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FOURIER COSINE AND SINE TRANSFORMS AND GENERALIZED LIPSCHITZ CLASSES IN UNIFORM METRIC*

КОСИНУС- І СИНУС-ПЕРЕТВОРЕННЯ ФУР'Є ТА УЗАГАЛЬНЕНІ КЛАСИ ЛІПШИЦЯ В РІВНОМІРНІЙ МЕТРИЦІ

For functions $f \in L^1(\mathbb{R}_+)$ with cosine (sine) Fourier transforms $\hat{f}_c$ $(\hat{f}_s)$ in $L^1(\mathbb{R})$, we give necessary and sufficient conditions in terms of $\hat{f}_c$ $(\hat{f}_s)$ for f to belong to generalized Lipschitz classes $H^{\omega,m}$ and $h^{\omega,m}$. Conditions for the uniform convergence of the Fourier integral and for the existence of the Schwartz derivative are also obtained.

Для функцій $f \in L^1(\mathbb{R}_+)$ із косинус-(синус-) перетвореннями Фур'є $\hat{f}_c$ $(\hat{f}_s)$ у $L^1(\mathbb{R})$ наведено (в термінах $\hat{f}_c$ $(\hat{f}_s)$) необхідні та достатні умови належності функцій f до узагальнених класів Ліпшиця $H^{\omega,m}$ та $h^{\omega,m}$. Також отримано умови рівномірної збіжності інтеграла Фур'є та існування похідної Шварца.

1. Introduction. Let $f\colon \mathbb{R} \to \mathbb{C}$ be a Lebesgue integrable function over $\mathbb{R}_+ = [0, +\infty)$, i.e., $f \in L^1(\mathbb{R}_+)$. Then the Fourier cosine and sine transforms of f are defined by

$$\hat{f}_c(x) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_{\mathbb{R}_+} f(t)\cos xt\,dt, \qquad \hat{f}_s(x) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_{\mathbb{R}_+} f(t)\sin xt\,dt, \quad x \in \mathbb{R}.$$

If, in addition, $\hat{f}_c \in L^1(\mathbb{R}_+)$ $(\hat{f}_s \in L^1(\mathbb{R}_+))$ and $f \in C(\mathbb{R}_+)$ (f is continuous on $\mathbb{R}_+$), then the inversion formula

$$f(t) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_{\mathbb{R}_+} \hat{f}_c(x)\cos xt\,dx \qquad \left(f(t) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_{\mathbb{R}_+} \hat{f}_s(x)\sin xt\,dx\right) \tag{1.1}$$

takes place for all $t \in \mathbb{R}_+$. A proof is similar to that of inversion formula for

$$\hat{f}(x) = (2\pi)^{-1/2} \int\limits_{\mathbb{R}} f(t) e^{-ixt}\,dt$$

and $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ (see [1, p. 192], Chapter 5). In this case we have also $\lim_{x\to+\infty} f(x) = 0$, that is $f \in C_0(\mathbb{R}_+)$. In all results connected with cosine (sine) Fourier transform we consider the even (odd) extension f_e (f_o) of a function $f \in C_0(\mathbb{R}_+)$ onto $\mathbb{R}$. For $m \in \mathbb{N}$ and f defined on $\mathbb{R}$ let introduce the m-th symmetric difference $\dot{\Delta}_h^m f(x) = \sum_{j=0}^m (-1)^{m-j}\binom{m}{j} f(x + (m-2j)h/2)$. If $f \in C_0(\mathbb{R}_+)$ (i.e., $f \in C(\mathbb{R})$ and $\lim_{x\to\pm\infty} f(x) = 0$) and $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$, then $\omega_m(f,\delta) :=$ $:= \sup\{\|\dot{\Delta}_h^m f\| : 0 \le h \le \delta\}$ is the m-th modulus of smoothness.

Denote by Φ the set of all continuous and increasing on $\mathbb{R}_+$ functions ω such that $\omega(0) = 0$ and $\omega(2t) \le C\omega(t)$, $t \in \mathbb{R}_+$. If $\omega \in \Phi$ and $\int_0^\delta t^{-1}\omega(t)\,dt = O(\omega(\delta))$, then ω belongs to the Bari class

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B; if $\omega \in \Phi$ and $\delta^m \int_\delta^\infty t^{-m-1}\omega(t)\,dt = O(\omega(\delta))$, $m > 0$, then ω belongs to the Bari–Stechkin class B_m (see [2]). If $\omega \in \Phi$ and $\omega(\lambda\delta) \le C\lambda^m\omega(\delta)$ for all $\lambda \ge 1$, $\delta > 0$, then $\omega \in N^m$. It is well known that $\omega_m(f,\delta) \in N^m$ (see [3], Chapter 3). By definition, $H^{\omega,m} = \{f \in C_0(\mathbb{R})\colon \omega_m(f,t) \le C\omega(t), t \in \mathbb{R}_+\}$ and $h^{\omega,m} = \{f \in C_0(\mathbb{R})\colon \omega_m(f,t) = o(\omega(t)), t \to 0\}$ for $\omega \in \Phi$. The class $H^{\omega,1}(h^{\omega,1})$ with $\omega(t) = t^\alpha$, $0 < \alpha \le 1$, will be denoted by $\mathrm{Lip}(\alpha)$ $(\mathrm{lip}(\alpha))$. There is a different notation for the class $H^{\omega,2}$ $(h^{\omega,2})$ with $\omega(t) = t^\alpha$, $0 < \alpha \le 2$. In the paper [4] it was denoted by $\mathrm{Zyg}(\alpha)$ $(\mathrm{zyg}(\alpha))$. F. Moricz [4] established several theorems connecting the behaviour of $\hat{f}$ and classes $\mathrm{Lip}(\alpha)$, $\mathrm{Zyg}(\alpha)$, $\mathrm{lip}(\alpha)$, $\mathrm{zyg}(\alpha)$. The main content of these results is represented in the following theorem.

