I_{M-1}} + \int_{I_{N-1} \times \overline{I}_{M-1}} + \int_{\overline{I}_{N-1} \times \overline{I}_{M-1}} \right) \times$$

$$\times \left(\frac{1}{A_{n-1}^{-\alpha}} \frac{1}{A_{m-1}^{-\beta}} \sum_{i=1}^{n} \sum_{j=1}^{m} A_{n-i}^{-\alpha-1} A_{m-j}^{-\beta-1} D_i(s) D_j(t) \Delta f(x,y,s,t) \right) :=$$

$$:= J_1 + J_2 + J_3 + J_4, \tag{17}$$

where

$$\Delta f\left(x,y,s,t\right):=f\left(x\dotplus s,y\dotplus t\right)-f\left(x,y\right).$$

From the condition of the Theorem 1 and Corollary 3 we conclude that

$$\left|J_{1}\right|\leq c\left(\alpha,\beta\right)nm\int\limits_{I_{N-1}\times I_{M-1}}\left|\Delta f\left(x,y,s,t\right)\right|dsdt=o\left(1\right)\tag{18}$$

as $n,m\rightarrow\infty$.

For J_2 we can write

$$\left|J_{2}\right|\leq\frac{c\left(\beta\right)m}{A_{n-1}^{-\alpha}}\int\limits_{I_{M-1}}\left|\int\limits_{\overline{I}_{N-1}}\sum_{i=1}^{2^{N-1}}A_{n-i}^{-\alpha-1}D_{i}\left(s\right)\Delta f\left(x,y,s,t\right)ds\right|dt+$$

$$+\frac{c\left(\beta\right)m}{A_{n-1}^{-\alpha}}\int\limits_{I_{M-1}}\left|\int\limits_{\overline{I}_{N-1}}\sum_{i=2^{N-1}+1}^{n}A_{n-i}^{-\alpha-1}D_{i}\left(s\right)\Delta f\left(x,y,s,t\right)ds\right|dt:=$$

$$:=J_{21}+J_{22}.\tag{19}$$

From Theorem DF we obtain

$$\left|J_{21}\right|\leq\frac{c\left(\beta\right)}{A_{n-1}^{-\alpha}}\sum_{l=0}^{N-1}\int\limits_{I_{M-1}}\left|\int\limits_{I_{l}\backslash I_{l+1}}\sum_{i=1}^{2^{N-1}}A_{n-i}^{-\alpha-1}D_{i}\left(s\right)\Delta\left(x,y,s,t\right)ds\right|dt\leq$$

$$\leq\frac{c\left(\beta\right)m}{A_{n-1}^{-\alpha}}\sum_{l=0}^{N-1}\omega\left(f;I_{M-1}\left(x\right)\times I_{l}\left(y\right)\right)\times\int\limits_{I_{l}\backslash I_{l+1}}\left|\sum_{i=1}^{2^{N-1}}A_{n-i}^{-\alpha-1}D_{i}\left(s\right)\right|ds\leq$$

$$\leq c\left(\alpha,\beta\right)\sum_{l=0}^{N-1}2^{(l-N)/2}\omega\left(f;I_{M-1}\left(x\right)\times I_{l}\left(y\right)\right)=$$

$$=c\left(\alpha,\beta\right)\left(\sum_{l\leq N/2}+\sum_{N/2<l<N}\right)2^{(l-N)/2}\omega\left(f;I_{M-1}\left(x\right)\times I_{l}\left(y\right)\right)\leq$$

$$\leq c\left(\alpha,\beta,f\right)\left\{2^{-N/4}+\omega\left(f;I_{M-1}\left(x\right)\times I_{[N/2]}\left(y\right)\right)\right\}=$$

$$=o\left(1\right)\quad\text{as}\quad n,\ m\rightarrow\infty.\tag{20}$$

For J_{22} we can write

$$\left|J_{22}\right|\quad\leq\frac{c\left(\beta\right)m}{A_{n-1}^{-\alpha}}\int\limits_{I_{M-1}}\left|\int\limits_{\overline{I}_{N-1}}\sum_{i=2^{N-1}+1}^{2^{N}}A_{n-i}^{-\alpha-1}D_{i}\left(s\right)\Delta f\left(x,y,s,t\right)ds\right|dt+$$

