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ON PROPERTIES OF n-TOTALLY PROJECTIVE ABELIAN p-GROUPS

ПРО ВЛАСТИВОСТІ n-ТОТАЛЬНО ПРОЕКЦІЙНИХ АБЕЛЕВИХ p-ГРУП

We prove some properties of n-totally projective abelian p-groups. Under some additional conditions for the group structure, we obtain an equivalence between the notions of n-total projectivity and strong n-total projectivity. We also show that n-totally projective A-groups are isomorphic if they have isometric p^n-socles.

Доведено деякі властивості n-тотально проекційних абелевих p-груп. При деяких додаткових умовах на будову груп встановлено еквівалентність понять n-тотальної проективності та сильної n-тотальної проективності. Також показано, що n-тотально проективні A-групи ізоморфні, якщо вони мають ізометричні p^n-цоколі.

Introduction. Throughout this paper, let us assume that all groups are additive p-primary groups and n is a fixed natural. Foremost, we recall some crucial notions from [7] and [8] respectively.

Definition 1. *A group G is said to be n-simply presented if there exists a p^n-bounded subgroup P of G such that G/P is simply presented. A summand of an n-simply presented group is called n-balanced projective.*

Definition 2. *A group G is said to be strongly n-simply presented = nicely n-simply presented if there exists a nice p^n-bounded subgroup N of G such that G/N is simply presented. A summand of a strongly n-simply presented group is called strongly n-balanced projective.*

Clearly, strongly n-simply presented groups are n-simply presented, while the converse fails (see, e.g., [7]).

Definition 3. *A group G is called n-totally projective if, for all ordinals λ, $G/p^\lambda G$ is $p^{\lambda+n}$-projective.*

Definition 4. *A group G is called strongly n-totally projective if, for any ordinal λ, $G/p^{\lambda+n}G$ is $p^{\lambda+n}$-projective.*

Apparently, strongly n-totally projective groups are n-totally projective, whereas the converse is wrong (see, for instance, [8]). Moreover, (strongly) n-simply presented groups are themselves (strongly) n-totally projective, but the converse is untrue (see, for example, [8]).

Definition 5. *A group G is called weakly n-totally projective if, for each ordinal λ, $G/p^\lambda G$ is $p^{\lambda+2n}$-projective.*

Evidently, n-totally projective groups are weakly n-totally projective with the exception of the reverse implication which is not valid.

The purpose of the present article is to explore some critical properties of n-totally projective groups, especially when some of the three variants of n-total projectivity do coincide. In fact, we show that if the group G is an A-group, then the concepts of being n-totally projective and strongly n-totally projective will be the same (Theorem 1). However, this is not the case for weakly n-totally projective groups (Example 1). We also establish that two n-totally projective A-groups are isomorphic if and only if they have isometric p^n-socles, i.e., isomorphic socles whose isomorphism preserves heights as computed in the whole group (Corollary 1). Likewise, we exhibit a concrete example of a strongly n-totally projective group with finite first Ulm subgroup that is not $\omega + n$-

totally $p^{\omega+n}$-projective (Example 2). Finally, some assertions about (strongly) n-simply presented and n-balanced projective groups are obtained as well (Proposition 3 and Corollaries 2 – 4).

We note for readers' convenience that all undefined explicitly notations and the terminology are standard and follow essentially those from [2 – 4]. Besides, for shortness, we will denote the torsion product Tor (G, H) of the groups G and H by $G \bigtriangledown H$. Also, for any group G and ordinal λ, $L_\lambda G$ is its completion in the p^λ-topology and let $E_\lambda G = (L_\lambda G)/G$.

Main results. We begin here with the equivalence of strong n-total projectivity and n-total projectivity under the extra assumption that the full group is an A-group. Specifically, the following holds:

Theorem 1. *Suppose G is an A-group. Then the following three conditions are equivalent*:

(a) *G is n-totally projective*;

(b) *G is strongly n-totally projective*;

(c) *for every limit ordinal λ of uncountable cofinality, we have $p^n E_\lambda G = \{0\}$.*

Proof. We first turn to a few thoughts on A-groups introduced in [4]. Let λ be a limit ordinal, and let

$$0 \to G \to H \to K \to 0 \tag{1}$$

be a p^λ-pure exact sequence with H a totally projective group of length λ and K a totally projective group. If λ has countable cofinality or $p^\lambda K = \{0\}$, then G is also totally projective. Otherwise, G is said to be a λ-elementary A-group. Note that $p^\lambda K$ is naturally isomorphic to $(L_\lambda G)/G = E_\lambda$ where $L_\lambda G$ is the completion in the p^λ-topology. An A-group G is then defined to be the direct sum of a collection of λ-elementary A-groups, for various ordinals of uncountable cofinality. Note that these groups G are classified in [4] up to an isomorphism using their Ulm invariants, together with the Ulm invariants of the totally projective groups $E_\lambda G$, over all limit ordinals λ of uncountable cofinality.

