UDC 519.21

Z.-J. Zong (School Math. Sci., Qufu Normal Univ., Shandong, China)

A COMONOTONIC THEOREM FOR BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS IN L^p AND ITS APPLICATIONS *

ТЕОРЕМА ПРО КОМОНОТОННІСТЬ ДЛЯ ЗВОРОТНИХ СТОХАСТИЧНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ У L^p ТА ЇЇ ЗАСТОСУВАННЯ

We study backward stochastic differential equations (BSDEs) under weak assumptions on the data. We obtain a comonotonic theorem for BSDEs in L^p, $1 < p \leq 2$. As applications of this theorem, we study the relation between Choquet expectations and minimax expectations and the relation between Choquet expectations and generalized Peng's g-expectations. These results generalize the known results of Chen et al.

Досліджено зворотні стохастичні диференціальні рівняння при слабких припущеннях щодо вихідних даних. Отримано теорему про комонотонність для зворотних стохастичних диференціальних рівнянь у просторі L^p, $1 < p \leq 2$. Як застосування цієї теореми, вивчено співвідношення між сподіваннями Шоке і мінімаксними сподіваннями та співвідношення між сподіваннями Шоке й узагальненими g-сподіваннями Пенга. Ці результати узагальнюють відомі результати Чена та ін.

1. Introduction. By Pardoux and Peng [14], we know that there exists a unique adapted and square integrable solution to a backward stochastic differential equation (BSDE for short) of type

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dW_s, \quad t \in [0, T], \tag{1.1}$$

providing that the function g is Lipschitz in both variables y and z, and that ξ and the process $(g(t, 0, 0))_{t\in[0,T]}$ are square integrable. We denote the unique solution of BSDE (1.1) by $\left(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)}\right)_{t\in[0,T]}$.

Since then, many researchers have been working on this subject and related properties of the solutions of BSDEs, due to the connection of this subject with mathematical finance, stochastic control, partial differential equation, stochastic game and stochastic geometry and mathematical economics; for example, see References [2–5, 7–13, 15–18]. Among these results, the comparison theorem of BSDEs with respect to $y_t^{(T,g,\xi)}$ plays an important role.

An interesting study is to obtain a comparison result applicable to the second part of the $z_t^{(T,g,\xi)}$ of the solution $\left(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)}\right)_{t\in[0,T]}$ of BSDE (1.1). In fact, because $z_t^{(T,g,\xi)}$ in BSDE (1.1) is a speed (volatility in mathematical finance), it is not easy to make comparisons regarding $z_t^{(T,g,\xi)}$ in the same way as to make comparisons regarding $y_t^{(T,g,\xi)}$.

Chen et al. [4] studied the comonotonicity of $z_t^{(T,g,\xi)}$. That is, let $\left(y_t^{(T,g,\xi_1)}, z_t^{(T,g,\xi_1)}\right)$ and $\left(y_t^{(T,g,\xi_2)}, z_t^{(T,g,\xi_2)}\right)$ be the solutions of BSDE (1.1) corresponding to terminal values $\xi = \xi_1$ and $\xi = \xi_2$, respectively. A sufficient condition on ξ_1 and ξ_2 has been given, under which

$$z_t^{(T,g,\xi_1)} \odot z_t^{(T,g,\xi_2)} \geq 0, \quad dP \times dt\text{-a.s.}$$

*This work was supported partially by the National Natural Science Foundation of China (No. 11171179) and the Research Foundation for the Doctoral Program of Higher Education (No. 20093705110002).

Here for any $z, x \in R^d$, denote $z \odot x = (z_1x_1, z_2x_2, \dots, z_dx_d)$, where z_i and x_i are the ith components of z and x, $i = 1, 2, \dots, d$. Furthermore, $z \odot x \geq 0$ means $z_ix_i \geq 0$, $i = 1, 2, \dots, d$. As applications of this result, Chen et al. provide a sufficient condition of Choquet expectations being equal to minimax expectations in [5] and give a necessary and sufficient condition of Choquet expectations being equal to g-expectations in [2].

In this paper, we investigate the comonotonicity of z_t under weak assumptions on the data. Furthermore, we give some applications of the comonotonic theorem. These results generalize the known results of Chen et al. [2, 4, 5].

This paper is organized as follows. In Section 2, we give some notations, lemmas and notions that are useful in this paper. In Section 3, we investigate the comonotonic theorem for BSDEs in L^p. In Section 4, using the comonotonic theorem, we give some results such as the relation between Choquet expectations and minimax expectations and the relation between Choquet expectations and generalized Peng's g-expectations.

2. Preliminaries. In this section, we shall present some notations, lemmas and notions that are used in this paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t\geq 0}$ be a d-dimensional standard Brownian motion with respect to filtration $(\mathcal{F}_t)_{t\geq 0}$ generated by the Brownian motion and all P-null subsets, i. e.,

$$\mathcal{F}_t = \sigma\{W_s; s \leq t\} \vee \mathcal{N},$$

where $\mathcal{N}$ is the set of all P-null subsets. Fix a real number $T > 0$. We assume that $\mathcal{F}_T = \mathcal{F}$.

Define

$L^p(\Omega, \mathcal{F}, P) := \{\xi\colon \xi$ is $\mathcal{F}$-measurable random variable such that $E[|\xi|^p] < \infty, p \geq 1\}$,

$\mathcal{L}(\Omega, \mathcal{F}, P) := \bigcup_{p>1} L^p(\Omega, \mathcal{F}, P)$,

$\mathcal{S}^p_T(R) := \Big\{V\colon (V_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted continuous R-valued process with $E\left[\sup_{0\leq t\leq T} |V_t|^p\right] < \infty, p \geq 1\Big\}$,

$\mathcal{S}_T(R) := \bigcup_{p>1} \mathcal{S}^p_T(R)$,

$L^p(0, T; P; R^n) := \Big\{V\colon (V_t)_{t\in[0,T]}$ is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted R^n-valued process with $E\Big[\Big(\int_0^T |V_s|^2 ds\Big)^{p/2}\Big] < \infty, p \geq 1\Big\}$,

$\mathcal{L}(0, T; P; R^n) := \bigcup_{p>1} L^p(0, T; P; R^n)$.

Throughout this paper, we assume that $1 < p \leq 2$.