Theorem A. (i) *If $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ and for some $\alpha \in (0,m]$, $m = 1,2$, we have*

$$\int\limits_{|t|<y} |t^m \hat{f}(t)|\,dt = O(y^{m-\alpha}) \quad \text{for all} \quad y > 0, \tag{1.2}$$

then $\hat{f} \in L^1(\mathbb{R})$ and $f \in \mathrm{Lip}(\alpha)$ for $m = 1$ and $f \in \mathrm{Zyg}(\alpha)$ for $m = 2$.

(ii) *If $f, \hat{f} \in L^1(\mathbb{R})$, $f \in \mathrm{Lip}(\alpha)$ for some $\alpha \in (0,1]$, $m = 1$, or $f \in \mathrm{Zyg}(\alpha)$ for some $\alpha \in (0,2]$, $m = 2$, and $t^m\hat{f}(t) \ge 0$ for all $t \in \mathbb{R}$, then (1.2) holds.*

(iii) *Both statements (i) and (ii) are valid for $0 < \alpha < m$, $m = 1,2$, if the right-hand side of (1.2) is replaced by $o(y^{m-\alpha})$, $y \to 0$, and the condition $f \in \mathrm{Lip}(\alpha)$ or $f \in \mathrm{Zyg}(\alpha)$ is replaced by $f \in \mathrm{lip}(\alpha)$ or $f \in \mathrm{zyg}(\alpha)$ correspondingly.*

In the paper [5] Theorem A was generalized to arbitrary $m \in \mathbb{N}$ and ω belonging to the class B or B_m. Such theorems in the case of trigonometric series are known as Boas-type results. Interesting survey of earlier results may be found in [6]. R. P. Boas, L. Leindler, J. Nemeth and S. Tikhonov [7, 8] considered the cases of cosine and sine series separately, while F. Moricz [9–11] and second author [12] studied such conditions in terms of complex Fourier coefficients (about papers of L. Leindler and J. Nemeth see Introduction and references in [7]). Let a_n, b_n are cosine and sine coefficients of $f \in L^1_{2\pi}$ and $\omega_\beta(f,\delta)$ is a modulus of continuity of order $\beta > 0$. Using our notations, we can formulate S. Tikhonov's results from [7] as follows.

Theorem B. *Let $\omega \in \Phi$ and $\beta > 0$, $f \in C_{2\pi}$ is even, $a_n \ge 0$ for all $n \in \mathbb{Z}_+$.*

(A) *If $\beta \ne 2l - 1$, $l \in \mathbb{N}$, and $\omega \in B$, then the conditions $\omega_\beta(f, 1/n) = O(\omega(1/n))$, $n \in \mathbb{N}$, and $\sum_{k=1}^n k^\beta a_k = O(n^\beta\omega(1/n))$ are equivalent.*

(B) *If $\beta = 2l - 1$, $l \in \mathbb{N}$, and $\omega \in B$, then the condition $\omega_\beta(f,1/n) = O(\omega(1/n))$ is equivalent to*

$$\sum_{k=1}^n k^{\beta+1} a_k = O(n^{\beta+1}\omega(1/n)), \quad n \in \mathbb{N},$$

and

$$\sum_{k=1}^n k^\beta a_k \sin kx = O(n^\beta\omega(1/n)), \quad n \in \mathbb{N},$$

uniformly in $x \in [0, 2\pi]$.

(C) *If $\omega \in B_\beta$, then the conditions $\omega_\beta(f,1/n) = O(\omega(1/n))$, $n \in \mathbb{N}$, and $\sum_{k=n}^\infty a_k = O(\omega(1/n))$, $n \in \mathbb{N}$, are equivalent.*

Parts (A) and (B) of Theorem B are valid for odd functions f, but exceptional values of β are $2l$, $l \in \mathbb{N}$ (see [7]). V. Fülöp [13] obtained analogs of the Theorem A for cosine and sine Fourier transforms.

By definition, a function f has the Schwartz derivative of order $m \in \mathbb{N}$ in the point x and this derivative equals to A if there exists $\lim_{h\to 0} h^{-m}\dot{\Delta}_h^m f(x) = A$. In [5] the following result is proved.

Theorem C. *Let $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, $m \in \mathbb{N}$ and*

$$\int\limits_{|t|>y} |\hat{f}(t)|\, dt = o(y^{-m}), \quad y \to +\infty.$$

Then the Schwartz derivative of order m exists at the point x and equals to A if and only if the principal value of the integral $(2\pi)^{-1/2} \int_{\mathbb{R}} (it)^m \hat{f}(t) e^{itx}\, dt$ exists and equals to A.

It is known the following theorem of R. Paley [15] (see also [16, p. 277], Ch. 4).

Theorem D. *Let the Fourier series $a_0/2 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ of a function $f \in C_{2\pi}$ has non-negative coefficients a_n, b_n. Then this series converges uniformly on $\mathbb{R}$.*

F. Moricz [11] proved a similar result.

Theorem E. *Let the Fourier series $\sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikx}$ of a function $f \in C_{2\pi}$ is such that $k\hat{f}(k) \ge 0$, $k \in \mathbb{Z}$. Then this series converges uniformly on $\mathbb{R}$.*

The aim of present paper is to obtain the sufficient conditions in order that functions to belong to the class $H^{\omega,m}$ or $h^{\omega,m}$ in terms of cosine and sine Fourier transforms. These conditions are necessary for functions with non-negative cosine and sine transforms. Also we obtain analogs of Theorems C and D (see Theorems 3 and 4). Theorem 5 is a generalization of Theorems 4, 5 and 8 from the paper [13].