$$+\frac{c(\beta)\, m}{A_{n-1}^{-\alpha}} \int\limits_{I_{M-1}} \left| \int\limits_{\overline{I}_{N-1}} \sum_{i=2^N+1}^{n} A_{n-i}^{-\alpha-1} D_i(s)\, \Delta f(x,y,s,t)\, ds \right| dt =$$

$$= J_{22}^1 + J_{22}^2. \tag{21}$$

From (2) we obtain

$$J_{22}^1 = \frac{c(\beta)\, m}{A_{n-1}^{-\alpha}} \int\limits_{I_{M-1}} \left| \int\limits_{\overline{I}_{N-1}} \sum_{i=1}^{2^{N-1}} A_{n-i-2^{N-1}}^{-\alpha-1} D_i(s)\, w_{2^{N-1}}(s)\, \Delta f(x,y,s,t)\, ds \right| dt =$$

$$= \frac{c(\beta)\, m}{A_{n-1}^{-\alpha}} \int\limits_{I_{M-1}} \left| \sum_{l=1}^{2^{N-1}-1} \sum_{i=1}^{2^{N-1}} A_{n-i-2^{N-1}}^{-\alpha-1} D_i\left(\frac{l}{2^{N-1}}\right) \times \right.$$

$$\left. \times \int\limits_{I_{N-1}^l} w_{2^{N-1}}(s)\, \Delta f(x,y,s,t)\, ds \right| dt. \tag{22}$$

Since (see [12])

$$\int\limits_{I_{N-1}^l} w_{2^{N-1}}(s)\, \Delta f(x,y,s,t)\, ds = \int\limits_{I_N^{2l}} \Delta_0^{N-1} f(x \dotplus s, y \dotplus t)_1\, ds$$

and

$$\sum_{i=1}^{2^{N-1}} A_{n-i-2^{N-1}}^{-\alpha-1} D_i(u) = \sum_{i=1}^{n-2^{N-1}} A_{n-i-2^{N-1}}^{-\alpha-1} D_i(u) - \sum_{i=1}^{n-2^N} A_{n-i-2^N}^{-\alpha-1} D_i(u) \tag{23}$$

from (8), (22) and Corollary 3 we can write

$$\left|J_{22}^1\right| \le \frac{c(\alpha,\beta)\, m n^{1-\alpha}}{n^{-\alpha}} \int\limits_{I_{M-1}\times I_N} \sum_{l=1}^{2^{N-1}-1} \frac{1}{l^{1-\alpha}} \left|\Delta_l^{N-1} f(x \dotplus s, y \dotplus t)_1\right| dsdt. \tag{24}$$

Set

$$\mu(n,m) := \left[\min\left\{N, \left(s(n,m)^{-1}\right)\right\}\right],$$

where

$$s(n,m) := \sup_{0<s<(N+1)2^{-N},\, 0<t<2^{-M+1}} \left|\Delta f(x,y,s,t)\right|.$$

Then from the condition of Theorem 1 and (24) we can write

$$\left|J_{22}^{1}\right| \le c\,(\alpha,\beta)\,nm \int\limits_{I_{M-1}\times I_N} \sum_{l=1}^{\mu(n,m)} \frac{1}{l^{1-\alpha}} \left|\Delta_l^{N-1} f\,(x \dotplus s, y \dotplus t)_1\right| dsdt+$$

$$+c\,(\alpha,\beta)\,nm \int\limits_{I_{M-1}\times I_N} \sum_{l=\mu(n,m)+1}^{2^{N-1}-1} \frac{1}{l^{1-\alpha}} \left|\Delta_l^{N-1} f\,(x \dotplus s, y \dotplus t)_1\right| dsdt \le$$

$$\le c\,(\alpha,\beta)\left\{s\,(n,m)\,(\mu\,(n,n))^{\alpha} + \left\{(i+\mu\,(n,m))^{1-\alpha}\right\} V_1\left(f;I^2\right)\right\} \le$$

$$\le c\,(\alpha,\beta,f)\left\{(s\,(n,m))^{1-\alpha} + \left\{(i+\mu\,(n,m))^{1-\alpha}\right\} V_1\left(f;I^2\right)\right\} =$$

