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CONFORMAL ISOPARAMETRIC SPACELIKE HYPERSURFACES IN CONFORMAL SPACES $\mathbb{Q}_1^4$ and $\mathbb{Q}_1^5$*

КОНФОРМНІ ІЗОПАРАМЕТРИЧНІ ПРОСТОРОПОДІБНІ ГІПЕРПОВЕРХНІ У КОНФОРМНИХ ПРОСТОРАХ $\mathbb{Q}_1^4$ І $\mathbb{Q}_1^5$

We study the conformal geometry of conformal spacelike hypersurfaces in the conformal spaces $\mathbb{Q}_1^4$ and $\mathbb{Q}_1^5$. We obtain a complete classification of conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^4$ and $\mathbb{Q}_1^5$.

Вивчено конформну геометрію конформних просторoподібних гіперповерхонь у конформних просторах $\mathbb{Q}_1^4$ і $\mathbb{Q}_1^5$. Отримано повну класифікацію конформних ізопараметричних просторoподібних гіперповерхонь у $\mathbb{Q}_1^4$ та $\mathbb{Q}_1^5$.

1. Introduction. Let $\langle\,,\rangle_s$ be the Lorentzian inner product with s negative index of the $(n+s)$-dimensional Euclidean space $\mathbb{R}^{n+s}$. Denoted by

$$\langle X, Y\rangle_s = \sum_{i=1}^{n} x_i y_i - \sum_{i=n+1}^{n+s} x_i y_i, \qquad X = (x_i), \qquad Y = (y_i) \in \mathbb{R}^{n+s}.$$

Let $\mathbb{R}P^{n+2}$ be $(n+2)$-dimensional real projective space. The quadric surface

$$\mathbb{Q}_1^{n+1} = \{[\xi] \in \mathbb{R}P^{n+2} | \langle \xi, \xi\rangle_2 = 0\},$$

is called *conformal space*. We define the Lorentzian space $\mathbb{R}_1^{n+1}$, de Sitter sphere $\mathbb{S}_1^{n+1}$ and anti-de Sitter sphere $\mathbb{H}_1^{n+1}$ by

$$\mathbb{R}_1^{n+1} = (\mathbb{R}^{n+1}, \langle\,,\rangle_1), \qquad \mathbb{S}_1^{n+1} = \{u \in \mathbb{R}^{n+2} | \langle u, u\rangle_1 = 1\},$$

$$\mathbb{H}_1^{n+1} = \{u \in \mathbb{R}^{n+2} | \langle u, u\rangle_2 = -1\}.$$

We call Lorentzian space $\mathbb{R}_1^{n+1}$, de Sitter sphere $\mathbb{S}_1^{n+1}$ and anti-de Sitter sphere $\mathbb{H}_1^{n+1}$ *Lorentzian space forms*.

Denote $\pi = \{[x] \in \mathbb{Q}_1^{n+1} | x_1 = x_{n+3}\}$, $\pi_+ = \{[x] \in \mathbb{Q}_1^{n+1} | x_{n+3} = 0\}$ and $\pi_- = \{[x] \in \in \mathbb{Q}_1^{n+1} | x_1 = 0\}$. Observe the conformal diffeomorphisms

$$\sigma_0 \colon \mathbb{R}_1^n \to \mathbb{Q}_1^{n+1} \backslash \pi, \quad u \mapsto \left[\left(\frac{\langle u,u\rangle - 1}{2}, u, \frac{\langle u,u\rangle + 1}{2}\right)\right],$$

$$\sigma_1 \colon \mathbb{S}_1^{n+1} \to \mathbb{Q}_1^{n+1} \backslash \pi_+, \quad u \mapsto [(u, 1)],$$

$$\sigma_{-1} \colon \mathbb{H}_1^{n+1} \to \mathbb{Q}_1^{n+1} \backslash \pi_-, \quad u \mapsto [(1, u)].$$

From [13], we may regard $\mathbb{Q}_1^{n+1}$ as the common compactified space of $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}$ and $\mathbb{H}_1^{n+1}$, while $\mathbb{R}_1^{n+1}$, $\mathbb{S}_1^{n+1}$ and $\mathbb{H}_1^{n+1}$ are regarded as the subsets of $\mathbb{Q}_1^{n+1}$.

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Suppose that $x\colon M \to \mathbb{Q}_1^{n+1}$ is a nondegenerated hypersurface, that is, $x_*(TM)$ is nondegenerated subbundle of $T\mathbb{Q}_1^{n+1}$. Let $y\colon U \to \mathbb{R}_2^{n+3}$ be a lift of $x\colon M \to \mathbb{Q}_1^{n+1}$ defined in an open subset U of M. We denote by Δ and κ Laplacian and the normalized scalar curvature of the local nondegerated metric $\langle dy, dy\rangle$. Then know that on M the 2-form $g = \varepsilon(\langle \Delta y, \Delta y\rangle - n^2\kappa)\langle dy, dy\rangle$ is a globally defined invariant of $x\colon M \to \mathbb{Q}_1^{n+1}$ under the conformal group transformations of $\mathbb{Q}_1^{n+1}$. When the 2-form $g = \varepsilon(\langle \Delta y, \Delta y\rangle - n^2\kappa)\langle dy, dy\rangle$ is nondegenerated, we call $x\colon M \to \mathbb{Q}_1^{n+1}$ a *conformal regular hypersurface* and $g = \varepsilon(\langle \Delta y, \Delta y\rangle - n^2\kappa)\langle dy, dy\rangle$ the *conformal metric* of x, where $\varepsilon = -1$ (spacelike) or $\varepsilon = 1$ (timelike). From [13], we know that there exists a unique lift $Y\colon U \to \mathbb{R}_2^{n+3}$ such that $g = \langle dY, dY\rangle$ up to a signature and we call Y the canonical lift of x. It is obvious that $g \equiv 0$ if and only if $x\colon M \to \mathbb{Q}_1^{n+1}$ is a umbilical hypersurface.