2. Auxiliary results. For $f \in L^1(\mathbb{R}_+)$ let us consider the Fejer operator

$$\sigma_t(f)_c(x) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_0^t \left(1 - \frac{|u|}{t}\right) \hat{f}_c(u) \cos xu\, du, \quad x \in \mathbb{R}_+,$$

and de La Vallee Poussin operator $v_t(f)_c = 2\sigma_{2t}(f) - \sigma_t(f)$. Similarly we define $\sigma_t(f)_s(x)$ and $v_t(f)_s(x)$. By definition $\sigma_t(f)_c(x)$ and $v_t(f)_c(x)$ are even while $\sigma_t(f)_s(x)$ and $v_t(f)_s(x)$ are odd. Let us remind that an entire function $f(z)$ has exponential type $t \ge 0$ ($f \in E_t$) if for each $\varepsilon > 0$ there exists $A = A(\varepsilon) > 0$ such that $|f(z)| \le Ae^{(t+\epsilon)|z|}$ for all $z \in \mathbb{C}$. By $UC(\mathbb{R})$ ($BUC(\mathbb{R})$) we denote the space of uniformly continuous (bounded uniformly continuous) functions on $\mathbb{R}$. For a function $f \in BUC(\mathbb{R})$ we set $A_t(f) = \inf\{\|f - g\|_\infty : g \in BUC(\mathbb{R}) \cap E_t\}$, $t \in \mathbb{R}_+$.

It is clear that $C_0(\mathbb{R}) \subset BUC(\mathbb{R})$. Lemma 1 connects the direct approximation theorems for $A_t(f)$ and properties of $v_t(f)_c$ ($v_t(f)_s$) (see [14], Ch. 5, § 5.1 and Ch. 8, § 8.6).

Lemma 1. *If $f \in BUC(\mathbb{R})$, $m \in \mathbb{N}$, $t > 0$ and f is even (odd), then*

$$\|f - v_t(f)_c\|_\infty \le C_1 A_t(f) \le C_2 \omega_m(f, 1/t)$$

$$(\|f - v_t(f)_s\|_\infty \le C_1 A_t(f) \le C_2 \omega_m(f, 1/t)).$$

A function $\gamma(t)$ will be called almost increasing (almost decreasing) if there exists a constant $k := k(\gamma) \ge 1$, such that $k\gamma(t) \ge \gamma(u)$ ($k\gamma(u) \ge \gamma(t)$) for $0 \le u \le t$.

Lemma 2 [2]. (i) *Let* $\omega \in \Phi$. *Then* $\omega \in B_k$, $k \in \mathbb{N}$, *if and only if there exists* $\alpha \in (0,k)$ *such that* $t^{\alpha-k}\omega(t)$ *is almost decreasing.*

(ii) *Let* $\omega \in \Phi$. *Then* $\omega \in B$ *if and only if there exists* $\alpha \in (0,1)$ *such that* $t^{-\alpha}\omega(t)$ *is almost increasing.*

Lemma 3. *Let* $F \in L^1(\mathbb{R}_+)$ *is differentiable on* $\mathbb{R}_+$ *and* $F' = f \in L^1(\mathbb{R}_+)$. *Then* $t\hat{F}_c(t) = -\hat{f}_s(t)$ *and* $t\hat{F}_s(t) - (2/\pi)^{1/2}F(0) = \hat{f}_c(t)$ *on* $\mathbb{R}_+$.

Proof. We have $F(x) = F(0) + \int_0^x f(t)\,dt$, $x \in \mathbb{R}_+$. Since $f \in L^1(\mathbb{R}_+)$, there exists $\lim_{x\to+\infty} F(x) = F(0) + \int_0^\infty f(t)\,dt$. But $F \in L^1(\mathbb{R}_+)$ implies $\lim_{x\to+\infty} F(x) = 0$. Using integration by parts, we obtain

$$\hat{f}_s(t) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_{\mathbb{R}_+} f(u)\sin tu\,du = \left(\frac{2}{\pi}\right)^{1/2} \left[F(u)\sin tu\,|_0^\infty - \int\limits_{\mathbb{R}_+} t\cos tuF(u)\,du\right] = -t\hat{F}_c(t).$$

Second identity is proved in a similar way.

Lemma 3 is proved.

Lemma 4 [5]. (i) *If* $\omega \in B_m$, $m \in \mathbb{N}$, $g(t)$ *is a non-negative measurable function and*

$$\int\limits_y^\infty g(t)dt = O\left(\omega\left(1/y\right)\right), \quad y > 0, \tag{2.1}$$

then $y^m g(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$ *and*

$$\int\limits_0^y t^m g(t)\,dt = O\left(y^m\omega\left(1/y\right)\right), \quad y > 0. \tag{2.2}$$

(ii) *If* $\omega \in B$, $g(t)$ *is a non-negative measurable function and* $t^m g(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$, *then* (2.2) *implies* (2.1).

Lemma 5 [5]. (i) *If* $\omega \in B_m$, $m \in \mathbb{N}$, $g(x)$ *is a non-negative, measurable function on* $\mathbb{R}_+$ *satisfying* (2.1) *and*

$$\int\limits_y^\infty g(t)\,dt = o(\omega(y^{-1})), \quad y \to +\infty, \tag{2.3}$$

then $t^m g(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$ *and*

$$\int\limits_0^y t^m g(t)\,dt = o(y^m\omega(y^{-1})), \quad y \to +\infty. \tag{2.4}$$

(ii) *If* $\omega \in B$, $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a measurable function such that* $t^m g(t) \in L^1_{\text{loc}}(\mathbb{R}_+)$ *and* (2.4) *holds, then* (2.3) *also holds.*

3. Main results.

Theorem 1. (i) *If* $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, $m \in \mathbb{N}$, $\omega \in B$ *and*

$$\int_0^y t^m |\hat{f}_c(t)|\,dt = O(y^m \omega(1/y)) \quad \text{for all} \quad y > 0, \tag{3.1}$$

or

$$\int_0^y t^m |\hat{f}_s(t)|\,dt = O(y^m \omega(1/y)) \quad \text{for all} \quad y > 0, \tag{3.2}$$

then $\hat{f}_c \in L^1(\mathbb{R})$ *(or* $\hat{f}_s \in L^1(\mathbb{R})$*) and* $f_e \in H^{\omega,m}$ *(or* $f_o \in H^{\omega,m}$*).*