$$= o\,(1) \quad \text{as} \quad n,m\to\infty. \tag{25}$$

Analogously, we can prove that

$$J_{22}^{2} = o\,(1) \quad \text{as} \quad n,m\to\infty. \tag{26}$$

Combining (21), (25) and (26) we obtain that

$$J_{22} = o\,(1) \quad \text{as} \quad n,m\to\infty. \tag{27}$$

From (19), (20) and (27) we conclude that

$$J_{2} = o\,(1) \quad \text{as} \quad n,m\to\infty. \tag{28}$$

Analogously, we can prove that

$$J_{3} = o\,(1) \quad \text{as} \quad n,m\to\infty. \tag{29}$$

For J_4, we can write

$$J_4 = \frac{1}{A_{n-1}^{-\alpha}}\frac{1}{A_{m-1}^{-\beta}} \int\limits_{\overline{I}_{N-1}\times\overline{I}_{M-1}} \sum_{(i,j)\le(2^{N-1},2^{M-1})} A_{n-i}^{-\alpha-1}A_{m-j}^{-\beta-1}\times$$

$$\times D_i\,(s)\,D_j\,(t)\,\Delta f\,(x,y,s,t)\,dsdt+$$

$$+\frac{1}{A_{n-1}^{-\alpha}}\frac{1}{A_{m-1}^{-\beta}} \int\limits_{\overline{I}_{N-1}\times\overline{I}_{M-1}} \sum_{(i,j)\nleq(2^{N-1},2^{M-1})} A_{n-i}^{-\alpha-1}A_{m-j}^{-\beta-1}\times$$

$$\times D_i\,(s)\,D_j\,(t)\,\Delta f\,(x,y,s,t)\,dsdt = J_{41}+J_{42}. \tag{30}$$

From Theorem DF we obtain

$$|J_{41}| \le \frac{1}{A_{n-1}^{-\alpha}}\frac{1}{A_{m-1}^{-\beta}} \sum_{q=0}^{N-2}\sum_{l=0}^{M-2} \Bigg| \int\limits_{I_q\setminus I_{q+1}} \int\limits_{I_l\setminus I_{l+1}} \sum_{i=1}^{2^{N-1}}\sum_{j=1}^{2^{M-1}} A_{n-i}^{-\alpha-1}A_{m-j}^{-\beta-1}\times$$

$$\times D_i(s)\,D_j(t)\,\Delta f(x,y,s,t)\,dsdt\Bigg| \le c(\alpha,\beta)\,n^{\alpha}m^{\beta}\sum_{q=0}^{N-2}\sum_{l=0}^{M-2}\omega\left(f;I_q(x)\times I_l(y)\right)\times$$

$$\times\int\limits_{I_q\setminus I_{q+1}}\left|\sum_{i=1}^{2^{N-1}}A_{n-i}^{-\alpha-1}D_i(s)\right|ds\int\limits_{I_l\setminus I_{l+1}}\left|\sum_{j=1}^{2^{M-1}}A_{m-j}^{-\beta-1}D_j(t)\right|dt\le$$

$$\le c(\alpha,\beta)\sum_{q=0}^{N-2}\sum_{l=0}^{M-2}\omega\left(f;I_q(x)\times I_l(y)\right)2^{(q-N)/2}2^{(l-M)/2}\le$$

$$\le c(\alpha,\beta)\left(\sum_{0\le q<N/2}\sum_{0\le l<M/2}+\sum_{0\le q<N/2}\sum_{M/2\le l<M}+\sum_{N/2\le q<N}\sum_{0\le l<M/2}+\right.$$

$$\left.+\sum_{N/2\le q<N}\sum_{M/2\le l<M}\right)\omega\left(f;I_q(x)\times I_l(y)\right)2^{(q-N)/2}2^{(l-M)/2}\le$$

$$\le c(\alpha,\beta,f)\left\{\frac{1}{2^{(N+M)/4}}+\frac{1}{2^{N/4}}+\frac{1}{2^{M/4}}+\omega\left(f;I_{[N/2]}(x)\times I_{[M/2]}(y)\right)\right\}=$$

$$=o(1)\quad \text{as}\quad n,m\to\infty. \tag{31}$$

Let $i\le 2^{N-1}$ and $2^{M-1}<j\le 2^M$. Then we can write

$$J_{42}=\frac{1}{A_{n-1}^{-\alpha}}\frac{1}{A_{m-1}^{-\beta}}\int\limits_{\overline{I}_{N-1}}\sum_{i=1}^{2^{N-1}}A_{n-i}^{-\alpha-1}D_i(s)\times$$