Let $x\colon M \to \mathbb{Q}_1^{n+1}$ be an n-dimensional immersed conformal regular spacelike hypersurface in conformal space $\mathbb{Q}_1^{n+1}$. We choose a local orthonormal basis $\{e_i\}$ for the induced metric $I = \langle dx, dx\rangle$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j} h_{ij}\theta_i \otimes \theta_j$ be the second fundamental form and $H = \dfrac{1}{n}\sum_i h_{ii}$ the mean curvature of the immersion x. From [7], we know that the conformal metric of the immersion x can be defined by $g = \dfrac{n}{n-1}\left\{\sum_{i,j} h_{ij}^2 - nH^2\right\}\langle dx, dx\rangle := e^{2\tau}\langle dx, dx\rangle$, which is a conformal invariant. Denote

$$\Phi = \sum_{i=1}^{n} e^{\tau} C_i\theta_i, \qquad \mathbf{A} = \sum_{i,j=1}^{n} e^{2\tau} A_{ij}\theta_i \otimes \theta_j, \qquad \mathbf{B} = \sum_{i,j=1}^{n} e^{2\tau} B_{ij}\theta_i \otimes \theta_j, \tag{1.1}$$

where C_i, A_{ij} and B_{ij} are defined by formulas (2.1)–(2.3) in Section 2. We call Φ, $\mathbf{A}$ and $\mathbf{B}$ *conformal form*, *conformal Blaschke tensor* and *conformal second fundamental form* of the immersion x, respectively. It is easy to prove that Φ, $\mathbf{A}$ and $\mathbf{B}$ are conformal invariants.

The conformal geometry of regular hypersurfaces in the conformal space is determined by conformal metric. The negative index of conformal space $\mathbb{Q}_1^{n+1}$ is 1. If the negative index is degenerate, we obtain the Möbius geometry in the unit sphere which had been studied by many authors (see [1–7, 9, 10, 16–18]). We call the eigenvalues of $\mathbf{B}$ the *conformal principal curvatures* of the immersion x, while the eigenvalues of $\mathbf{A}$ are called the *conformal Blaschke eigenvalues* of x. A regular spacelike hypersurface $x\colon M \to \mathbb{Q}_1^{n+1}$ is called a *conformal isoparametric spacelike hypersurface*, if $\Phi \equiv 0$ and the conformal principal curvatures of the immersion x are constant.

Let $\mathbb{S}^k(a)$ and $\mathbb{H}^k(a)$ denote k-dimensional sphere and k-dimensional hyperbolic surface with radius $\dfrac{1}{a}$, $\mathbb{S}_1^k(a)$ and $\mathbb{H}_1^k(a)$ denote k-dimensional de Sitter sphere and k-dimensional anti-de Sitter sphere with radius $\dfrac{1}{a}$, where a is a constant parametric. Recently, C. X. Nie et al. [11–14] studied the conformal geometry of regular spacelike hypersurfaces in the conformal space $\mathbb{Q}_1^{n+1}$ and obtained the following results:

Theorem 1.1 [12]. *If $x\colon M \to \mathbb{Q}_1^{n+1}$ is a conformal regular spacelike hypersurface in $\mathbb{Q}_1^{n+1}$ with parallel conformal second fundamental form, then M is conformal equivalent to an open part of these standard embeddings*:

(i) *the Riemannian product* $\mathbb{S}^m(a) \times \mathbb{H}^{n-m} - \left(\sqrt{a^2 - \dfrac{n-1}{m(n-m)}}\right)$ *in* $\mathbb{S}_1^{n+1}\left(\sqrt{\dfrac{n-1}{m(n-m)}}\right)$, $a > \sqrt{\dfrac{n-1}{m(n-m)}}$;

(ii) *the Riemannian product* $\mathbb{R}^m \times \mathbb{H}^{n-m}\left(\sqrt{\dfrac{n-1}{m(n-m)}}\right)$ *in* $\mathbb{R}_1^{n+1}$;

(iii) *the Riemannian product* $\mathbb{H}^m(a) \times \mathbb{H}^{n-m}\left(\sqrt{\dfrac{n-1}{m(n-m)} - a^2}\right)$ *in* $\mathbb{H}_1^{n+1}\left(\sqrt{\dfrac{n-1}{m(n-m)}}\right)$, $0 < a < \sqrt{\dfrac{n-1}{m(n-m)}}$;

(iv) *the spacelike hypersurface* $x = \sigma_0 \circ u \colon \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(b) \to \mathbb{Q}_1^{n+1}$ *with* $b = \sqrt{a^2-1}$, $p \geq 1$, $q \geq 1$, $p+q < n$, *where* $u \colon \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{n-p-q-1} \times \mathbb{H}^q(b) \to \mathbb{R}_1^{n+2} \subset \mathbb{R}_1^{n+1}$:

$$u(u', t, u'', u''') = (tu', u'', tu'''), \quad u' \in \mathbb{S}^p(a), \quad t \in \mathbb{R}^+, \quad u'' \in \mathbb{R}^{n-p-q-1}, \quad u''' \in \mathbb{H}^q(b).$$

Theorem 1.2 [13]. *If* $x \colon M \to \mathbb{Q}_1^{n+1}$ *is a conformal isoparametric spacelike hypersurface with two distinct principle curvatures, then* M *is conformal equivalent to an open part of these standard embeddings*:

(i) *the Riemannian product* $\mathbb{S}^m(a) \times \mathbb{H}^{n-m}\left(\sqrt{a^2 - \dfrac{n-1}{m(n-m)}}\right)$ *in* $\mathbb{S}_1^{n+1}\left(\sqrt{\dfrac{n-1}{m(n-m)}}\right)$, $a > \sqrt{\dfrac{n-1}{m(n-m)}}$;

(ii) *the Riemannian product* $\mathbb{R}^m \times \mathbb{H}^{n-m}\left(\sqrt{\dfrac{n-1}{m(n-m)}}\right)$ *in* $\mathbb{R}_1^{n+1}$;

(iii) *the Riemannian product* $\mathbb{H}^m(a) \times \mathbb{H}^{n-m}\left(\sqrt{\dfrac{n-1}{m(n-m)} - a^2}\right)$ *in* $\mathbb{H}_1^{n+1}\left(\sqrt{\dfrac{n-1}{m(n-m)}}\right)$, $0 < a < \sqrt{\dfrac{n-1}{m(n-m)}}$.