Suppose function $g : \Omega \times [0, T] \times R \times R^d \longmapsto R$ satisfies the following conditions:

(H.1) $g(\cdot, 0, 0) \in L^p(0, T; P; R)$;

(H.1$'$) $g(\cdot, 0, 0) \in \mathcal{L}(0, T; P; R)$;

(H.2) g satisfies a uniform Lipschitz condition, that is: there exists a constant $\mu > 0$ such that for any $y_1, y_2 \in R$, $z_1, z_2 \in R^d$, $|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq \mu(|y_1 - y_2| + |z_1 - z_2|)$, $t \in [0, T]$;

(H.3) $g(\cdot, y, 0) = 0$ $\forall y \in R$.

Lemma 2.1 (see Briand et al. [1]). *Suppose g satisfies* (H.1) *and* (H.2). *Then for any $\xi \in$ $\in L^p(\Omega, \mathcal{F}, P)$, the BSDE*

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dW_s, \tag{2.1}$$

has a unique pair of adapted processes $\left(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)}\right)_{t\in[0,T]} \in \mathcal{S}_T^p(R) \times L^p(0,T;P;R^d)$.

***Remark* 2.1.** From Lemma 2.1, we have: suppose g satisfies (H.1′) and (H.2), then for each given $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, BSDE (2.1) has a unique pair of adapted processes $\left(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)}\right)_{t\in[0,T]} \in$ $\in \mathcal{S}_T(R) \times \mathcal{L}(0,T;P;R^d)$.

We give the a priori estimate for BSDEs which is of standard type and taken from [1].

Lemma 2.2. *Suppose g satisfies* (H.1) *and* (H.2). *For any ξ_1, $\xi_2 \in L^p(\Omega, \mathcal{F}, P)$, let $\left(y_t^{(T,g,\xi_1)}, z_t^{(T,g,\xi_1)}\right)$ and $\left(y_t^{(T,g,\xi_2)}, z_t^{(T,g,\xi_2)}\right)$ be the solutions of BSDE* (2.1) *corresponding to $\xi = \xi_1$ and $\xi = \xi_2$, respectively. Then there exists a constant $C_p > 0$ depending only on p, T and Lipschitz constant μ such that*

$$E\left[\sup_{0\le t\le T}\left|y_t^{(T,g,\xi_1)} - y_t^{(T,g,\xi_2)}\right|^p\right] + E\left[\left(\int_0^T \left|z_s^{(T,g,\xi_1)} - z_s^{(T,g,\xi_2)}\right|^2 ds\right)^{p/2}\right] \le C_p E\left[|\xi_1 - \xi_2|^p\right].$$

The following comparison theorem is very useful.

Lemma 2.3 (Comparison theorem, see Hu and Chen [10])**.** *Suppose g and $\overline{g}$ satisfy* (H.1) *and* (H.2). *For any $\xi_1, \xi_2 \in L^p(\Omega, \mathcal{F}, P)$, let $\left(y_t^{(T,g,\xi_1)}, z_t^{(T,g,\xi_1)}\right)$ and $\left(y_t^{(T,g,\xi_2)}, z_t^{(T,g,\xi_2)}\right)$ be the solutions of the following two BSDEs*:

$$y_t^1 = \xi_1 + \int_t^T g\left(s, y_s^1, z_s^1\right) ds - \int_t^T z_s^1 \cdot dW_s, \quad t \in [0,T],$$

$$y_t^2 = \xi_2 + \int_t^T \overline{g}\left(s, y_s^2, z_s^2\right) ds - \int_t^T z_s^2 \cdot dW_s, \quad t \in [0,T].$$

If

$$\xi_1 \ge \xi_2 \quad \text{a.e.,} \qquad \hat{g}_t = g(t,y,z) - \overline{g}(t,y,z) \ge 0 \quad \text{a.e.,}$$

then for each $t \in [0,T]$,

$$y_t^{(T,g,\xi_1)} \ge y_t^{(T,\overline{g},\xi_2)} \quad \text{a.e.}$$

In the case, we have

$$y_t^{(T,g,\xi_2)} = y_t^{(T,\overline{g},\xi_2)} \quad \text{a.e. if and only if} \quad \xi_1 = \xi_2 \quad \text{a.e.,} \qquad \hat{g}_t = 0 \quad \text{a.e.}$$

Definition 2.1 (Generalized Peng's g-expectation, see [10])**.** *Suppose g satisfies* (H.2) *and* (H.3). *For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, let $\left(y_t^{(T,g,\xi)}, z_t^{(T,g,\xi)}\right)$ be the solution of BSDE* (2.1) *with terminal value ξ. Consider the mapping $\mathcal{E}_g[\cdot]\colon \mathcal{L}(\Omega, \mathcal{F}, P) \longmapsto R$, denoted by $\mathcal{E}_g[\xi] = y_0^{(T,g,\xi)}$. We call $\mathcal{E}_g[\xi]$ the generalized Peng's g-expectation of ξ.*

Definition 2.2 (Generalized Peng's g-expectation, see [10]). *Suppose g satisfies* (H.2) *and* (H.3). *The generalized Peng's conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by*

$$\mathcal{E}_g[\xi|\mathcal{F}_t] = y_t^{(T,g,\xi)}.$$

The generalized Peng's conditional g-expectation has the following property.

Proposition 2.1 (see [10]). *$\mathcal{E}_g[\xi|\mathcal{F}_t]$ is the unique random variable η in $\mathcal{L}(\Omega, \mathcal{F}_t, P)$ such that*

$$\mathcal{E}_g[1_A\xi] = \mathcal{E}_g[1_A\eta] \quad \forall A \in \mathcal{F}_t.$$

3. A comonotonic theorem for BSDEs in L^p. From now on, we further assume that the function g is deterministic, i.e.,

$$g\colon [0,T] \times R \times R^d \longmapsto R.$$

Then (H.1) can be rewritten as follows:

$$\int_0^T |g(s,0,0)|^2 ds < \infty. \tag{H}$$

More specifically, we suppose that g_1 and g_2 satisfy the assumption (H) and (H.2). For any ξ_1 and $\xi_2 \in \mathcal{L}(\Omega, \mathcal{F}, P)$, let $\left(y_t^{(T,g_i,\xi_i)}, z_t^{(T,g_i,\xi_i)}\right)$ be the solutions of the following BSDEs:

$$y_t^i = \xi_i + \int_t^T g_i\left(s, y_s^i, z_s^i\right) ds - \int_t^T z_s^i \cdot dW_s, \quad t \in [0,T], \quad i = 1,2. \tag{3.1}$$

Now we consider the case where random variables ξ_1 and ξ_2 satisfy that there exist two functions ϕ_1 and ϕ_2 such that ξ_1 and ξ_2 are of the form $\xi_i = \phi_i\left(X_T^i\right)$, where (X_t^i) are the solutions of the following SDEs, respectively,

$$dX_s^i = b_i\left(s, X_s^i\right) ds + \sigma_i\left(s, X_s^i\right) \cdot dW_s,$$

$$X_0^i = x, \quad x \in R, \quad i = 1,2,$$

and b_i and σ_i satisfy the following assumption for each $i = 1,2$.