$$\times\left(\int\limits_{\overline{I}_{M-1}}\sum_{j=1}^{2^{M-1}}A_{m-j-2^{M-1}}^{-\beta-1}D_j(t)\,w_{2^{M-1}}(t)\,\Delta f(x,y,s,t)\,dt\right)ds=$$

$$=\frac{1}{A_{n-1}^{-\alpha}}\frac{1}{A_{m-1}^{-\beta}}\int\limits_{\overline{I}_{N-1}}\sum_{i=1}^{2^{N-1}}A_{n-i}^{-\alpha-1}D_i(s)\sum_{l=1}^{2^{M-1}-1}\sum_{j=1}^{2^{M-1}}A_{m-j-2^{M-1}}^{-\beta-1}D_j\left(\frac{l}{2^{M-1}}\right)\times$$

$$\times\left(\int\limits_{I_M^{2l}}\Delta_0^{M-1}f\left(x\dotplus s,y\dotplus t\right)_2dt\right)ds.$$

Consequently, from Corollary 3 and (23) we obtain

$$|J_{42}| \leq \frac{c(\beta)\, m}{A_{n-1}^{-\alpha}} \int\limits_{2^{-N+1}}^{2^{-[N/2]}} \left| \sum_{i=1}^{2^{N-1}} A_{n-i}^{-\alpha-1} D_i(s) \right| \left(\int\limits_{I_M} \sum_{l=1}^{2^{M-1}-1} \frac{\Delta_l^{M-1} f(x \dotplus s, y \dotplus t)_2}{l^{1-\beta}} dt \right) ds +$$

$$+ \frac{c(\beta)\, m}{A_{n-1}^{-\alpha}} \int\limits_{2^{-[N/2]}}^{1} \left| \sum_{i=1}^{2^{N-1}} A_{n-i}^{-\alpha-1} D_i(s) \right| \left(\int\limits_{I_M} \sum_{l=1}^{2^{M-1}-1} \frac{\Delta_l^{M-1} f(x \dotplus s, y \dotplus t)_2}{l^{1-\beta}} dt \right) ds =$$

$$= J_{42}^1 + J_{42}^2. \tag{32}$$

Set

$$r(n,m) := \sup_{0<s<2^{-N/2},\, 0<t<(2M+1)2^{-M}} |\Delta f(x,y,s,t)|$$

and

$$\theta(n,m) := \left[\min \left\{ M, r(n,m)^{-1} \right\} \right].$$

Then applying Theorem DF for J_{42}^1 we have

$$J_{42}^1 \quad \leq \frac{c(\beta)\, m}{A_{n-1}^{-\alpha}} \int\limits_{2^{-N+1}}^{2^{-[N/2]}} \left| \sum_{i=1}^{2^{N-1}} A_{n-i}^{-\alpha-1} D_i(s) \right| \left(\int\limits_{I_M} \sum_{l=1}^{\theta(n,m)} \frac{\Delta_l^{M-1} f(x \dotplus s, y \dotplus t)_2}{l^{1-\beta}} dt \right) ds +$$

$$+ \frac{c(\beta)\, m}{A_{n-1}^{-\alpha}} \int\limits_{2^{-N+1}}^{2^{-[N/2]}} \left| \sum_{i=1}^{2^{N-1}} A_{n-i}^{-\alpha-1} D_i(s) \right| \left(\int\limits_{I_M} \sum_{l=\theta(n,m)}^{2^{M-1}-1} \frac{\Delta_l^{M-1} f(x \dotplus s, y \dotplus t)_2}{l^{1-\beta}} dt \right) ds \leq$$

$$\leq c(\alpha,\beta) \left\{ r(n,m)\, \theta^{\beta}(n,m) + \left\{ (l + \theta(n,m))^{1-\beta} V_2\left(f; I^2\right) \right\} \right\} \leq$$

$$\leq c(\alpha,\beta) \left\{ r^{1-\beta}(n,m) + \left\{ (l + \theta(n,m))^{1-\beta} V_2\left(f; I^2\right) \right\} \right\} =$$

$$= o(1) \quad \text{as} \quad n, m \to \infty, \tag{33}$$

$$J_{42}^2 \leq \frac{c(\alpha,\beta) \left\{ i^{1-\beta} \right\} V_2\left(f; I^2\right)}{2^{N/4}} = o(1) \quad \text{as} \quad n, m \to \infty. \tag{34}$$