We notice that in [4] and [5], the authors classified the Möbius isoparametric hypersurfaces in the unit spheres $\mathbb{S}^4$ and $\mathbb{S}^5$. In this paper, we obtain the complete classification of conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^4$ and $\mathbb{Q}_1^5$.

Theorem 1.3. *Let* $x \colon M \to \mathbb{Q}_1^4$ *be a conformal isoparametric spacelike hypersurface in* $\mathbb{Q}_1^4$. *Then* M *is conformal equivalent to an open part of these standard embeddings*:

(i) *the Riemannian product* $\mathbb{S}^m(a) \times \mathbb{H}^{3-m}\left(\sqrt{a^2 - \dfrac{2}{m(3-m)}}\right)$ *in* $\mathbb{S}_1^4\left(\sqrt{\dfrac{2}{m(3-m)}}\right)$, $a > \sqrt{\dfrac{2}{m(3-m)}}$, $m = 1, 2$;

(ii) *the Riemannian product* $\mathbb{R}^m \times \mathbb{H}^{3-m}\left(\sqrt{\dfrac{2}{m(3-m)}}\right)$ *in* $\mathbb{R}_1^4$, $m = 1, 2$;

(iii) *the Riemannian product* $\mathbb{H}^m(a) \times \mathbb{H}^{3-m}\left(\sqrt{\dfrac{2}{m(3-m)} - a^2}\right)$ *in* $\mathbb{H}_1^4\left(\sqrt{\dfrac{2}{m(3-m)}}\right)$, $0 < a < \sqrt{\dfrac{2}{m(3-m)}}$, $m = 1, 2$;

(iv) *the spacelike hypersurface* $x = \sigma_0 \circ u \colon \mathbb{S}^1(a) \times \mathbb{R}^+ \times \mathbb{H}^1(b) \to \mathbb{Q}_1^4$ *with* $b = \sqrt{a^2-1}$, *where* $u \colon \mathbb{S}^1(a) \times \mathbb{R}^+ \times \mathbb{H}^1(b) \to \mathbb{R}_1^5 \subset \mathbb{R}_1^4$:

$$u(u', t, u''') = (tu', tu'''), \qquad u' \in \mathbb{S}^1(a), \quad t \in \mathbb{R}^+, \quad u''' \in \mathbb{H}^1(b).$$

Theorem 1.4. *Let $x\colon M \to \mathbb{Q}_1^5$ be a conformal isoparametric spacelike hypersurface in $\mathbb{Q}_1^5$. Then M is conformal equivalent to an open part of these standard embeddings*:

(i) *the Riemannian product* $\mathbb{S}^m(a) \times \mathbb{H}^{4-m}\left(\sqrt{a^2 - \dfrac{3}{m(4-m)}}\right)$ *in* $\mathbb{S}_1^5\left(\sqrt{\dfrac{3}{m(4-m)}}\right)$, $a > \sqrt{\dfrac{3}{m(4-m)}}$, $m = 1, 2, 3$;

(ii) *the Riemannian product* $\mathbb{R}^m \times \mathbb{H}^{4-m}\left(\sqrt{\dfrac{3}{m(4-m)}}\right)$ *in* $\mathbb{R}_1^5$, $m = 1, 2, 3$;

(iii) *the Riemannian product* $\mathbb{H}^m(a) \times \mathbb{H}^{4-m}\left(\sqrt{\dfrac{3}{m(4-m)} - a^2}\right)$ *in* $\mathbb{H}_1^5\left(\sqrt{\dfrac{3}{m(4-m)}}\right)$, $0 < a < \sqrt{\dfrac{3}{m(4-m)}}$, $m = 1, 2, 3$;

(iv) *the spacelike hypersurface* $x = \sigma_0 \circ u\colon \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{4-p-q-1} \times \mathbb{H}^q(b) \to \mathbb{Q}_1^5$ *with* $b = \sqrt{a^2 - 1}$, $p \geq 1$, $q \geq 1$, $p + q < 4$, *where* $u\colon \mathbb{S}^p(a) \times \mathbb{R}^+ \times \mathbb{R}^{4-p-q-1} \times \mathbb{H}^q(b) \to \mathbb{R}_1^6 \subset \mathbb{R}_1^5$:

$$u(u', t, u'', u''') = (tu', u'', tu'''), \quad u' \in \mathbb{S}^p(a), \quad t \in \mathbb{R}^+, \quad u'' \in \mathbb{R}^{4-p-q-1}, \quad u''' \in \mathbb{H}^q(b).$$

2. Fundamental formulas on conformal geometry. In this section, we review the conformal invariants and fundamental formulas on conformal geometry of spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$, for more details (see [14]).

Let $x\colon M \to \mathbb{Q}_1^{n+1}$ be an n-dimensional conformal regular spacelike hypersurface with $\Phi \equiv 0$ in $\mathbb{Q}_1^{n+1}$. We have (see [13])

$$\langle \Delta Y, \Delta Y \rangle = (n^2\kappa - 1),$$

where Y is the canonical lift of x defined in Section 1 and $n(n-1)\kappa$ is the conformal scalar curvature of x. Let $\{E_1, \ldots, E_n\}$ denote a local orthonormal frame on (M, g) with dual frame $\{\omega_1, \ldots, \omega_n\}$. Putting $Y_i = E_i(Y)$, then we have

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y \rangle Y,$$

$$\langle N, Y \rangle = 1, \qquad \langle N, N \rangle = 0, \qquad \langle Y_i, N \rangle = 0, \qquad \langle Y_i, Y_j \rangle = \delta_{ij}, \quad 1 \leq i, \quad j \leq n.$$