(ii) *If* $m \in \mathbb{N}$ *be even,* $f_e \in L^1(\mathbb{R}) \cap H^{\omega,m}$ *and* $\hat{f}_c(t)$ *keeps its sign on* $\mathbb{R}_+$*, then* (3.1) *holds. If* $m \in \mathbb{N}$ *be odd,* $\omega \in B_m$*,* $f_e \in L^1(\mathbb{R}) \cap H^{\omega,m}$ *and* $\hat{f}_c(t)$ *keeps its sign on* $\mathbb{R}_+$*, then* (3.1) *holds.*

(iii) *If* $m \in \mathbb{N}$ *be odd,* $\omega \in \Phi$*,* $f_o \in L^1(\mathbb{R}) \cap H^{\omega,m}$ *and* $\hat{f}_s(t)$ *keeps its sign on* $\mathbb{R}_+$*, then* (3.2) *holds. If* $m \in \mathbb{N}$ *be even and* $f_o \in L^1(\mathbb{R}) \cap H^{\omega,m}$ *and* $\hat{f}_s(t)$ *keeps its sign on* $\mathbb{R}_+$*, then* (3.2) *holds.*

Proof. (i) By Lemma 4(i) the integral $\int_y^\infty |\hat{f}_c(t)|\,dt$ is finite for all $y > 0$ and it is well known that $\hat{f}_c \in C_0(\mathbb{R}_+)$. Therefore, $\hat{f}_c \in L^1(\mathbb{R}_+)$. Further,

$$\dot{\Delta}_h^m \cos xt = \operatorname{Re} \dot{\Delta}_h^m e^{ixt} = \operatorname{Re}\left[e^{ixt}\left(2i \sin\frac{ht}{2}\right)^m\right], \qquad m \in \mathbb{N}, \quad h > 0.$$

For even m we have $\dot{\Delta}_h^m \cos xt = (-1)^{m/2} \cos xt (2 \sin ht/2)^m$ and for odd m we see that $\dot{\Delta}_h^m \cos xt = (-1)^{(m+1)/2} \sin xt (2\sin ht/2)^m$. Similar formulas are valid for $\dot{\Delta}_h^m \sin xt$. By the inversion formula (1.1) we find that

$$\dot{\Delta}_h^m f_e(x) = \begin{cases} \left(\dfrac{2}{\pi}\right)^{1/2} (-1)^{m/2} \displaystyle\int_{\mathbb{R}_+} \hat{f}_c(t) \cos xt \left(2 \sin\frac{ht}{2}\right)^m dt, & m \text{ is even}, \\ \left(\dfrac{2}{\pi}\right)^{1/2} (-1)^{(m+1)/2} \displaystyle\int_{\mathbb{R}_+} \hat{f}_c(t) \sin xt \left(2 \sin\frac{ht}{2}\right)^m dt, & m \text{ is odd}, \end{cases} \tag{3.3}$$

and

$$\dot{\Delta}_h^m f_o(x) = \begin{cases} \left(\dfrac{2}{\pi}\right)^{1/2} (-1)^{m/2} \displaystyle\int_{\mathbb{R}_+} \hat{f}_s(t) \sin xt \left(2 \sin\frac{ht}{2}\right)^m dt, & m \text{ is even}, \\ \left(\dfrac{2}{\pi}\right)^{1/2} (-1)^{(m+1)/2} \displaystyle\int_{\mathbb{R}_+} \hat{f}_s(t) \cos xt \left(2 \sin\frac{ht}{2}\right)^m dt, & m \text{ is odd}. \end{cases} \tag{3.4}$$

Thus, in all cases $\dot{\Delta}_h^m f_e(x)$ $\left(\dot{\Delta}_h^m f_o(x)\right)$ is either even or odd function of x. From (3.3) we deduce

$$|\dot{\Delta}_h^m f_e(x)| \le \left(\frac{2}{\pi}\right)^{1/2} \left(\int_0^{1/h} + \int_{1/h}^{\infty}\right) |\hat{f}_c(t)| \left|2 \sin\frac{ht}{2}\right|^m dt =: \left(\frac{2}{\pi}\right)^{1/2} (I_h + J_h)$$

for $h > 0$. By (3.1) and inequality $|\sin t| \le t$, $t \in \mathbb{R}_+$, we have

$$|I_h| \leq \int\limits_0^{1/h} h^m t^m |\hat{f}_c(t)|\, dt \leq C_1 h^m h^{-m} \omega(h) = C_1 \omega(h). \tag{3.5}$$

On the other hand, by Lemma 4(ii) we see that

$$|J_h| \leq 2^m \int\limits_{1/h}^{\infty} |\hat{f}(t)|\, dt \leq C_2 \omega(h). \tag{3.6}$$

Combining (3.1) and (3.2) yields $f_e \in H^{\omega,m}$. For $\hat{f}_s$ and f_o the proof is similar.

(ii) Let $\hat{f}_c(t) \geq 0$ for $t \geq 0$ and m is even. Then from the condition $f_e \in H^{\omega,m}$ and inequality $\sin t \geq 2t/\pi$, $t \in [0, \pi/2]$, we obtain

$$C_3 \omega(h) \geq |\dot{\Delta}_h^m f(0)| = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_{\mathbb{R}_+} \hat{f}_c(t) \left(2 \sin \frac{ht}{2}\right)^m dt \geq C_4 \int\limits_0^{1/h} \hat{f}_c(t)(ht)^m\, dt$$

or $\displaystyle\int_0^{1/h} t^m \hat{f}_c(t)\, dt \leq C_5 h^{-m} \omega(h)$, that is equivalent to (3.1).

If $\hat{f}_c(t) \geq 0$ for $t \geq 0$ and m is odd, then by Lemma 1 we have

$$f_e(0) - v_t(f_e)(0) = \left(\frac{2}{\pi}\right)^{1/2} \left(\int\limits_t^{2t} \left(\frac{u}{t} - 1\right) \hat{f}_c(u)\, du + \int\limits_{2t}^{\infty} \hat{f}_c(u)\, du \right) \leq C_6 \omega\left(\frac{1}{t}\right),$$

whence

$$\int\limits_t^{\infty} \hat{f}_c(u)\, du \leq C_7 \omega\left(\frac{2}{t}\right) \leq C_8 \omega \frac{1}{t}.$$

Using condition $\omega \in B_m$ and Lemma 4(i), we obtain (3.1).