Combining (32), (33) and (34) we conclude that

$$J_{42} = o(1) \quad \text{as} \quad n, m \to \infty. \tag{35}$$

Analogously, we can prove that (35) holds in the cases when

$$(i,j) \in \left\{ (i,j) : 0 \leq i \leq 2^{N-1}, 2^M < j \leq m \right\} \bigcup$$

$$\bigcup \left\{(i,j) : 2^{N-1} < i \le 2^N, 0 \le j \le 2^{M-1}\right\} \bigcup \left\{(i,j) : 2^N < i \le n, 0 \le j \le 2^{M-1}\right\}.$$

Let $2^{N-1} < i \le 2^N$ and $2^M < j \le m$. Then we can write

$$J_{42} = \frac{1}{A_{n-1}^{-\alpha}} \frac{1}{A_{m-1}^{-\beta}} \sum_{k=1}^{2^{N-1}-1} \sum_{l=1}^{2^M-1} \sum_{i=1}^{2^{N-1}} \sum_{j=1}^{m'} A_{n-i-2^{N-1}}^{-\alpha-1} A_{m'-j}^{-\beta-1} D_i \left(\frac{k}{2^{N-1}}\right) D_j \left(\frac{l}{2^M}\right) \times$$

$$\times \int\limits_{I_N^{2k} \times I_{M+1}^{2l}} \Delta_{00}^{N-1,M} f(x \dotplus s, y \dotplus t)\, ds dt.$$

Set

$$p(n,m) := \left[\min\left\{N, M, (\psi(n,m))^{-1/(2(\alpha+\beta))}\right\}\right],$$

where

$$\psi(n,m) := \sup_{0<s<\frac{N+1}{2^N},\, 0<t<\frac{2M+1}{2^{M+1}}} |\Delta f(x,y,s,t)|.$$

Then from the condition of the theorem we can write

$$|J_{42}| \le c(\alpha,\beta)\, nm \int\limits_{I_N \times I_{M+1}} \sum_{k=1}^{2^{N-1}-1} \sum_{l=1}^{2^M-1} \frac{1}{k^{1-\alpha}} \frac{1}{l^{1-\beta}} \left|\Delta_{kl}^{N-1,M} f(x \dotplus s, y \dotplus t)\right| ds dt \le$$

$$\le c(\alpha,\beta)\, nm \int\limits_{I_N \times I_{M+1}} \sum_{(k,l) < (p(n,m),p(n,m))} \frac{1}{k^{1-\alpha}} \frac{1}{l^{1-\beta}} \left|\Delta_{kl}^{N-1,M} f(x \dotplus s, y \dotplus t)\right| ds dt +$$

$$+ c(\alpha,\beta)\, nm \int\limits_{I_N \times I_{M+1}} \sum_{(k,l) \not< (p(n,m),p(n,m))} \frac{1}{k^{1-\alpha}} \frac{1}{l^{1-\beta}} \left|\Delta_{kl}^{N-1,M} f(x \dotplus s, y \dotplus t)\right| ds dt \le$$

$$\le c(\alpha,\beta) \left\{\psi(n,m)(p(n,m))^{\alpha+\beta} + \left(\left\{k^{1-\alpha}\right\} \left\{(l + p(n,m))^{1-\beta}\right\}\right) V_{1,2}\left(f, I^2\right) + \right.$$

$$\left. + \left(\left\{(k + p(n,m))^{1-\alpha}\right\} \left\{l^{1-\beta}\right\}\right) V_{1,2}\left(f, I^2\right)\right\} =$$

$$= o(1) \quad \text{as} \quad n, m \to \infty. \tag{36}$$

Analogously, we can prove that (36) holds in the cases when

$$(i,j) \in \left\{(i,j) : 2^{N-1} < i \le 2^N, 2^{M-1} < j \le 2^M\right\} \bigcup$$

$$\bigcup \left\{(i,j) : 2^N < i \le n, 2^{M-1} < j \le 2^M\right\} \bigcup \left\{(i,j) : 2^N < i \le n, 2^M < j \le m\right\}.$$

From (30), (31), (35) and (36) we have

$$J_4 = o(1) \quad \text{as} \quad n, m \to \infty. \tag{37}$$

Combining (17), (18), (28), (29) and (37) we complete the proof of Theorem 1.

Proof of Theorem **2.** The proof of the part a) of the Theorem 2 follows from Theorem B, Theorems 1 and 3. Now, we prove the part b).