Let $\mathbb{V}$ be the orthogonal complement to the subspace Span$\{Y, N, Y_1, \ldots, Y_n\}$ in $\mathbb{R}_1^{n+2}$. Along M, we have the following orthogonal decomposition:

$$\mathbb{R}_1^{n+2} = \text{Span}\{Y, N\} \oplus \text{Span}\{Y_1, \ldots, Y_n\} \oplus \mathbb{V},$$

where $\mathbb{V}$ is called *conformal normal bundle* of the immersion x. Let ξ be a unit basis of $\mathbb{V}$ and $\langle \xi, \xi \rangle = -1$. Then $\{Y, N, Y_1, \ldots, Y_n, \xi,\}$ forms a moving frame in $\mathbb{R}_1^{n+2}$ along M. We use the following range of indices throughout this paper:

$$1 \leq i, j, k, l, m \leq n.$$

The structure equations on M with respect to the conformal metric g can be written as

$$dY = \sum_i \omega_i Y_i,$$

$$dN = \sum_i \psi_i Y_i + \phi\xi,$$

$$dY_i = -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{in+1}\xi,$$

$$d\xi = \phi Y + \sum_i \omega_{in+1} Y_i,$$

where $\{\psi_i, \omega_{ij}, \omega_{in+1}, \phi\}$ are 1-forms on M with

$$\omega_{ij} + \omega_{ji} = 0.$$

By exterior differentiation of these equations, we get

$$\sum_i \omega_i \wedge \psi_i = 0, \qquad \sum_i \omega_{in+1} \wedge \omega_i = 0,$$

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j,$$

$$d\psi_i = \sum_j \omega_{ij} \wedge \psi_j + \omega_{in+1} \wedge \phi,$$

$$d\phi = \sum_i \omega_{in+1} \wedge \psi_i,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{in+1} \wedge \omega_{jn+1} - \omega_i \wedge \psi_j - \psi_i \wedge \omega_j,$$

$$d\omega_{in+1} = \sum_j \omega_{ij} \wedge \omega_{jn+1} + \omega_i \wedge \phi,$$

where

$$\psi_i = \sum_j A_{ij}\omega_j, \qquad A_{ij} = A_{ji}, \qquad \omega_{in+1} = \sum_j B_{ij}\omega_j, \qquad B_{ij} = B_{ji}, \qquad \phi = \sum_i C_i \omega_i.$$

Let the conformal metric $g = e^{2\tau} I$. Then the local orthonormal frame $\{E_1, \ldots, E_n\}$ on (M, g) and the dual frame $\{\omega_1, \ldots, \omega_n\}$ satisfy $E_i = e^{-\tau} e_i$ and $\omega_i = e^{\tau}\theta_i$. A_{ij}, B_{ij} and C_i are locally defined functions and satisfy

$$e^{2\tau} C_i = H\tau_i - H_i - \sum_j h_{ij}\tau_j, \tag{2.1}$$

$$e^{2\tau} A_{ij} = \tau_i\tau_j - \tau_{i,j} - Hh_{ij} - \frac{1}{2}\left(\sum_k \tau^k\tau_k - H^2 - \epsilon\right) I_{ij}, \tag{2.2}$$

$$e^{\tau}B_{ij} = h_{ij} - HI_{ij}, \tag{2.3}$$

where $\tau_{i,j}$ is Hessian of τ with respect to the first fundamental form I, $\tau^i = \sum_j I^{ij}\tau_j$, $(I^{ij}) = (I_{ij})^{-1}$, $H_i = e_i(H)$ and $\epsilon = 0$ for $\mathbb{R}_1^{n+1}$, $\epsilon = 1$ for $\mathbb{S}_1^{n+1}$ and $\epsilon = -1$ for $\mathbb{H}_1^{n+1}$ (see [14])

$$\sum_i B_{ii} = 0, \qquad \sum_{i,j} B_{ij}^2 = \frac{n-1}{n}, \qquad \mathrm{tr}\mathbf{A} = \frac{1}{2n}(n^2\kappa - 1). \tag{2.4}$$

Defining the covariant derivative of C_i, A_{ij}, B_{ij} by

$$\sum_j C_{i,j}\omega_j = dC_i + \sum_j C_j\omega_{ji}, \tag{2.5}$$

$$\sum_k A_{ij,k}\omega_k = dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki}, \tag{2.6}$$

$$\sum_k B_{ij,k}\omega_k = dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki}, \tag{2.7}$$

$$d\omega_{ij} = \sum_k \omega_{ik}\wedge\omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l, \qquad R_{ijkl} = -R_{jikl}, \tag{2.8}$$

we have

$$A_{ij,k} - A_{ik,j} = B_{ij}C_k - B_{ik}C_j, \tag{2.9}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{kj}A_{ki}), \tag{2.10}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.11}$$

$$R_{ijkl} = -(B_{ik}B_{jl} - B_{il}B_{jk}) + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2.12}$$

where R_{ijkl} denotes the curvature tensor with respect to the conformal metric g on M. Since the conformal form $\Phi \equiv 0$, we have for all indices i, j, k

$$A_{ij,k} = A_{ik,j}, \qquad B_{ij,k} = B_{ik,j}, \qquad \sum_k B_{ik}A_{kj} = \sum_k B_{kj}A_{ki}. \tag{2.13}$$

Defining the second covariant derivative of B_{ij} by

$$\sum_l B_{ij,kl}\omega_l = dB_{ij,k} + \sum_l B_{lj,k}\omega_{li} + \sum_l B_{il,k}\omega_{lj} + \sum_l B_{ij,l}\omega_{lk}, \tag{2.14}$$

we have the following Ricci identities:

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj}R_{mikl} + \sum_m B_{im}R_{mjkl}. \tag{2.15}$$

3. Some examples and propositions. We cite some examples of conformal regular spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$:

***Example* 3.1.** Spacelike hypersurface $x\colon \mathbb{S}^m(a)\times\mathbb{H}^{n-m}(\sqrt{a^2-r^2})\to\mathbb{S}_1^{n+1}(r)$, $r<a$. Let

$$x=(x_1,x_2)\in\mathbb{S}^m(a)\times\mathbb{H}^{n-m}\left(\sqrt{a^2-r^2}\right)\subset\mathbb{R}_1^{m+1}\times\mathbb{R}_1^{n-m+1},$$

$$\langle x_1,x_1\rangle=a^2,\qquad \langle x_2,x_2\rangle=-\left(a^2-r^2\right),$$

and

$$e_{n+1}=\left(-\frac{\sqrt{a^2-r^2}}{a}\frac{x_1}{r},-\frac{a}{\sqrt{a^2-r^2}}\frac{x_2}{r}\right)$$

be the unit normal vector of x such that $\langle e_{n+1},e_{n+1}\rangle=-1$. By a direct calculation, we know that x has two distinct conformal principal curvatures $\frac{c}{r}$ and $\frac{1}{rc}$ with multiplicities m and $n-m$, respectively, where $c=\frac{\sqrt{a^2-r^2}}{a}$. The conformal second fundamental form of x is parallel.