(iii) If $\hat{f}_s(t) \geq 0$ for $t \geq 0$ and m is odd, then the proof is similar to that of the item (ii) for even m. Let $\hat{f}_s(t) \geq 0$ for $t \geq 0$, m is even and $f \in H^{\omega,m}$. Then for $t > 0$ by (3.4) we have

$$C_9 \omega(t) \geq |\dot{\Delta}_t^m f(x)| = \left(\frac{2}{\pi}\right)^{1/2} \left| \int\limits_{\mathbb{R}_+} \hat{f}_s(u) \sin xu \left(2 \sin \frac{tu}{2}\right)^m du \right|.$$

Integrating previous inequality by $x \in [0, t]$, we obtain

$$\left| \int\limits_0^t \int\limits_{\mathbb{R}_+} \hat{f}_s(u) \sin xu \left(2 \sin \frac{tu}{2}\right)^m du\, dx \right| \leq C_9 \int\limits_0^t \omega(t)\, du = C_9 t \omega(t)$$

or

$$C_{10} \int\limits_0^{1/t} u^{-1} \hat{f}_s(u)(tu)^{m+2}\, du \leq \int\limits_0^{1/t} \hat{f}_s(u) u^{-1}(1 - \cos tu) \left(2 \sin \frac{tu}{2}\right)^m du =$$

$$= \int\limits_0^{1/t} \int_0^t \sin xu\, dx \hat{f}_s(u) \left(2 \sin \frac{tu}{2}\right)^m du \leq C_9 t \omega(t).$$

From last inequality in the form $\int_0^{1/t} \hat{f}_s(u)u^{m+1}\,du = O(t^{-m-1}\omega(t))$, $t > 0$, the condition $\omega \in B$ and Lemma 4(ii) we deduce that $\int_y^{\infty} \hat{f}_s(t)\,dt = O(\omega(1/y))$, $y > 0$. Using $\omega \in B_m$ and Lemma 4(i), we obtain (3.2).

Theorem 1 is proved.

***Remark* 1.** In parts (ii) and (iii) of Theorem 1 one may assume non-negativity or non-positivity of $\operatorname{Re}\hat{f}_c$, $\operatorname{Im}\hat{f}_c$, $\operatorname{Re}\hat{f}_s$, $\operatorname{Im}\hat{f}_s$ instead of $\hat{f}_c$ and $\hat{f}_s$. Theorem 1 is a generalization of Theorems 1, 2, 6 and 7 from [13] and a non-periodic analog of theorem B and its sine counterpart (see Theorems 3.1 and 3.2 in [7]).

***Corollary* 1.** *Let $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, $\hat{f}_c(t)$ keeps its sign on $\mathbb{R}_+$, $m \in \mathbb{N}$, $\omega \in B_m \cap B$. Then the following three conditions are equivalent*:

1) $f_e \in H^{\omega,m}$;

2) (3.1), *and*

3)
$$\int_y^{\infty} \hat{f}_c(t)\,dt = O(\omega(1/y)), \quad y > 0. \tag{3.7}$$

Analogous proposition is valid for $\hat{f}_s$ and f_0.

Theorem 2. (i) *If $m \in \mathbb{N}$ is odd, $\omega \in B \cap N^m$, $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$ and $\hat{f}_c(t) \ge 0$ on $\mathbb{R}_+$, then $f_e \in H^{\omega,m}$ if and only if*

$$\int_0^{y} t^{m+1}\hat{f}_c(t)\,dt = O(y^{m+1}\omega(1/y)), \quad y > 0, \tag{3.8}$$

and

$$\int_0^{y} t^{m}\hat{f}_c(t)\sin xt\,dt = O(y^{m}\omega(1/y)), \quad y > 0, \tag{3.9}$$

uniformly in $x \in \mathbb{R}_+$.

(ii) *If $m \in \mathbb{N}$ is even, $\omega \in B \cap N^m$, $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$ and $\hat{f}_s(t) \ge 0$ on $\mathbb{R}_+$, then $f_o \in H^{\omega,m}$ if and only if*

$$\int_0^{y} t^{m+1}\hat{f}_s(t)\,dt = O(y^{m+1}\omega(1/y)), \quad y > 0, \tag{3.10}$$

and

$$\int_0^{y} t^{m}\hat{f}_s(t)\sin xt\,dt = O(y^{m}\omega(1/y)), \quad y > 0, \tag{3.11}$$

uniformly in $x \in \mathbb{R}_+$.

Proof. (i) By Lemma 4(ii), (3.8) implies (3.7). Using (3.3), we have for $h > 0$

$$|\dot{\Delta}_h^m f(x)| \leq \left(\frac{2}{\pi}\right)^{1/2} \left(\left| \int\limits_0^{1/h} \hat{f}_c(t) \sin xt \left(2 \sin \frac{th}{2}\right)^m dt \right| + \int\limits_{1/h}^{\infty} \hat{f}_c(t)\, dt \right) =:$$

$$=: \left(\frac{2}{\pi}\right)^{1/2} (I_h(x) + J_h(x))$$

and $J_h(x) = O(\omega(h))$, $h > 0$, by (3.7). From Taylor's formula we obtain $2 \sin th/2 = th +$ $+ \alpha(th)(th)^3$, where $|\alpha(t)| \leq C$, $t \in \mathbb{R}$, whence

$$I_h(x) \leq C_1 \left| \int\limits_0^{1/h} \hat{f}_c(t) \sin xt (th)^m\, dt \right| +$$

$$+C_1 \left| \int\limits_0^{1/h} \sum_{j=1}^m \binom{m}{j} (th)^{m-j} (\alpha(th))^j (th)^{3j} \hat{f}_c(t) \sin xt\, dt \right| =: I_h^{(1)}(x) + I_h^{(2)}(x).$$

It is clear that

$$I_h^{(1)}(x) \leq C_1 h^m \left| \int\limits_0^{1/h} \hat{f}_c(t) t^m \sin xt\, dt \right| = O(\omega(h)), \quad h > 0,$$

uniformly in $x \in \mathbb{R}_+$ according to (3.9). On the other hand,

$$I_h^{(2)}(x) \leq C_2 \sum_{j=1}^m h^{m+2j} \int\limits_0^{1/h} t^{m+2j} \hat{f}_c(t)\, dt. \tag{3.12}$$

Since $N^m \subset B_{m+2j}$ by Lemma 2 for all $1 \leq j \leq m$, each term from the right-hand side of (3.12) is $O(\omega(h))$ according to (3.7) and Lemma 4(i). Thus, $I_h(x) = O(\omega(h))$, $h > 0$, and $|\dot{\Delta}_h^m f(x)| =$ $= O(\omega(h))$, $h > 0$.