Assumption A. Let $b_i(t,x)\colon [0,T] \times R \longmapsto R$, $\sigma_i(t,x)\colon [0,T] \times R \longmapsto R^d$ be continuous in (t,x) and uniformly Lipshictz continuous in $x \in R$, for each $i = 1,2$.

Definition 3.1. *The functions ϕ and ψ are said to be comonotonic, if both ϕ and ψ are of the same monotonicity, that is, if ϕ is increasing (or decreasing), so is ψ.*

The following theorem is called comonotonic theorem for BSDEs, which plays an important role in our paper.

Theorem 3.1. *Suppose that $\left(y_t^{(T,g_1,\xi_1)}, z_t^{(T,g_1,\xi_1)}\right)$ and $\left(y_t^{(T,g_2,\xi_2)}, z_t^{(T,g_2,\xi_2)}\right)$ are the solutions of BSDE* (3.1) *corresponding to terminal values $\xi_1 = \phi_1(X_T^1)$ and $\xi_2 = \phi_2(X_T^2)$, respectively. If ϕ_1 and ϕ_2 are comonotonic and*

$$\sigma_1\left(t, X_t^1\right) \odot \sigma_2\left(t, X_t^2\right) \geq 0, \quad dP \times dt\text{-a.s.},$$

then

$$z_t^{(T,g_1,\xi_1)} \odot z_t^{(T,g_2,\xi_2)} \geq 0, \quad dP \times dt\text{-a.s.}$$

Proof. If $\phi_1(X_T^1)$ and $\phi_2(X_T^2) \in L^2(\Omega, \mathcal{F}, P)$, Chen et al. have proved Theorem 3.1 holds in [5]. Otherwise, there exists $1 < p < 2$, such that $\phi_1(X_T^1)$ and $\phi_2(X_T^2) \in L^p(\Omega, \mathcal{F}, P)$. Set $\xi_i^n = \phi_i^n(X_T^i) = (\phi_i(X_T^i) \wedge n) \vee (-n)$, $i = 1, 2$, then $\phi_i^n(X_T^i) \in L^2(\Omega, \mathcal{F}, P)$ and both $\phi_i^n(X_T^i)$ and $\phi_i(X_T^i)$ are of the same monotonicity, for each $i = 1, 2$. Let $z_t^{(T,g_1,\xi_1^n)}$ and $z_t^{(T,g_2,\xi_2^n)}$ be the solutions of BSDE (3.1) corresponding to $\phi_1^n(X_T^1)$ and $\phi_2^n(X_T^2)$, by Chen et al. [5], we have

$$z_t^{(T,g_1,\xi_1^n)} \odot z_t^{(T,g_2,\xi_2^n)} \geq 0, \quad dP \times dt\text{-a.s.} \tag{3.2}$$

Applying Lemma 2.2, we can obtain $z_t^{(T,g_1,\xi_1^n)} \to z_t^{(T,g_1,\xi_1)}$ and $z_t^{(T,g_2,\xi_2^n)} \to z_t^{(T,g_2,\xi_2)}$ in $L^p(0,T;P;R^d)$ as $n \to \infty$. This with (3.2) implies that

$$z_t^{(T,g_1,\xi_1)} \odot z_t^{(T,g_2,\xi_2)} \geq 0, \quad dP \times dt\text{-a.s.} \tag{3.3}$$

Theorem 3.1 is proved.

Using Theorem 3.1, immediately, we can obtain the following theorem.

Theorem 3.2. *Suppose that b and σ satisfy Assumption A. Let (X_s) be the solution of SDE*

$$dX_s = b(s, X_s)ds + \sigma(s, X_s) \cdot dW_s, X_0 = x, \quad s \in [0, T].$$

Assume ϕ is a function such that $\phi(X_T) \in \mathcal{L}(\Omega, \mathcal{F}, P)$ and g satisfies (H) *and* (H.2). *Let $\left(y_t^{(T,g,\phi(X_T))}, z_t^{(T,g,\phi(X_T))}\right)$ be the solution of the BSDE*

$$y_t = \phi(X_T) + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s \cdot dW_s, \quad t \in [0, T].$$

(i) *If ϕ is an increasing function, then*

$$z_t^{(T,g,\phi(X_T))} \odot \sigma(t, X_t) \geq 0, \quad dP \times dt\text{-a.s.} \tag{3.4}$$

(ii) *If ϕ is a decreasing function, then*

$$z_t^{(T,g,\phi(X_T))} \odot \sigma(t, X_t) \leq 0, \quad dP \times dt\text{-a.s.}. \tag{3.5}$$

4. Some applications of the comonotonic theorem. ***4.1. Additivity of generalized Peng's g-expectations.*** We know that if $g(t, y, z)$ is nonlinear in (y, z), then $\mathcal{E}_g[\cdot]$ is usually nonlinear on $\mathcal{L}(\Omega, \mathcal{F}, P)$. In this subsection, applying the comonotonic theorem, we give that for some special random variables, $\mathcal{E}_g[\cdot]$ still has the additivity property even when g is nonlinear.

Definition 4.1. (i) *A function $g(t, y, z) \colon [0, T] \times R \times R^d \longmapsto R$ is called positively additive, if for any (y_1, z_1) and $(y_2, z_2) \in R \times R^d$, then*

$$g(t, y_1 + y_2, z_1 + z_2) = g(t, y_1, z_1) + g(t, y_2, z_2),$$

whenever $y_1 y_2 \geq 0, z_1 \odot z_2 \geq 0 \; \forall t \in [0, T]$.

(ii) *A function $g(t, y, z) \colon [0, T] \times R \times R^d \longmapsto R$ is called semipositively additive, if for any (y_1, z_1) and $(y_2, z_2) \in R \times R^d$, then*

$$g(t, y_1 + y_2, z_1 + z_2) = g(t, y_1, z_1) + g(t, y_2, z_2),$$

whenever $z_1 \odot z_2 \geq 0 \; \forall t \in [0, T]$.

***Remark* 4.1.** (i) If g is a positively additive (or semipositively additive) function, it is easy to check that $g(t,0,0)=0\ \forall t\in[0,T]$.