***Example* 3.2.** Spacelike hypersurface $x\colon\mathbb{R}^m\times\mathbb{H}^{n-m}(r)\to\mathbb{R}_1^{n+1}$.

Let $x=(x_1,x_2)$, $x_1\in\mathbb{R}^m$, $x_2\in\mathbb{H}^{n-m}(r)\subset\mathbb{R}_1^{n-m+1}$, $\langle x_2,x_2\rangle=-r^2$ and $e_{n+1}=\left(0,\frac{x_2}{r}\right)$ be the unit normal vector of x such that $\langle e_{n+1},e_{n+1}\rangle=-1$. By a direct calculation, we know that x has two distinct conformal principal curvatures 0 and $-\frac{1}{r}$ with multiplicities m and $n-m$, respectively. The conformal second fundamental form of x is parallel.

***Example* 3.3.** Spacelike hypersurface $x\colon\mathbb{H}^m(a)\times\mathbb{H}^{n-m}(\sqrt{r^2-a^2})\to\mathbb{H}_1^{n+1}(r)$, $0<a<r$. Let

$$x=(x_1,x_2)\in\mathbb{H}^m(a)\times\mathbb{H}^{n-m}\left(\sqrt{r^2-a^2}\right)\subset\mathbb{R}_1^{m+1}\times\mathbb{R}_1^{n-m+1},$$

$$\langle x_1,x_1\rangle=-a^2,\qquad \langle x_2,x_2\rangle=-\left(r^2-a^2\right),$$

and

$$e_{n+1}=\left(-\frac{\sqrt{r^2-a^2}}{a}\frac{x_1}{r},\frac{a}{\sqrt{r^2-a^2}}\frac{x_2}{r}\right)$$

be the unit normal vector of x such that $\langle e_{n+1},e_{n+1}\rangle=-1$. By a direct calculation, we know that x has two distinct conformal principal curvatures $\frac{c}{r}$ and $-\frac{1}{rc}$ with multiplicities m and $n-m$, respectively, where $c=\frac{\sqrt{r^2-a^2}}{a}$. The conformal second fundamental form of x is parallel.

***Example* 3.4** [12]**.** For any natural number p, q, $p+q<n$ and real number $a\in(1,+\infty)$ and $b=\sqrt{a^2-1}$, consider the immersed hypersurface $u\colon\mathbb{S}^p(a)\times\mathbb{R}^+\times\mathbb{R}^{n-p-q-1}\times\mathbb{H}^q(b)\to\mathbb{R}_1^{n+2}\subset$ $\subset\mathbb{R}_1^{n+1}:$

$$u(u',t,u'',u''')=(tu',u'',tu'''),\quad u'\in\mathbb{S}^p(a),\quad t\in\mathbb{R}^+,\quad u''\in\mathbb{R}^{n-p-q-1},\quad u'''\in\mathbb{H}^q(b).$$

Then $x=\sigma_0\circ u\colon\mathbb{S}^p(a)\times\mathbb{R}^+\times\mathbb{R}^{n-p-q-1}\times\mathbb{H}^q(b)\to\mathbb{Q}_1^{n+1}$ is a conformal regular spacelike hypersurface in $\mathbb{Q}_1^{n+1}$, it is denoted by $WP(p,q,a)=x(\mathbb{S}^p(a)\times\mathbb{R}^+\times\mathbb{R}^{n-p-q-1}\times\mathbb{H}^q(b))$. From [12], by a direct calculation, we know that $WP(p,q,a)$ has three distinct conformal principal curvatures and the conformal second fundamental form is parallel.

From Nomizu [15], Li and Xie [8], we know that the following:

Proposition 3.1 [15, 8]**.** *Let x be Euclidean isoparametric spacelike hypersurfaces in Lorentzian space forms. Then x can have at most two distinct Euclidean principal curvatures.*

Proposition 3.2. *Let $x\colon M \to \mathbb{Q}_1^{n+1}$ be an n-dimensional conformal isoparametric spacelike hypersurfaces in $\mathbb{Q}_1^{n+1}$ with constant normalized conformal scalar curvature κ and $\kappa \neq 1$. Then x is an n-dimensional Euclidean isoparametric spacelike hypersurfaces.*

Proof. Let κ and R be the normalized conformal scalar curvature and the normalized Euclidean scalar curvature. From [14], we know that $\kappa = R$. Let B_i and λ_i be the conformal principal curvatures and the Euclidean principal curvatures of x. Since (2.3) implies that the matrix (B_{ij}) and (h_{ij}) are commutative, we can choose a local orthonormal basis such that $B_{ij} = B_i\delta_{ij}$ and $h_{ij} = \lambda_i\delta_{ij}$. From (2.3), we have

$$e^{\tau}B_i = \lambda_i - H. \tag{3.1}$$

From (2.1), we have

$$0 = H\tau_i - H_i - \lambda_i\tau_i = (H - \lambda_i)\tau_i - H_i. \tag{3.2}$$

From the Gaussian equation of x, we have $n(n-1)(R-1) = \sum_{i,j} h_{ij}^2 - n^2H^2$. Thus

$$e^{2\tau} = \frac{n}{n-1}\left(\sum_{i,j} h_{ij}^2 - nH^2\right) = n^2(R - 1 + H^2). \tag{3.3}$$