Conversely, it is easy to see that $H^{\omega,m} \subset H^{\omega,m+1}$ by definition and $N^m \subset B^{m+1}$ by Lemma 2. Hence, under conditions of theorem we have $f \in H^{\omega,m+1}$ with $\omega \in B_{m+1}$. Since $m+1$ is even, by Theorem 1(ii) we obtain (3.8). Using above notations, we have $I_h(x) \leq J_h(x) + C_3|\dot{\Delta}_h^m f(x)|$ and $I_h^{(1)}(x) \leq C_4(I_h^{(2)}(x) + J_h(x) + |\dot{\Delta}_h^m f(x)|$. By Lemma 4(ii) and condition $\omega \in B$, (3.8) implies (3.7). Finally, $\omega \in N^m \subset B_{m+2j}$ and (3.7) implies $I_h^{(2)}(x) = O(\omega(h))$, $h > 0$, as above. Thus, $I_h^{(1)}(x) = O(\omega(h))$, $h > 0$, unformly in $x \in \mathbb{R}_+$, that is equivalent to (3.9).

(ii) The proof is similar to that of (i).

Theorem 2 is proved.

Corollary 2. (i) *If $m \in \mathbb{N}$ is odd, $\omega(t) = t^m$, $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$ and $\hat{f}_c(t) \geq 0$ on $\mathbb{R}_+$, then $f_e \in H^{\omega,m}$ if and only if*

$$\int\limits_0^y t^{m+1} \hat{f}_c(t)\, dt = O(y), \quad y > 0, \qquad \textit{and} \qquad \int\limits_0^y t^m \hat{f}_c(t) \sin xt\, dt = O(1), \quad y > 0,$$

uniformly in $x \in \mathbb{R}_+$.

(ii) *Similar assertion is valid for* $\hat{f}_s$, f_o *and even* $m \in \mathbb{N}$.

***Remark* 2.** Theorem 2 is an analog of Theorems 3.1 and 3.2, part (B), in [7] (see the item (B) in Theorem B). Corollary 2 is an extensoin of Theorem 3 in [13], where the necessary and sufficent condition for $f \in \mathrm{Lip}(1)$ in terms of $\hat{f}_c$ is given.

Theorem 3. (i) *Let* $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, $m \in \mathbb{N}$ *and*

$$\int\limits_y^\infty |\hat{f}_c(t)|\,dt = o(y^{-m}), \quad y \to +\infty. \tag{3.13}$$

Then the Schwartz derivative of f *of order* m *exists in the point* $x > 0$ *and equals to* $A(x)$ *if and only if the integral* $(2/\pi)^{1/2} \int_{\mathbb{R}_+} t^m \hat{f}_c(t) \cos(xt + m\pi/2)\,dt$ *converges and equals to* $A(x)$.

(ii) *Similar assertion is valid for* $\hat{f}_s(t)$.

Proof. By (3.3) we have

$$\dot{\Delta}_h^m f(x) = \left(\frac{2}{\pi}\right)^{1/2} \left(\int\limits_0^{1/h} + \int\limits_{1/h}^\infty\right) \hat{f}_c(t) \cos\left(xt + m\frac{\pi}{2}\right) \left(2\sin\frac{ht}{2}\right)^m dt =$$

$$=: \left(\frac{2}{\pi}\right)^{1/2} (A_h(x) + B_h(x)).$$

According to (3.13) we have $B_h(x) = o(h^m)$, $h \to 0$. Using identity $2\sin th/2 = th + \alpha(th)(th)^3$, where $\alpha(t) = O(1)$, $t \in \mathbb{R}$ (see the proof of Theorem 2), we write

$$A_h(x) = \int\limits_0^{1/h} \hat{f}_c(t)(ht)^m \cos\left(xt + m\frac{\pi}{2}\right) dt +$$

$$+ \sum_{j=1}^m \binom{m}{j} \int\limits_0^{1/h} \hat{f}_c(t) \cos\left(xt + m\frac{\pi}{2}\right) (ht)^{m+2j} (\alpha(ht))^j\,dt =: A_h^{(1)}(x) + A_h^{(2)}(x).$$

Since $\int_y^\infty |\hat{f}_c(t)|\,dt = o(\omega(1/y))$, $y \to +\infty$, for $\omega(t) = t^m$ and $t^m \in N^m \subset B_{m+2j}$ for all $1 \le j \le m$, by Lemma 5(i) we obtain

$$A_h^{(2)}(x) = O\left(\sum_{j=1}^m h^{m+2j} \int\limits_0^{1/h} |\hat{f}_c(t)| t^{m+2j}\,dt\right) = o(h^{m+2j} h^{-m-2j} h^m) = o(h^m), \quad h \to 0.$$

Therefore, the existence of the limit

$$B(x) := \lim_{h\to 0} h^{-m} A_h^{(1)}(x) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_{\mathbb{R}_+} \hat{f}_c(t) t^m \cos\left(xt + m\frac{\pi}{2}\right) dt$$

is equivalent to the existence of $\lim_{h\to 0} h^{-m} \dot{\Delta}_h^m f(x) =: A(x)$ and in the last case $B(x) = A(x)$.

(ii) The proof of this item is similar to that of (i).

Theorem 3 is proved.