(ii) The following two functions are positively additive and semipositively additive, respectively,

$$g(t,y,z)=a_t|y|+\sum_{i=1}^{d}\mu_t^i|z^i|,$$

$$g(t,y,z)=b_t y+\sum_{i=1}^{d}\nu_t^i|z^i|,$$

where z^i is the i th component of z.

Theorem 4.1. *Suppose that $\phi_1(X_T^1)$ and $\phi_2(X_T^2)$ are the random variables defined in Theorem* 3.1 *and that g satisfies* (H.2) *and* (H.3).

(i) *Suppose ϕ_1 and ϕ_2 are comonotonic with $\phi_1(X_T^1)\geq 0$, $\phi_2(X_T^2)\geq 0$ (or $\phi_1(X_T^1)\leq 0$, $\phi_2(X_T^2)\leq 0$). If*

$$\sigma_1\left(t,X_t^1\right)\odot\sigma_2\left(t,X_t^2\right)\geq 0,\quad dP\times dt\text{-a.s.},$$

and g is a positively additive function, then

$$\mathcal{E}_g[\phi_1(X_T^1)+\phi_2(X_T^2)|\mathcal{F}_t]=\mathcal{E}_g[\phi_1(X_T^1)|\mathcal{F}_t]+\mathcal{E}_g[\phi_2(X_T^2)|\mathcal{F}_t]\quad \text{a.e.},\quad t\in[0,T].$$

In particular,

$$\mathcal{E}_g[\phi_1(X_T^1)+\phi_2(X_T^2)]=\mathcal{E}_g[\phi_1(X_T^1)]+\mathcal{E}_g[\phi_2(X_T^2)].$$

(ii) *If g is a semipositively additive function, then the assumptions $\phi_1(X_T^1)\geq 0$, $\phi_2(X_T^2)\geq 0$ (or $\phi_1(X_T^1)\leq 0$, $\phi_2(X_T^2)\leq 0$) in* (i) *can be dropped.*

Proof. (i) For each $i=1,2$, let $\left(y_t^{(T,g,\phi_i(X_T^i))},z_t^{(T,g,\phi_i(X_T^i))}\right)$ be the solution of BSDEs

$$y_t^i=\phi_i(X_T^i)+\int_t^T g\left(s,y_s^i,z_s^i\right)ds-\int_t^T z_s^i\cdot dW_s,\quad t\in[0,T].$$

Since ϕ_1 and ϕ_2 are comonotinic and

$$\sigma_1\left(t,X_t^1\right)\odot\sigma_2\left(t,X_t^2\right)\geq 0,\quad dP\times dt\text{-a.s.},$$

by Theorem 3.1, we have

$$z_t^{(T,g,\phi_1(X_T^1))}\odot z_t^{(T,g,\phi_2(X_T^2))}\geq 0,\quad dP\times dt\text{-a.s.}\tag{4.1}$$

We next show

$$y_t^{(T,g,\phi_1(X_T^1))}y_t^{(T,g,\phi_2(X_T^2))}\geq 0\quad\text{a.e.},\quad t\in[0,T].$$

Indeed, if $\phi_1(X_T^1)\geq 0$, $\phi_2(X_T^2)\geq 0$, then applying Lemma 2.3,

$$y_t^{(T,g,\phi_1(X_T^1))}\geq 0\quad\text{a.e.},\qquad y_t^{(T,g,\phi_2(X_T^2))}\geq 0\quad\text{a.e.}$$

If $\phi_1(X_T^1) \leq 0$, $\phi_2(X_T^2) \leq 0$, applying Lemma 2.3 again,

$$y_t^{(T,g,\phi_1(X_T^1))} \leq 0 \quad \text{a.e.}, \qquad y_t^{(T,g,\phi_2(X_T^2))} \leq 0 \quad \text{a.e.}$$

Hence

$$y_t^{(T,g,\phi_1(X_T^1))} y_t^{(T,g,\phi_2(X_T^2))} \geq 0 \quad \text{a.e.}, \quad t \in [0,T]. \tag{4.2}$$

(4.1), (4.2) and the assumption that $g(t,y,z)$ is a positively additive function imply

$$y_t^{(T,g,\phi_1(X_T^1))} + y_t^{(T,g,\phi_2(X_T^2))} = \phi_1(X_T^1) + \phi_2(X_T^2) +$$

$$+ \int_t^T g\left(s, y_s^{(T,g,\phi_1(X_T^1))} + y_s^{(T,g,\phi_2(X_T^2))}, z_s^{(T,g,\phi_1(X_T^1))} + z_s^{(T,g,\phi_2(X_T^2))}\right) ds -$$

$$- \int_t^T \left(z_s^{(T,g,\phi_1(X_T^1))} + z_s^{(T,g,\phi_2(X_T^2))}\right) \cdot dW_s.$$

It follows that

$$\mathcal{E}_g[\phi_1(X_T^1) + \phi_2(X_T^2)|\mathcal{F}_t] = \mathcal{E}_g[\phi_1(X_T^1)|\mathcal{F}_t] + \mathcal{E}_g[\phi_2(X_T^2)|\mathcal{F}_t] \quad \text{a.e.}, \quad t \in [0,T].$$

Choose $t = 0$, then

$$\mathcal{E}_g[\phi_1(X_T^1) + \phi_2(X_T^2)] = \mathcal{E}_g[\phi_1(X_T^1)] + \mathcal{E}_g[\phi_2(X_T^2)].$$

The proof of (i) is complete.

(ii) is obvious.

Theorem 4.1. is proved.

4.2. Choquet expectations, minimax expectations and generalized Peng's g-expectations. 4.2.1. Minimax expectations versus generalized Peng's g-expectations. Let

$$\mathcal{P} := \left\{ Q^\theta : \frac{dQ^\theta}{dP} = e^{-\frac{1}{2}\int_0^T |\theta_s|^2 \mathrm{d}s + \int_0^T \theta_s \cdot dW_s}, \quad |\theta_t^i| \leq \mu, dP \times dt\text{-a.s.} \right\}, \tag{4.3}$$

where θ_t^i is the ith component of θ_t.

Referring to [3, 5, 8], for any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, we define $\overline{\mathcal{E}}[\xi] = \sup_{Q \in \mathcal{P}} E_Q[\xi]$, $\underline{\mathcal{E}}[\xi] = \inf_{Q \in \mathcal{P}} E_Q[\xi]$. We further define conditional minimax expectations by

$$\overline{\mathcal{E}}[\xi|\mathcal{F}_t] = \text{ess} \sup_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t], \qquad \underline{\mathcal{E}}[\xi|\mathcal{F}_t] = \text{ess} \inf_{Q \in \mathcal{P}} E_Q[\xi|\mathcal{F}_t].$$

Obviously, $\overline{\mathcal{E}}[\xi|\mathcal{F}_0] = \overline{\mathcal{E}}[\xi]$, $\underline{\mathcal{E}}[\xi|\mathcal{F}_0] = \underline{\mathcal{E}}[\xi]$, where ess is essential.