Since κ is constant, we know that R is constant. From (3.3), $\tau_i = \dfrac{HH_i}{R-1+H^2}$. From (3.2),

$$0 = \frac{R - 1 + \lambda_i H}{R - 1 + H^2}H_i. \tag{3.4}$$

If H is not constant, then there is some i such that $H_{,i} \neq 0$. Thus $R - 1 + \lambda_i H = 0$ for such i. From (3.1), we have that $R - 1 + (e^{\tau}B_i + H)H = 0$ for such i. Combining with (3.3), we see that for such i

$$R - 1 + (n\sqrt{R - 1 + H^2}B_i + H)H = 0.$$

Thus, we see that for such i

$$(n^2B_i^2 - 1)H^4 + (n^2B_i^2 - 2)(R-1)H^2 - (R-1)^2 = 0. \tag{3.5}$$

Since B_i is constant, if $n^2B_i^2 - 1 = 0$, from (3.5) and $R \neq 1$, we infer that $H^2 = 1 - R$ is constant, this is a contradiction. If $n^2B_i^2 - 1 \neq 0$, by (3.5) and $R \neq 1$, we see that $H^2 = 1 - R$ or $H^2 = \dfrac{R-1}{n^2B_i^2 - 1}$, also a contradiction. We conclude that H must be constant. From (3.1), we know that λ_i are constant for all i .

Proposition 3.2 is proved.

4. Proofs of theorems. ***Proof of Theorem* 1.3.** From (2.4), we know that the number γ of distinct conformal principal curvatures can only take the values $\gamma = 2, 3$. From (2.13), we know that we can choose the local orthonormal basis E_i to diagonalize the matrix (B_{ij}) and (A_{ij}), that is, $B_{ij} = B_i\delta_{ij}$ and $A_{ij} = A_i\delta_{ij}$.

Let B_1, B_2, B_3 be the constant conformal principal curvatures of x. From (2.7), we have

$$\sum_k B_{ij,k}\omega_k = (B_i - B_j)\omega_{ij}. \tag{4.1}$$

We consider the following cases:

(1) If $\gamma = 2$, from Theorem 1.2, we know that Theorem 1.3 is true.

(2) If $\gamma = 3$ and the conformal second fundamental form is parallel. From Theorem 1.1, we know that Theorem 1.3 is true. If $\gamma = 3$ and the conformal second fundamental form is not parallel. We can prove that this case does not occur. In fact, since $B_1 \neq B_2 \neq B_3$, from (4.1), we have

$$B_{ii,k} = 0, \quad \text{for all} \quad i, k, \tag{4.2}$$

and

$$\omega_{ij} = \sum_k \frac{B_{ij,k}}{B_i - B_j}\omega_k, \quad \text{for} \quad i \neq j. \tag{4.3}$$

Since the conformal second fundamental form is not parallel, combining with (4.2), we know that $B_{12,3} \neq 0$. We may prove that $B_{12,3}$ is constant. In fact, from (2.14), (4.2) and (4.3), we have

$$\sum_k B_{12,3k}\omega_k = dB_{12,3}, \tag{4.4}$$

$$\sum_k B_{ii,jk}\omega_k = 2\sum_{l\neq i,j} B_{li,j}\omega_{li} = 2\sum_k\sum_{l\neq i,j}\frac{B_{li,j}B_{li,k}}{B_l - B_i}\omega_k. \tag{4.5}$$

Thus,

$$B_{ii,jk} = 2\sum_{l\neq i,j}\frac{B_{li,j}B_{li,k}}{B_l - B_i}. \tag{4.6}$$

From (4.2) and (4.6), we know that

$$B_{ii,ji} = B_{ii,jl} = 0, \quad \text{for distinct} \quad i, j, l. \tag{4.7}$$

From (2.15), we have

$$B_{ij,kl} - B_{ij,lk} = (B_i - B_j)R_{ijkl}.$$

From (2.12), we know that if three of $\{i, j, k, l\}$ are either the same or distinct, then $R_{ijkl} = 0$. Thus, if three of $\{i, j, k, l\}$ are either the same or distinct, then

$$B_{ij,kl} = B_{ij,lk}. \tag{4.8}$$

From (4.7), (4.8) and (2.13), we have $B_{12,31} = B_{11,23} = 0$, $B_{12,32} = B_{22,13} = 0$, $B_{12,33} = B_{33,12} = 0$. Thus, (4.4) implies that $dB_{12,3} = 0$. Therefore, we know that $B_{12,3}$ is constant. From (4.3) and (2.8),

$$-\frac{1}{2}\sum_{k,l} R_{12kl}\omega_k \wedge \omega_l = d\omega_{12} - \omega_{13}\wedge\omega_{32} = -\frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)}\omega_1 \wedge \omega_2,$$

$$-\frac{1}{2}\sum_{k,l}R_{13kl}\omega_k\wedge\omega_l = d\omega_{13}-\omega_{12}\wedge\omega_{23} = -\frac{2B_{12,3}^2}{(B_1-B_2)(B_3-B_2)}\omega_1\wedge\omega_3,$$

$$-\frac{1}{2}\sum_{k,l}R_{23kl}\omega_k\wedge\omega_l = d\omega_{23}-\omega_{21}\wedge\omega_{13} = -\frac{2B_{12,3}^2}{(B_2-B_1)(B_3-B_1)}\omega_2\wedge\omega_3.$$

Thus,

$$R_{1212} = \frac{2B_{12,3}^2}{(B_1-B_3)(B_2-B_3)},$$

$$R_{1313} = \frac{2B_{12,3}^2}{(B_1-B_2)(B_3-B_2)},$$

$$R_{2323} = \frac{2B_{12,3}^2}{(B_2-B_1)(B_3-B_1)}.$$

We have

$$\kappa = \frac{1}{6}\sum_{i\neq j}R_{ijij} = R_{1212}+R_{1313}+R_{2323} = 0.$$

From (3.1) and Proposition 3.2, we know that x is a 3-dimensional Euclidean isoparametric spacelike hypersurfaces with three distinct Euclidean principal curvatures. This is in contradiction with Proposition 3.1.

Theorem 1.3 is proved.