Next, since a direct sum of groups is (strongly) n-totally projective if and only if each of its terms has that property, and since the functor $E_\lambda G$ also respects direct sums (because λ has uncountable cofinality), we may assume that G is a λ-elementary A-group and that we possess a representing sequence as in (1). Notice that for any limit ordinal $\beta < \lambda$, we have a balanced-exact sequence implied via (1)

$$0 \to G/p^\beta G \to H/p^\beta H \to K/p^\beta K \to 0.$$

On the other hand, since K is totally projective, $K/p^\beta K$ is p^β-projective, so that this sequence splits. It now follows that $G/p^\beta G$ is a summand of the totally projective group $H/p^\beta H$, and hence it is p^β-projective too. Our result will therefore follow from the statement:

Claim. If λ is a limit ordinal of uncountable cofinality and G is a λ-elementary A-group, then $G \cong G/p^\lambda G \cong G/p^{\lambda+n} G$ is $p^{\lambda+n}$-projective if and only if $p^n E_\lambda G \cong p^{\lambda+n} K = \{0\}$.

In order to prove that Claim, observe that (1) can actually be viewed as a p^λ-pure projective resolution of K. Compare this with the standard p^λ-pure projective resolution of K given by

$$0 \to M_\lambda \bigtriangledown K \to H_\lambda \bigtriangledown K \to K \to 0$$

where M_λ is a λ-elementary S-group of length λ and H_λ is the Prüfer group of length λ (see [8]). By virtue of the Schanuel's lemma (cf. [3]), there is an isomorphism

$$(M_\lambda \bigtriangledown K) \oplus H \cong (H_\lambda \bigtriangledown K) \oplus G.$$

Since H and $H_\lambda \bigtriangledown K$ are obviously p^λ-projective, it suffices to show that $M_\lambda \bigtriangledown K$ is $p^{\lambda+n}$-projective if and only if $p^{\lambda+n}K = \{0\}$.

To this aim, suppose first that $p^{\lambda+n}K = \{0\}$; so in particular, K is $p^{\lambda+n}$-projective, whence $M_\lambda \bigtriangledown K$ is $p^{\lambda+n}$-projective (see [9]). For the converse, we see that $H_\lambda \bigtriangledown K$ will also be complete in the p^λ-topology. Consequently, $E_\lambda(M_\lambda \bigtriangledown K) \cong E_\lambda G \cong p^\lambda K$. Supposing $p^{\lambda+n}K \neq \{0\}$, we need to demonstrate that $M_\lambda \bigtriangledown K$ is not $p^{\lambda+n}$-projective. Considering a direct summand of K, it suffices to assume that $p^\lambda K$ is cyclic of order p^m, where $m > n$. Let M be a p^λ-high subgroup of K. It follows that M is also $p^{\lambda+n}$-high in K and hence it is $p^{\lambda+n+1}$-pure in K. In addition, $K/M \cong \mathbf{Z}(p^\infty)$, so that $M_\lambda \bigtriangledown (K/M) \cong M_\lambda$. It would then follow that the sequence

$$0 \to M_\lambda \bigtriangledown M \to M_\lambda \bigtriangledown K \to M_\lambda \to 0$$

is $p^{\lambda+n+1}$-pure. If $M_\lambda \bigtriangledown K$ actually were $p^{\lambda+n}$-projective, then Lemma 2.1 (g) from [8] would imply that the sequence splits. Therefore, M_λ is isomorphic to a summand of $M_\lambda \bigtriangledown K$. However, $E_\lambda(M_\lambda \bigtriangledown K) \cong p^\lambda K$ is reduced, whereas $E_\lambda M_\lambda \cong \mathbf{Z}(p^\infty)$ is divisible. This contradiction proves the entire Claim and hence the theorem.

As a consequence, we yield the following result concerning the isomorphism characterization of n-totally projective A-groups.