The following lemma shows that $\overline{\mathcal{E}}[\xi]$, $\underline{\mathcal{E}}[\xi]$, $\overline{\mathcal{E}}[\xi|\mathcal{F}_t]$ and $\underline{\mathcal{E}}[\xi|\mathcal{F}_t]$ are well-defined for all $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$.

Lemma 4.1. *For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, then $\overline{\mathcal{E}}[\xi|\mathcal{F}_t] \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$ and $\underline{\mathcal{E}}[\xi|\mathcal{F}_t] \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$.*

Proof. For any $Q \in \mathcal{P}$, then there exists an adapted process $\{a_t\}$ bounded by μ such that

$$\frac{dQ^a}{dP} = e^{-\frac{1}{2}\int_0^T |a_s|^2 ds + \int_0^T a_s \cdot dW_s}.$$

For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, then there exists $p > 1$ such that $\xi \in L^p(\Omega, \mathcal{F}, P)$. By Hölder's inequality, we obtain

$$E_{Q^a}[|\xi||\mathcal{F}_t] = \frac{E\left[|\xi|\frac{dQ^a}{dP}|\mathcal{F}_t\right]}{E\left[\frac{dQ^a}{dP}|\mathcal{F}_t\right]} \le \frac{(E[|\xi|^p|\mathcal{F}_t])^{1/p}\left(E\left[\left(\frac{dQ^a}{dP}\right)^q|\mathcal{F}_t\right]\right)^{1/q}}{E\left[\frac{dQ^a}{dP}|\mathcal{F}_t\right]},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Since $\left(e^{-\frac{1}{2}\int_0^t |a_s|^2 ds + \int_0^t a_s \cdot dW_s}\right)_{t\in[0,T]}$ and $\left(e^{-\frac{1}{2}\int_0^t |qa_s|^2 ds + \int_0^t qa_s \cdot dW_s}\right)_{t\in[0,T]}$ are both martingales with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, hence

$$\frac{\left(E\left[\left(\frac{dQ^a}{dP}\right)^q|\mathcal{F}_t\right]\right)^{1/q}}{E\left[\frac{dQ^a}{dP}|\mathcal{F}_t\right]} \le e^{\frac{1}{2}(q-1)d\mu^2 T}\frac{\left(e^{-\frac{1}{2}\int_0^t |qa_s|^2 ds + \int_0^t qa_s \cdot dW_s}\right)^{1/q}}{\mathrm{e}^{-\frac{1}{2}\int_0^t |a_s|^2 ds + \int_0^t a_s \cdot dW_s}} \le e^{\frac{1}{2}(q-1)d\mu^2 T}.$$

Thus

$$E_{Q^a}[|\xi||\mathcal{F}_t] \le e^{\frac{1}{2}(q-1)d\mu^2 T}(E[|\xi|^p|\mathcal{F}_t])^{1/p},$$

which implies $\overline{\mathcal{E}}[\xi|\mathcal{F}_t] \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$ and $\underline{\mathcal{E}}[\xi|\mathcal{F}_t] \in \mathcal{L}(\Omega, \mathcal{F}_t, P)$.

In the following, for simplicity, we write in the sequel $\mathcal{E}^\mu[\cdot|\mathcal{F}_t] \equiv \mathcal{E}_g[\cdot|\mathcal{F}_t]$ for $g = \mu\sum_{i=1}^d |z^i|$ and $\mathcal{E}^{-\mu}[\cdot|\mathcal{F}_t] \equiv \mathcal{E}_g[\cdot|\mathcal{F}_t]$ for $g = -\mu\sum_{i=1}^d |z^i|$, where z^i is the ith component of z.

The following theorem shows a relation between minimax expectations and generalized Peng's g-expectations.

Theorem 4.2 (Martingale representation theorem for minimum and maximum expectations)**.** *If $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, then $\mathcal{E}^\mu[\xi|\mathcal{F}_t] = \overline{\mathcal{E}}[\xi|\mathcal{F}_t]$, $\mathcal{E}^{-\mu}[\xi|\mathcal{F}_t] = \underline{\mathcal{E}}[\xi|\mathcal{F}_t]$. In particular, $\mathcal{E}^\mu[\xi] = \overline{\mathcal{E}}[\xi]$, $\mathcal{E}^{-\mu}[\xi] = \underline{\mathcal{E}}[\xi]$.*

By Lemma 2.3 and Girsanov's theorem, it is easy to prove Theorem 4.2. The proof is very similar to that of Theorem 2.2 in Chen and Epstein [3]. We omit it.

4.2.2. Choquet expectations versus minimax expectations.

Definition 4.2. *A capacity is a real valued set function $V\colon \mathcal{F} \mapsto [0,1]$ satisfying*:

(i) $V(\varnothing) = 0$, $V(\Omega) = 1$;

(ii) $V(A) \le V(B)$ *for any* $A \subseteq B$.

The related Choquet expectation is denoted by

$$\mathcal{C}[\xi] := \int_{-\infty}^{0} (V(\xi \ge t) - 1)dt + \int_{0}^{\infty} V(\xi \ge t)dt.$$

From Definition 4.2, we may verify that $\mathcal{C}[\cdot]$ satisfies (see [6]):

(1) monotonicity: If $\xi \ge \eta$, then $\mathcal{C}[\xi] \ge \mathcal{C}[\eta]$,

(2) positive homogeneity: If $\lambda \geq 0$, then $\mathcal{C}[\lambda\xi] = \lambda\mathcal{C}[\xi]$,

(3) translation invariance: If $c \in R$, then $\mathcal{C}[\xi + c] = \mathcal{C}[\xi] + c$.

Define

$$V_g(A) := \mathcal{E}_g(1_A) \quad \forall A \in \mathcal{F}.$$

It is easy to check that V_g is a capacity. The related Choquet expectation is denoted by

$$\mathcal{C}_g[\xi] := \int_{-\infty}^{0} (V_g(\xi \geq t) - 1)dt + \int_{0}^{\infty} V_g(\xi \geq t)dt.$$

We next show that for any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, $\mathcal{C}_g[\xi] < \infty$.