***Proof of Theorem* 1.4.** From (2.4), we know that the number γ of distinct conformal principal curvatures can only take the values $\gamma = 2, 3, 4$. From (2.13), we know that we can choose the local orthonormal basis E_i to diagonalize the matrix (B_{ij}) and (A_{ij}), that is, $B_{ij} = B_i\delta_{ij}$ and $A_{ij} = A_i\delta_{ij}$.

Let B_1, B_2, B_3, B_4 be the constant conformal principal curvatures of x. We consider the following cases:

(1) If $\gamma = 2$, from Theorem 1.2, we know that Theorem 1.4 is true.

(2) If $\gamma = 3$ and the conformal second fundamental form is parallel. From Theorem 1.1, we know that Theorem 1.4 is true. If $\gamma = 3$ and the conformal second fundamental form is not parallel. We can prove that this case does not occur. In fact, without loss of generality, we may assume that $B_1 \neq B_2 \neq B_3 = B_4$. From (4.1), we have

$$B_{ii,k} = 0, \qquad B_{34,k} = 0, \quad \text{for all} \quad i, k, \tag{4.9}$$

and

$$\omega_{ij} = \sum_k \frac{B_{ij,k}}{B_i - B_j}\omega_k, \quad \text{for} \quad B_i \neq B_j. \tag{4.10}$$

From (4.9), (4.10) and (2.14), we have

$$\sum_l B_{13,4l}\omega_l = B_{12,4}\omega_{23} + B_{12,3}\omega_{24} = \frac{2B_{12,3}B_{12,4}}{B_2 - B_3}\omega_1, \tag{4.11}$$

$$\sum_l B_{11,3l}\omega_l = 2B_{12,3}\omega_{21} = \frac{2B_{12,3}^2}{B_2 - B_1}\omega_3 + \frac{2B_{12,3}B_{12,4}}{B_2 - B_1}\omega_4. \tag{4.12}$$

Comparing two side of (4.11) and (4.12), we have

$$B_{13,41} = \frac{2B_{12,3}B_{12,4}}{B_2 - B_3}, \qquad B_{13,42} = B_{13,43} = B_{13,44} = 0, \tag{4.13}$$

$$B_{11,33} = \frac{2B_{12,3}^2}{B_2 - B_1}, \qquad B_{11,34} = \frac{2B_{12,3}B_{12,4}}{B_2 - B_1}, \qquad B_{11,32} = 0. \tag{4.14}$$

From (4.8), (2.13), (4.13) and (4.14), we have $B_{12,3}B_{12,4} = 0$. Since the conformal second fundamental form is not parallel, without loss of generality, we may assume that $B_{12,3} \neq 0$ and $B_{12,4} = 0$. We may also prove that $B_{12,3}$ is constant. In fact, from (2.14), (4.9) and (4.10), we have

$$\sum_k B_{12,3k}\omega_k = dB_{12,3}, \tag{4.15}$$

$$\sum_k B_{ii,jk}\omega_k = 2\sum_{l\neq i,j} B_{li,j}\omega_{li} = 2\sum_k \sum_{l\neq i,j} \frac{B_{li,j}B_{li,k}}{B_l - B_i}\omega_k, \quad \text{for} \quad B_l \neq B_i. \tag{4.16}$$

Thus,

$$B_{ii,jk} = 2\sum_{l\neq i,j} \frac{B_{li,j}B_{li,k}}{B_l - B_i}, \quad \text{for} \quad B_l \neq B_i. \tag{4.17}$$

From (4.9) and (4.17), we know that

$$B_{ii,ji} = B_{ii,jl} = 0, \quad \text{for distinct} \quad i, j, l. \tag{4.18}$$

From (4.18), (4.8) and (2.13), we have

$$B_{12,31} = B_{11,23} = 0, \qquad B_{12,32} = B_{22,13} = 0, \qquad B_{12,33} = B_{33,12} = 0. \tag{4.19}$$

On the other hand, from (4.9), (4.10) and $B_{12,4} = 0$, we have

$$\sum_k B_{34,1k}\omega_k = B_{12,3}\omega_{24} = \sum_k \frac{B_{12,3}B_{24,k}}{B_2 - B_4}\omega_k.$$

Thus,

$$B_{34,1k} = \frac{B_{12,3}B_{24,k}}{B_2 - B_4},$$

and we have $B_{34,12} = 0$. From (4.8) and (2.13), we have

$$B_{12,34} = B_{34,12} = 0. \tag{4.20}$$

(4.19) and (4.20) imply that $dB_{12,3} = 0$. Therefore, we know that $B_{12,3}$ is constant.

From (4.9) and (4.10), we have

$$\omega_{12} = \frac{B_{12,3}}{B_1 - B_2}\omega_3, \qquad \omega_{13} = \frac{B_{12,3}}{B_1 - B_3}\omega_2, \qquad \omega_{23} = \frac{B_{12,3}}{B_2 - B_3}\omega_1, \qquad \omega_{14} = \omega_{24} = 0. \tag{4.21}$$

From (4.21) and (2.8), by a simple calculation, we have

$$-\frac{1}{2}\sum_{k,l} R_{12kl}\omega_k \wedge \omega_l = d\omega_{12} - \omega_{13} \wedge \omega_{32} =$$

$$= -\frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)}\omega_1 \wedge \omega_2 - \frac{B_{12,3}}{B_1 - B_2}\omega_4 \wedge \omega_{34}, \tag{4.22}$$

$$-\frac{1}{2}\sum_{k,l} R_{13kl}\omega_k \wedge \omega_l = d\omega_{13} - \omega_{12} \wedge \omega_{23} = -\frac{2B_{12,3}^2}{(B_1 - B_2)(B_3 - B_2)}\omega_1 \wedge \omega_3, \tag{4.23}$$

$$-\frac{1}{2}\sum_{k,l} R_{14kl}\omega_k \wedge \omega_l = -\omega_{13} \wedge \omega_{34} = -\frac{B_{12,3}}{B_1 - B_3}\omega_2 \wedge \omega_{34}, \tag{4.24}$$

$$-\frac{1}{2}\sum_{k,l} R_{23kl}\omega_k \wedge \omega_l = d\omega_{23} - \omega_{21} \wedge \omega_{13} = -\frac{2B_{12,3}^2}{(B_2 - B_1)(B_3 - B_1)}\omega_2 \wedge \omega_3, \tag{4.25}$$

$$-\frac{1}{2}\sum_{k,l} R_{24kl}\omega_k \wedge \omega_l = d\omega_{24} - \omega_{23} \wedge \omega_{34} = -\frac{B_{12,3}}{B_2 - B_3}\omega_1 \wedge \omega_{34}. \tag{4.26}$$