***Remark* 3.** Theorem 3 is an analog of Theorem C.

Theorem 4. *Let* $f \in L^1(\mathbb{R}_+) \cap UC(\mathbb{R}_+)$, $\hat{f}_s(t) \geq 0$ $(\hat{f}_c(t) \geq 0)$ *on* $\mathbb{R}_+$. *If* $F(x) = \int_0^x f(t)\,dt \in L^1(\mathbb{R}_+)$, *then*

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} \lim_{y\to\infty} \int_0^y \hat{f}_s(t) \sin xt\,dt \qquad \left(f(x) = \left(\frac{2}{\pi}\right)^{1/2} \lim_{y\to\infty} \int_0^y \hat{f}_c(t) \cos xt\,dt\right)$$

uniformly in $x \in \mathbb{R}_+$.

Proof. If $f \in L^1(\mathbb{R})$ is even, then $F(x) = \int_0^x f(t)\,dt$ is odd on $\mathbb{R}$ and vice versa. As it is noted in [5], for $f \in L^1(\mathbb{R}) \cap UC(\mathbb{R})$ we have $|\dot{\Delta}_h^2 F(x)| = o(h)$, $h \to 0$, i.e., $F \in h^{\omega,2}$ for $\omega(t) = t$. Now we consider odd f $(f \equiv f_o)$ and even F. By Theorem 8 in [13] or Theorem 5 below we have

$$\int_0^y t^2 |\hat{F}_c(t)|\,dt = o(y^2 y^{-1}) = o(y), \quad y \to +\infty, \tag{3.14}$$

and by Lemma 5

$$\int_y^\infty |\hat{F}_c(t)|\,dt = o(y^{-1}), \quad y \to +\infty, \tag{3.15}$$

since $\omega(t) = t \in B_2$. Using the fact that $\hat{F}_c(t) \in C_0(\mathbb{R}_+)$ and (3.15), we obtain $\hat{F}_c(t) \in L^1(\mathbb{R}_+)$ and by inversion formula (1.1)

$$F(x+h) - F(x) = -\left(\frac{2}{\pi}\right)^{1/2} \left(\int_0^{1/h} + \int_{1/h}^\infty\right) \hat{F}_c(t)(\cos xt - \cos(x+h)t)\,dt =:$$

$$=: -\left(\frac{2}{\pi}\right)^{1/2} (A_h(x) + B_h(x)).$$

By virtue of (3.15) we have $B_h(x) = o(h)$, $h \to 0$, uniformly in $x \in \mathbb{R}_+$. On the other hand, using identity $\cos xt - \cos(x+h)t = \cos xt(1 - \cos ht) + \sin xt \sin ht$, we see that

$$A_h(x) = \int_0^{1/h} \hat{F}_c(t) 2\sin^2\left(\frac{ht}{2}\right) \cos xt\,dt + \int_0^{1/h} \hat{F}_c(t) \sin xt \sin ht\,dt =: A_h^{(1)}(x) + A_h^{(2)}(x).$$

By (3.14) and inequality $|\sin t| \leq t$, $t \geq 0$, we obtain

$$|A_h^{(1)}(x)| \leq h^2 \int_0^{1/h} |\hat{F}_c(t)| t^2\,dt = o(h), \quad h \to 0,$$

uniformly in $x \in \mathbb{R}_+$, while

$$A_h^{(2)}(x) = h \int_0^{1/h} \hat{F}_c(t) t \sin xt\,dt + \int_0^{1/h} \hat{F}_c(t) \alpha^3(ht)(ht)^3\,dt =: A_h^{(3)}(x) + A_h^{(4)}(x)$$

(see the proof of Theorem 2). From (3.15) and condition $\omega \in B_3$ for $\omega(t) = t$ due to Lemma 5(i) we have

$$|A_h^{(4)}(x)| = O\left(h^3 \int\limits_0^{1/h} t^3 |\hat{F}_c(t)|\, dt\right) = o(h^3 h^{-3} h) = o(h), \quad h \to 0,$$

also uniformly in $x \in \mathbb{R}_+$. Thus, by Lemma 3

$$\frac{F(x+h) - F(x)}{h} = -\left(\frac{2}{\pi}\right)^{1/2} \int\limits_0^{1/h} \hat{F}_c(t) t \sin xt\, dt + o(1) =$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \int\limits_0^{1/h} \hat{f}_s(t) \sin xt\, dt + o(1), \quad h \to 0.$$

Similar relation holds for $(F(x) - F(x-h))/h$ and tending h to zero yields

$$f(x) = \left(\frac{2}{\pi}\right)^{1/2} \int\limits_0^{\infty} \hat{f}_s(t) \sin xt\, dt$$

uniformly in $x \in \mathbb{R}_+$. The proof of the second statement of Theorem 4 is similar to that of the first one.

Theorem 4 is proved.

***Remark* 4.** Theorem 4 is a non-periodic analog of Theorem D of R. Paley [15].

Theorem 5. (i) *If* $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, $m \in \mathbb{N}$, $\omega \in B$ *and*

$$\int\limits_0^y t^m |\hat{f}_c(t)|\, dt = o(y^m \omega(1/y)), \quad y \to +\infty, \tag{3.16}$$

or

$$\int\limits_0^y t^m |\hat{f}_s(t)|\, dt = o(y^m \omega(1/y)), \quad y \to +\infty, \tag{3.17}$$

and (3.1) *or* (3.2) *respectively hold for all* $y > 0$, *then* $\hat{f}_c \in L^1(\mathbb{R}_+)$ (*or* $\hat{f}_s \in L^1(\mathbb{R}_+)$) *and* $f_e \in h^{\omega,m}$ (*or* $f_o \in h^{\omega,m}$).

(ii) *If* $m \in \mathbb{N}$ *and* f_e (*or* f_o) *satisfy conditions of Theorem* 1 (ii) (*or Theorem* 1 (iii)), *then* $f_e \in h^{\omega,m}$ *implies* (3.16) (*or* $f_e \in h^{\omega,m}$ *implies* (3.17)).