Lemma 4.2. *Suppose that g satisfies* (H.2) *and* (H.3), *then* $\mathcal{C}_g[\xi] < \infty$ *for each* $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$.

Proof. We set $\overline{g}(t, y, z) := -g(t, 1 - y, -z)$ for any $(t, y, z) \in [0, T] \times R \times R^d$. Obviously $\overline{g}$ satisfies (H.2) and (H.3) with the same Lipschitz constant as g. It is easy to check that $V_{\overline{g}}(A) = 1 - V_g(A^C)$ for each $A \in \mathcal{F}$ and

$$\mathcal{C}_g[\xi] = \mathcal{C}_g[\xi^+] + \mathcal{C}_g[-\xi^-] = \mathcal{C}_g[\xi^+] - \mathcal{C}_{\overline{g}}[\xi^-] \quad \forall \xi \in \mathcal{L}(\Omega, \mathcal{F}, P).$$

For each $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, there exists $1 < p < 2$ such that $\xi \in L^p(\Omega, \mathcal{F}, P)$. From Lemma 2.2, for fixed $p' \in (1, p)$, we have $\mathcal{E}_g[\xi] \leq L(E[|\xi|^{p'}])^{1/p'}$, where $L > 0$ is a constant depending only on p', T and Lipschitz constant μ. Thus,

$$\int_{1}^{\infty} V(\xi \geq t)dt \leq L\int_{1}^{\infty} (P(\xi \geq t))^{1/p'}dt \leq L\left(\int_{1}^{\infty} t^{p-1}P(\xi \geq t)dt\right)^{1/p'}\left(\int_{1}^{\infty} t^{-\frac{(p-1)q'}{p'}}dt\right)^{1/q'},$$

where $\dfrac{1}{p'} + \dfrac{1}{q'} = 1$. Since $\int_1^\infty t^{p-1}P(\xi \geq t)dt \leq E[|\xi|^p] < \infty$ and $\int_1^\infty t^{-\frac{(p-1)q'}{p'}}dt < \infty$, we get $\mathcal{C}_g[\xi^+] < \infty$. Similarly, $\mathcal{C}_{\overline{g}}[\xi^-] < \infty$. This concludes the proof of the lemma.

For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, we define

$$\overline{\mathcal{C}}[\xi] := \int_{-\infty}^{0} (\overline{V}(\xi \geq t) - 1)dt + \int_{0}^{\infty} \overline{V}(\xi \geq t)dt,$$

$$\underline{\mathcal{C}}[\xi] := \int_{-\infty}^{0} (\underline{V}(\xi \geq t) - 1)dt + \int_{0}^{\infty} \underline{V}(\xi \geq t)dt,$$

where $\overline{V}$ and $\underline{V}$ are upper and lower probabilities defined by

$$\overline{V}(A) := \sup_{Q \in \mathcal{P}} Q(A), \qquad \underline{V}(A) := \inf_{Q \in \mathcal{P}} Q(A),$$

where $\mathcal{P}$ is the same as in (4.3).

Obviously, $\overline{V}(A) = \overline{\mathcal{E}}[1_A] = \mathcal{E}^{\mu}[1_A]$, $\underline{V}(A) = \underline{\mathcal{E}}[1_A] = \mathcal{E}^{-\mu}[1_A]$.

We have the following theorem.

Theorem 4.3. *Suppose (X_s) be the solution of SDE in Theorem 3.2. Let ϕ be a monotonic function such that $\phi(X_T) \in \mathcal{L}(\Omega, \mathcal{F}, P)$. Assuming that for all $t \geq 0$ and $x \in R$, $\sigma^i(t,x) > 0$ ($\sigma^i(t,x)$ is the ith component of $\sigma(t,x)$), $i = 1, 2, \ldots, d$, then there exist probability measures Q_1 and Q_2 such that*

(a) *for any ϕ that is increasing, then*

$$\overline{\mathcal{C}}[\phi(X_T)] = \overline{\mathcal{E}}[\phi(X_T)] = E_{Q_1}[\phi(X_T)], \qquad \underline{\mathcal{C}}[\phi(X_T)] = \underline{\mathcal{E}}[\phi(X_T)] = E_{Q_2}[\phi(X_T)];$$

(b) *for any ϕ that is decreasing, then*

$$\overline{\mathcal{C}}[\phi(X_T)] = \overline{\mathcal{E}}[\phi(X_T)] = E_{Q_2}[\phi(X_T)], \qquad \underline{\mathcal{C}}[\phi(X_T)] = \underline{\mathcal{E}}[\phi(X_T)] = E_{Q_1}[\phi(X_T)].$$

The probability measures Q_1 and Q_2 are defined by

$$\frac{dQ_1}{dP} = e^{-\frac{1}{2}d\mu^2 T + \mu \sum_{i=1}^d W_T^i}, \qquad \frac{dQ_2}{dP} = e^{-\frac{1}{2}d\mu^2 T - \mu \sum_{i=1}^d W_T^i}.$$

Proof. We only prove (a). The rest of this theorem can be proved in a similar manner.

Proof of part (a). For any $\phi(X_T) \in \mathcal{L}(\Omega, \mathcal{F}, P)$, there exists $1 < p < 2$, such that $\phi(X_T) \in \in L^p(\Omega, \mathcal{F}, P)$. Consider the following BSDE:

$$y_t = \phi(X_T) + \int_t^T \mu \sum_{i=1}^d |z_s^i| ds - \int_t^T z_s \cdot dW_s, \quad t \in [0, T]. \tag{4.4}$$

Let (y_t^μ, z_t^μ) be the unique solution of BSDE (4.4). By Theorem 3.2, noting that ϕ is an increasing function, we have

$$z_t^\mu \odot \sigma(t, X_t) \geq 0, \quad dP \times dt\text{-a.s.}$$

Since $\sigma^i(t,x) > 0$, we can deduce

$$z_t^{\mu,i} \geq 0, \quad dP \times dt\text{-a.s.}, \quad i = 1, 2, \ldots, d.$$

Therefore, (y_t^μ, z_t^μ) is also the unique solution of the following linear BSDE:

$$y_t = \phi(X_T) + \int_t^T \mu \sum_{i=1}^d z_s^i ds - \int_t^T z_s \cdot dW_s, \quad t \in [0, T]. \tag{4.5}$$