Let $\omega_{34} = \sum_k \Gamma_{k4}^3 \omega_k$, $\Gamma_{k4}^3 = -\Gamma_{k3}^4$. Comparing two side of (4.22)–(4.26), we have

$$R_{1212} = \frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)},$$

$$R_{1313} = \frac{2B_{12,3}^2}{(B_1 - B_2)(B_3 - B_2)},$$

$$R_{2323} = \frac{2B_{12,3}^2}{(B_2 - B_1)(B_3 - B_1)}, \qquad R_{1414} = R_{2424} = 0.$$

From (4.22), (4.24) and (4.26), we know that

$$\frac{1}{2}R_{12k4} = \frac{B_{12,3}}{B_2 - B_1}\Gamma_{k4}^3, \qquad \frac{1}{2}R_{142k} = \frac{B_{12,3}}{B_1 - B_3}\Gamma_{k4}^3, \qquad \frac{1}{2}R_{24k1} = \frac{B_{12,3}}{B_3 - B_2}\Gamma_{k4}^3. \tag{4.27}$$

Since we know that the Bianchi identities of curvature tensors R_{ijkl} are $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ and $R_{ijkl} = R_{klij}$, $R_{ijlk} = R_{jikl}$, we have $R_{142k} + R_{12k4} + R_{24k1} = 0$. Thus, from (4.27), we have $\Gamma_{k4}^3 = 0$ for all k. Thus $\omega_{34} = 0$. From (4.21) and (2.8)

$$-\frac{1}{2}\sum_{k,l} R_{34kl}\omega_k \wedge \omega_l = d\omega_{34} - \sum_k \omega_{3k} \wedge \omega_{k4} = 0.$$

This implies that $R_{3434} = 0$. We have

$$\kappa = \frac{1}{12}\sum_{i \neq j} R_{ijij} = R_{1212} + R_{1313} + R_{1414} + R_{2323} + R_{2424} + R_{3434} = 0.$$

From (3.1) and Proposition 3.2, we know that x is a 4-dimensional Euclidean isoparametric spacelike hypersurfaces with three distinct Euclidean principal curvatures. This is in contradiction with Proposition 3.1. Thus, we know that Theorem 1.4 is true.

(3) If $\gamma = 4$, from [12], we know that the conformal second fundamental form is not parallel. We can prove that this case does not occur. In fact, we may assume that $B_1 \neq B_2 \neq B_3 \neq B_4$. Denote by i, j, k, l the four distinct elements of $\{1, 2, 3, 4\}$ with order arbitrarily given, then from (2.7), we have

$$\omega_{ij} = \frac{B_{ij,k}\omega_k + B_{ij,l}\omega_l}{B_i - B_j}, \quad \text{for} \quad i \neq j. \tag{4.28}$$

From (4.28) and (2.8), by a simple calculation (see [5]), we have

$$-\frac{1}{2}\sum_{s,t} R_{ijst}\omega_s \wedge \omega_t = d\omega_{ij} - \omega_{ik} \wedge \omega_{kj} - \omega_{il} \wedge \omega_{lj} \equiv$$

$$\equiv -\left(\frac{2B_{ij,k}^2}{(B_i - B_k)(B_j - B_k)} + \frac{2B_{ij,l}^2}{(B_i - B_l)(B_j - B_l)}\right)\omega_i \wedge \omega_j$$

$$\mathrm{mod}\left(\omega_s \wedge \omega_t, (s,t) \neq (i,j), (j,i)\right).$$

Comparing two side of the above equation, we have

$$R_{ijij} = \frac{2B_{ij,k}^2}{(B_i - B_k)(B_j - B_k)} + \frac{2B_{ij,l}^2}{(B_i - B_l)(B_j - B_l)}.$$

Thus,

$$R_{1212} = \frac{2B_{12,3}^2}{(B_1 - B_3)(B_2 - B_3)} + \frac{2B_{12,4}^2}{(B_1 - B_4)(B_2 - B_4)},$$

$$R_{1313} = \frac{2B_{12,3}^2}{(B_1 - B_2)(B_3 - B_2)} + \frac{2B_{13,4}^2}{(B_1 - B_4)(B_3 - B_4)},$$

$$R_{1414} = \frac{2B_{13,4}^2}{(B_1 - B_3)(B_4 - B_3)} + \frac{2B_{12,4}^2}{(B_1 - B_2)(B_4 - B_2)},$$

$$R_{2323} = \frac{2B_{12,3}^2}{(B_2 - B_1)(B_3 - B_1)} + \frac{2B_{23,4}^2}{(B_2 - B_4)(B_3 - B_4)},$$

$$R_{2424} = \frac{2B_{12,4}^2}{(B_2 - B_1)(B_4 - B_1)} + \frac{2B_{23,4}^2}{(B_2 - B_3)(B_4 - B_3)},$$

$$R_{3434} = \frac{2B_{13,4}^2}{(B_3 - B_1)(B_4 - B_1)} + \frac{2B_{23,4}^2}{(B_3 - B_2)(B_4 - B_2)}.$$

We have

$$\kappa = \frac{1}{12}\sum_{i\neq j} R_{ijij} = R_{1212} + R_{1313} + R_{1414} + R_{2323} + R_{2424} + R_{3434} = 0.$$

From (3.1) and Proposition 3.2, we know that x is a 4-dimensional Euclidean isoparametric spacelike hypersurfaces with four distinct Euclidean principal curvatures. This is in contradiction with Proposition 3.1.

Theorem 1.4 is proved.

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