Consider the function φ_N^m defined by

$$\varphi_N^m(x) := \begin{cases} 2^{N+1}x - 2j, \quad x \in \left[2j2^{-N-1}, (2j+1)\,2^{-N-1}\right) - \\ -\left(2^{N+1}x - 2j - 2\right), \quad x \in \left[(2j+1)\,2^{-N-1}, (2j+2)\,2^{-N-1}\right), \\ j = 2^{m-1}, \ldots, 2^m - 1. \end{cases}$$

Let

$$f_N(x,y) := \sum_{m=1}^{N} t_{2^m} \varphi_N^m(x)\, \varphi_N^m(y)\, \operatorname{sgn}\left(K_{2^N}^{-\alpha}(x)\right) \operatorname{sgn}\left(K_{2^N}^{-\beta}(y)\right),$$

where

$$t_n := \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)^{-1}.$$

It is easy to show that $f_N \in P\Lambda BV\left(I^2\right)$. Indeed, let $y \in \left[2^{m-N-1}, 2^{m-N}\right)$ for some $m = 1, 2, ..., N$. Then from the construction of the function f_N we can write

$$\sum_i \frac{|f_N(\Delta_i, y)|}{\lambda_i} \leq c t_{2^m} \sum_{i=1}^{2^m} \frac{1}{\lambda_i} \leq c < \infty.$$

Consequently

$$\Lambda V_1(f_N) < \infty. \tag{38}$$

Analogously, we can prove that

$$\Lambda V_2(f_N) < \infty. \tag{39}$$

Combining (38) and (39) we conclude that $f_N \in P\Lambda BV\left(I^2\right)$.

We can write

$$\sigma_{2^N,2^N}^{-\alpha,-\beta} f_N(0,0) = \int_{I^2} f_N(x,y)\, K_{2^N}^{-\alpha}(x)\, K_{2^N}^{-\beta}(y)\, dxdy =$$

$$= \sum_{m=1}^{N} t_{2^m} \int_{[2^{m-N-1}, 2^{m-N})^2} \varphi_N^m(x)\, \varphi_N^m(y) \left|K_{2^N}^{-\alpha}(x)\right| \left|K_{2^N}^{-\beta}(y)\right| dxdy \geq$$

$$\geq c \sum_{m=1}^{N} t_{2^m} \int_{[2^{m-N-1}, 2^{m-N})^2} \left|K_{2^N}^{-\alpha}(x)\right| \left|K_{2^N}^{-\beta}(y)\right| dxdy.$$

Since [22]

$$\int\limits_{[2^{m-N-1},2^{m-N})} \left|K_{2^N}^{-\alpha}(x)\right| dx \geq c(\alpha)\, 2^{m\alpha}$$

we have

$$\left|\sigma_{2^N,2^N}^{-\alpha,-\beta} f_N(0,0)\right| \geq c(\alpha,\beta) \sum_{m=1}^{N} t_{2^m} 2^{m(\alpha+\beta)}. \tag{40}$$

Let $\lambda_j := \gamma_j j^{1-(\alpha+\beta)}$. The from the condition of the Theorem 2 we obtain that $\gamma_j \geq \gamma_{j+1}$. Hence, we have

$$\frac{1}{t_{2^m}} = \sum_{i=1}^{2^m} \frac{1}{\lambda_i} = \sum_{i=1}^{2^m} \frac{1}{i^{1-(\alpha+\beta)}\gamma_i} \leq c(\alpha,\beta) \frac{2^{m(\alpha+\beta)}}{\gamma_{2^m}},$$

$$t_{2^m} 2^{m(\alpha+\beta)} \geq c(\alpha,\beta)\, \gamma_{2^m}.$$

Consequently, from (40) we have

$$\left|\sigma_{2^N,2^N}^{-\alpha,-\beta} f_N(0,0)\right| \geq c(\alpha,\beta) \sum_{m=1}^{N} \gamma_{2^m} = c(\alpha,\beta) \sum_{m=1}^{N} \frac{\lambda_{2^m}}{2^{m(1-(\alpha+\beta))}} \to \infty \quad \text{as} \quad N \to \infty.$$

Applying the Banach – Steinhaus theorem, we obtain that there exists a continuous function $f \in$ $\in P\Lambda BV\left(I^2\right)$ such that

$$\sup_n \left|\sigma_{2^n,2^n}^{-\alpha,-\beta} f(0,0)\right| = \infty.$$

Theorem 2 is proved.

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