Let $\overline{W}_t := W_t - \mu(1, \ldots, 1)^T t$. By Girsanov's theorem, $(\overline{W}_t)_{t \in [0,T]}$ is a Q_1-Brownian motion, where $\dfrac{dQ_1}{dP} = e^{-\frac{1}{2}d\mu^2 T + \mu \sum_{i=1}^d W_T^i}$. Moreover, BSDE (4.5) can be rewritten as

$$y_t = \phi(X_T) - \int_t^T z_s \cdot d\overline{W}_s, \quad t \in [0, T].$$

It is easy to check that $\left(\int_0^t z_s \mathrm{d}\overline{W}_s \right)_{t \in [0,T]}$ is a martingale with respect to Q_1. Indeed, we have, by the BDG inequality and Hölder's inequality,

$$E_{Q_1}\left[\left|\int\limits_0^t z_s d\overline{W}_s\right|\right] \leq E_{Q_1}\left[\left(\int\limits_0^T |z_s|^2 ds\right)^{1/2}\right] \leq$$

$$\leq \left(E\left[\left(\int\limits_0^T |z_s|^2 ds\right)^{p/2}\right]\right)^{1/p} \left(E\left[\left(\frac{dQ_1}{dP}\right)^q\right]\right)^{1/q} < \infty,$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Thus,

$$\overline{\mathcal{E}}[\phi(X_T)] = E_{Q_1}[\phi(X_T)]. \tag{4.6}$$

On the other hand, let $\phi^n(X_T) := (\phi(X_T) \wedge n) \vee (-n)$. From [5], we have

$$\overline{\mathcal{E}}[\phi^n(X_T)] = \overline{\mathcal{C}}[\phi^n(X_T)].$$

Note that

$$\overline{V}(\phi^n(X_T) \geq t) = \overline{\mathcal{E}}\left[1_{(\phi^n(X_T)\geq t)}\right] = \mathcal{E}^\mu\left[1_{(\phi^n(X_T)\geq t)}\right],$$

$$V(\phi(X_T) \geq t) = \overline{\mathcal{E}}\left[1_{(\phi(X_T)\geq t)}\right] = \mathcal{E}^\mu\left[1_{(\phi(X_T)\geq t)}\right]$$

and hence

$$\overline{V}(\phi^n(X_T) \geq t) \to \overline{V}(\phi(X_T) \geq t), \quad \text{as} \quad n \to \infty.$$

Applying the monotonic convergence theorem, we obtain

$$\overline{\mathcal{C}}[\phi^n(X_T)] \to \overline{\mathcal{C}}[\phi(X_T)], \quad \text{as} \quad n \to \infty.$$

From Lemma 2.2, we obtain

$$\overline{\mathcal{E}}[\phi^n(X_T)] \to \overline{\mathcal{E}}[\phi(X_T)], \quad \text{as} \quad n \to \infty.$$

Therefore

$$\overline{\mathcal{E}}[\phi(X_T)] = \overline{\mathcal{C}}[\phi(X_T)]. \tag{4.7}$$

From (4.6) and (4.7), we have

$$\overline{\mathcal{C}}[\phi(X_T)] = E_{Q_1}[\phi(X_T)].$$

In a similar manner, we can obtain

$$\underline{\mathcal{C}}[\phi(X_T)] = E_{Q_2}[\phi(X_T)].$$

Now we give an example to illustrate how our result allows one to calculate Choquet expectations.

***Example* 4.1.** For simplicity, let $T = 1$, $d = 1$. Suppose $b = 0$, $\sigma = 1$ and $x = 0$, then $X_1 = W_1$, $W_1 \sim N(0,1)$.

(i) Let $\phi(x) = \exp\left(\dfrac{x^2}{2p_1} - x\right) 1_{(x \geq p_1)}$, where $1 < p_1 < 2$.

Obviously, ϕ is an increasing function and $\phi(X_1) = \phi(W_1) = \exp\left(\dfrac{W_1^2}{2p_1} - W_1\right) 1_{(W_1 \geq p_1)}$.

It is easy to check out

$$E[|\phi(W_1)|^{p_1}] = \int\limits_{p_1}^{\infty} \exp\left(\frac{x^2}{2} - p_1 x\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi} p_1} e^{-p_1^2} < \infty,$$

and

$$E[|\phi(W_1)|^{p}] = \infty \quad \forall p > p_1.$$

Hence, $\phi(W_1) \in \mathcal{L}(\Omega, \mathcal{F}, P)$, $\phi(W_1) \notin L^2(\Omega, \mathcal{F}, P)$.

Using Theorem 4.3, we obtain

$$\overline{\mathcal{C}}[\phi(W_1)] = E_{Q_1}[\phi(W_1)] = \int\limits_{p_1}^{\infty} \exp\left(\frac{x^2}{2p_1} - x\right) e^{-\frac{1}{2}\mu^2 + \mu x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2} \int\limits_{p_1}^{\infty} e^{-\frac{1}{2}\left(1 - \frac{1}{p_1}\right)x^2} e^{-(1-\mu)x} dx < \infty,$$

$$\underline{\mathcal{C}}[\phi(W_1)] = E_{Q_2}[\phi(W_1)] = \int\limits_{p_1}^{\infty} \exp\left(\frac{x^2}{2p_1} - x\right) e^{-\frac{1}{2}\mu^2 - \mu x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2} \int\limits_{p_1}^{\infty} e^{-\frac{1}{2}\left(1 - \frac{1}{p_1}\right)x^2} e^{-(1+\mu)x} dx < \infty.$$

(ii) Let $\phi(x) = \exp\left(\dfrac{x^2}{2p_1} + x\right) 1_{(x \leq -p_1)}$, where $1 < p_1 < 2$.

Obviously, ϕ is a decreasing function and $\phi(X_1) = \phi(W_1) = \exp\left(\dfrac{W_1^2}{2p_1} + W_1\right) 1_{(W_1 \leq -p_1)}$.

It is easy to check out

$$E[|\phi(W_1)|^{p_1}] = \int\limits_{-\infty}^{-p_1} \exp\left(\frac{x^2}{2} + p_1 x\right) \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}x^2} dx = \frac{1}{\sqrt{2\pi} p_1} \mathrm{e}^{-p_1^2} < \infty,$$

and

$$E[|\phi(W_1)|^{p}] = \infty \quad \forall p > p_1.$$

Hence, $\phi(W_1) \in \mathcal{L}(\Omega, \mathcal{F}, P)$, $\phi(W_1) \notin L^2(\Omega, \mathcal{F}, P)$.