Proof. (i) By condition of Theorem for every $\varepsilon > 0$ there exists $y_0(\varepsilon)$, such that

$$\left(\frac{2}{\pi}\right)^{1/2} \int\limits_0^y t^m |\hat{f}_c(t)|\, dt < \varepsilon y^m \omega(1/y) \quad \text{for all} \quad y > y_0.$$

If I_h and J_h are defined in the proof of Theorem 1, then similarly to (3.5) we have $|I_h| \le \le \varepsilon h^m h^{-m}\omega(h) = \varepsilon\omega(h)$ for $0 < h < y_0^{-1}$. On the other hand, by Lemma 5 (ii) we have $|J_h| = = o(\omega(h))$, $h \to 0$. Thus, $|\dot{\Delta}_h^m f(x)| = O(I_h + J_h) = o(\omega(h))$ and $f_e \in h^{\omega,m}$ $(f_o \in h^{\omega,m})$.

(ii) Let m be even and $\hat{f}_c(t) \ge 0$ on $\mathbb{R}_+$. If $f \in h^{\omega,m}$, then

$$\varepsilon\omega(h) \ge |\dot{\Delta}_h^m f(0)| \ge C_1 \int\limits_0^{1/h} \hat{f}_c(t)(ht)^m\, dt, \quad 0 < h < h_0(\varepsilon),$$

whence $\int_0^{1/h} |t^m \hat{f}_c(t)|\, dt = o(h^{-m}\omega(h))$, $h \to 0$, and (3.16) is proved.

Let m be odd, $\hat{f}_c(t) \ge 0$ on $\mathbb{R}_+$ and $\omega \in B_m$. Similarly to the proof of Theorem 1 (ii) we find that $\int_{2t}^{\infty} \hat{f}_c(u)\, du < \varepsilon\omega(1/t)$ for $t > t_0(\varepsilon)$ and $\int_t^{\infty} \hat{f}_c(u)\, du = o(\omega(1/t))$, $t \to +\infty$. Using condition $\omega \in B_m$ and Lemma 5 (i), we obtain (3.16).

The case of odd m and $\hat{f}_s \ge 0$ is similar to the case of even m and $\hat{f}_c \ge 0$. Finally, if m is even, $\omega \in B$ and $\hat{f}_s(t) \ge 0$ on $\mathbb{R}_+$, then similarly to the proof of Theorem 1 (iii) we have $\int_0^{1/t} u^{-1}\hat{f}_s(u)(tu)^{m+2}\, du \le \varepsilon t\omega(1/t)$ for $t > t_0(\varepsilon)$ and by Lemma 5 (ii) we deduce that

$$\int\limits_y^{\infty} \hat{f}_s(t)\, dt = o(\omega(1/y)), \quad y \to +\infty. \tag{3.18}$$

Using $\omega \in B_m$ and Lemma 5 (ii), we obtain (3.17).

Theorem 5 is proved.

***Remark* 5.** Theorem 5 is a generalization of Theorems 4, 5 and 8 from [13].

***Corollary* 3.** *Let $f \in L^1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+)$, $\hat{f}_c(t)$ keeps its sign on $\mathbb{R}_+$, $m \in \mathbb{N}$, $\omega \in B_m \cap B$. Then three conditions $f \in h^{\omega,m}$,* (3.16) *and* (3.18) *are equivalent. Similar assertion is valid for $\hat{f}_s$.*

1. *Butzer P. L., Nessel R. J.* Fourier analysis and approximation. – Basel; Stuttgart: Birkhäuser, 1971. – Vol. 1.
2. *Bari N. K., Stechkin S. B.* Best approximation and differential properties of two conjugate functions // Trudy Mosk. Mat. Obshch. – 1956. – **5**. – P. 483 – 522 (in Russian).
3. *Timan A. F.* Theory of approximation of functions of a real variable (in Russian). – Moscow: Fizmatgiz, 1960 (English transl.: Pergamon Press, N.Y., 1963).
4. *Moricz F.* Absolutely convergent Fourier integrals and classical function spaces // Arch. Math. – 2008. – **91**, № 1. – P. 49 – 62.
5. *Volosivets S. S.* Fourier transforms and generalized Lipschitz classes in uniform metric // J. Math. Anal. and Appl. – 2011. – **383**, № 1. – P. 344 – 352.
6. *Boas (Jr.) R. P.* Integrability theorems for trigonometric transforms. – New York: Springer, 1967.
7. *Tikhonov S.* Smoothness conditions and Fourier series // Math. Ineq. Appl. – 2007. – **10**, № 2. – P. 229 – 242.
8. *Tikhonov S.* On generalized Lipschitz classes and Fourier series // Z. Anal. Anwend. – 2004. – **23**, № 4. – S. 745 – 764.
9. *Moricz F.* Absolutely convergent Fourier series and function classes // J. Math. Anal. and Appl. – 2006. – **324**, № 2. – P. 1168 – 1177.

10. *Moricz F.* Higher order Lipschitz classes of functions and absolutely convergent Fourier series // Acta math. hung. – 2008. – **120**, № 4. – P. 355 – 366.
11. *Moricz F.* Absolutely convergent Fourier series, classical function spaces and Paley's theorem // Anal. Math. – 2008. – **34**, № 4. – P. 261 – 276.
12. *Volosivets S. S.* Fourier coefficients and generalized Lipschitz classes in uniform metric // Real Anal. Exch. – 2009. – **34**, № 1. – P. 219 – 226.
13. *Fülöp V.* Sine, cosine transforms and classical function classes // Anal. Math. – 2009. – **35**, № 3. – P. 199 – 212.
14. *Nikol'skii S. M.* Approximation of several variables and embedding theorems (in Russian). – Moscow: Nauka, 1977 (English transl.: Berlin: Springer, 1975).
15. *Paley R. E. A. C.* On Fourier series with positive coefficients // J. London Math. Soc. – 1932. – **7**. – P. 205 – 208.
16. *Bari N. K.* Trigonometric series (in Russian). – Moscow: Fizmatgiz, 1961.

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