Using Theorem 4.3, we obtain

$$\overline{\mathcal{C}}[\phi(W_1)] = E_{Q_2}[\phi(W_1)] = \int\limits_{-\infty}^{-p_1} \exp\left(\frac{x^2}{2p_1} + x\right) e^{-\frac{1}{2}\mu^2 - \mu x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2} \int\limits_{-\infty}^{-p_1} e^{-\frac{1}{2}\left(1 - \frac{1}{p_1}\right)x^2} e^{-(\mu-1)x} dx < \infty,$$

$$\underline{\mathcal{C}}[\phi(W_1)] = E_{Q_1}[\phi(W_1)] = \int\limits_{-\infty}^{-p_1} \exp\left(\frac{x^2}{2p_1} + x\right) e^{-\frac{1}{2}\mu^2 + \mu x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\mu^2} \int\limits_{-\infty}^{-p_1} e^{-\frac{1}{2}\left(1 - \frac{1}{p_1}\right)x^2} e^{(\mu+1)x} dx < \infty.$$

***Remark* 4.2.** The Choquet expectations $\overline{\mathcal{C}}[\phi(W_1)]$, $\underline{\mathcal{C}}[\phi(W_1)]$ in Example 4.1 can not be calculated by Chen and Kulperger [5], because $\phi(W_1) \notin L^2(\Omega, \mathcal{F}, P)$. But thanks to Theorem 4.3, since $\phi(W_1) \in \mathcal{L}(\Omega, \mathcal{F}, P)$, one can easily calculate them.

4.2.3. Choquet expectations versus generalized Peng's g-expectations. In this subsection, we provide a necessary and sufficient condition of Choquet expectations being equal to generalized Peng's g-expectations.

We have the following theorem.

Theorem 4.4. *Suppose that g satisfies* (H.2) *and* (H.3). *Then there exists a Choquet expectation whose restriction to $\mathcal{L}(\Omega, \mathcal{F}, P)$ is equal to a generalized Peng's g-expectation if and only if g does not depend on y and is linear in z, i.e.,*

$$g(t, y, z) = v_t \cdot z = \sum_{i=1}^{d} v_t^i z^i.$$

Proof. Since $L^2(\Omega, \mathcal{F}, P) \subset \mathcal{L}(\Omega, \mathcal{F}, P)$, the proof of necessity can be seen in Chen et al. [2] and Hu [11]. We only prove the sufficiency.

For any $\xi \in \mathcal{L}(\Omega, \mathcal{F}, P)$, there exists $1 < p < 2$, such that $\xi \in L^p(\Omega, \mathcal{F}, P)$. If $g(t, y, z) = v_t \cdot z$, let us consider the BSDE

$$y_t = \xi + \int\limits_t^T v_s \cdot z_s ds - \int\limits_t^T z_s \cdot dW_s.$$

Set $\overline{W}_t = W_t - \int_0^t v_s ds$, then

$$y_t = \xi - \int\limits_t^T z_s \cdot d\overline{W}_s.$$

By Girsanov's theorem, $(\overline{W}_t)_{t\in[0,T]}$ is a Q-Brownian motion under Q defined by

$$\frac{dQ}{dP} = \exp\left[-\frac{1}{2}\int_0^T |v_s|^2 ds + \int_0^T v_s \cdot dW_s\right].$$

Thus

$$\mathcal{E}_g[\xi] = E_Q[\xi].$$

This implies the generalized Peng's g-expectation is a classical mathematical expectation. Obviously, the classical mathematical expectation can be represented by the Choquet expectation. So the proof of sufficiency is complete.

1. *Briand P., Delyon B., Hu Y., Pardoux E., Stoica L.* L^p solutions of backward stochastic differential equations // Stochast. Process. and Appl. – 2003. – **108**. – P. 109 – 129.
2. *Chen Z., Chen T., Davison M.* Choquet expectation and Peng's g-expectation // Ann. Probab. – 2005. – **33**, № 3. – P. 1179 – 1199.
3. *Chen Z., Epstein L.* Ambiguity, risk and asset returns in continuous time // Econometrica. – 2002. – **70**, № 4. – P. 1403 – 1443.
4. *Chen Z., Kulperger R., Wei G.* A comonotonic theorem for BSDEs // Stochast. Process. and Appl. – 2005. – **115**. – P. 41 – 54.
5. *Chen Z., Kulperger R.* Minimax pricing and Choquet pricing // Insurance: Math. Econ. – 2006. – **38**, № 3. – P. 518 – 528.
6. *Choquet G.* Theory of capacities // Ann. Inst. Fourier. – 1953. – **5**. – P. 131 – 195.
7. *Darling R. W. R.* Constructing gamma martingales with prescribed limits, using BSDEs // Ann. Probab. – 1995. – **23**. – P. 1234 – 1261.
8. *El Karoui N., Peng S., Quenez M. C.* Backward stochastic differential equations in finance // Math. Finance. – 1997. – **7**, № 1. – P. 1 – 71.
9. *Hamadene S., Lepeltier J.* Zero-sum stochastic differential games and BSDEs // Stochastics and Stochast. Repts. – 1995. – **54**. – P. 221 – 231.
10. *Hu F., Chen Z.* Generalized Peng's g-expectation and related properties // Statist. Probab. Lett. – 2010. – **80**. – P. 191 – 195.
11. *Hu M.* On the integral representation of g-expectations // Comptes Rend. Math. – 2010. – **348**. – P. 571 – 574.
12. *Hu Y.* Probabilistic interpretation for a systems of quasi elliptic PDE with Niemann boundary conditions // Stochast. Process. and Appl. – 1993. – **48**. – P. 107 – 121.
13. *Ma J., Protter J., Yong J.* Solving forward-backward stochastic differential equations-a four step scheme // Probab. Theory Relat. Fields. – 1994. – **98**. – P. 339 – 359.
14. *Pardoux E., Peng S.* Adapted solution of a backward stochastic differential equation // Systems Control Lett. – 1990. – **14**, № 1. – P. 55 – 61.
15. *Pardoux E., Zhang S.* Generalized BSDEs and nonlinear Neumann boundary value problems // Probab. Theory Relat. Fields. – 1998. – **110**. – P. 535 – 558.
16. *Peng S.* robabilistic interpretation for systems of quasilinear parabolic partial differential equations // Stochastics and Stochast. Repts. – 1991. – **37**. – P. 61 – 74.
17. *Peng S.* Backward SDE and related g-expectation // Pitman Res. Notes in Math. Ser. – Vol. 364. Backward Stochastic Differential Equations / Eds El N. Karoui, L. Mazliak. – Harlow: Longman, 1997. – P. 141 – 159.
18. *Peng S.* Monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob – Meyer's type // Probab. Theory Relat. Fields. – 1999. – **113**. – P. 473 – 499.

Received 10.01.11