***Corollary* 1.** *Suppose G and G' are n-totally projective A-groups. Then G and G' are isomorphic if and only if $G[p^n]$ and $G'[p^n]$ are isometric.*

Proof. Applying Theorem 1, G and G' are both strongly n-totally projective and both $E_\lambda G, E_\lambda G'$ are p^n-bounded for each limit ordinal λ of uncountable cofinality. Since G and G' clearly possess identical Ulm invariants, we need to illustrate that for for any λ as above we have $E_\lambda G \cong E_\lambda G'$. It is readily checked that every element of $E_\lambda G$ can be represented by a neat Cauchy net $\{x_i\}_{i<\alpha}$ where each $x_i \in G[p^n]$. This means that $E_\lambda G$ can also be described as $L_\lambda(G[p^n])/(G[p^n])$, where the numerator of this expression consists of the inverse limit of $G[p^n]/(p^\alpha G)[p^n]$ over all $\alpha < \lambda$. Since $G[p^n]$ and $G'[p^n]$ are isometric, by what we have shown above it follows that $E_\lambda G$ and $E_\lambda G'$ are isomorphic for all λ. But employing [5], we can conclude that $G \cong G'$, as claimed.

Corollary 1 is proved.

The following example shows that Theorem 1 is not longer true for weakly n-totally projective groups.

***Example* 1.** There exists a weakly n-totally projective A-group which is not n-totally projective.

Proof. Construct any A-group G of length ω_1 which is proper p^{ω_1+2}-projective, that is, p^{ω_1+2}-projective but not p^{ω_1+1}-projective. For example, if M_{ω_1} is an elementary S-group of length ω_1, and H_{ω_1+2} is the Prüfer group of length ω_1+2, then $G = H_{\omega_1+2} \bigtriangledown M_{\omega_1}$ will be such a group. Furthermore, it follows immediately that G is weakly 1-totally projective but it is not 1-totally projective as desired.

The next example shows that the class of strongly n-totally projective groups is not contained in the class of $\omega+n$-totally $p^{\omega+n}$-projective groups. Recall that in [1] a group G is said to be *$\omega+n$-totally $p^{\omega+n}$-projective group* if each $p^{\omega+n}$-bounded subgroup is $p^{\omega+n}$-projective.

***Example* 2.** There exists a strongly n-totally projective group with finite inseparable first Ulm subgroup which is not $\omega + n$-totally $p^{\omega+n}$-projective.

Proof. Suppose A is a separable $p^{\omega+1}$-projective group whose socle $A[p]$ is not $\aleph_0$-coseparable (such a group exists even in ZFC and is common to construct) and H is a countable group with $p^\omega H$ being finite and $p^{\omega+n}H \neq 0$. Letting $G = A \oplus H$, then G is strongly n-totally projective. Indeed, it is pretty easy to see that $G/p^{\lambda+n}G$ is $p^{\lambda+n}$-projective for any (limit) ordinal λ because both A and H are n-totally projective. Since G is neither a direct sum of countable groups nor a $p^{\omega+n}$-projective group, if it were $\omega + n$-totally $p^{\omega+n}$-projective, it would be proper. However, appealing to Theorem 3.1 of [1], this cannot be happen.

Another example in this way can be found in ([6], Example 2.5).

On the other hand, $\omega + n$-totally $p^{\omega+n}$-projective groups are contained in the class of n-totally projective groups. In fact, by a plain combination of Proposition 3.1 and Theorem 1.2 (a_1) in [6] along with [7], $\omega + n$-totally $p^{\omega+n}$-projective groups are themselves n-simply presented and thus they are n-totally projective, as asserted.

In this way the following statement is true as well. Imitating [1], recall that a group is said to be *ω-totally $p^{\omega+n}$-projective* if every its separable subgroup is $p^{\omega+n}$-projective.

Proposition 1. *Each n-totally projective group with countable first Ulm subgroup is ω-totally $p^{\omega+n}$-projective.*

Proof. If G is n-totally projective, then with the aid of Definition 3 we obtain that the quotient $G/p^\omega G$ will actually be $p^{\omega+n}$-projective, and so ω-totally $p^{\omega+n}$-projective. Since $p^\omega G$ is countable and the ω-totally $p^{\omega+n}$-projective groups are closed under ω_1-bijections (see [6]), G will be ω-totally $p^{\omega+n}$-projective, as expected.

We will be next concentrated to some characteristic properties of (strongly) n-totally projective groups.

Proposition 2. *Let $P \leq G[p]$.*

(a) *If G is (strongly) n-totally projective, then G/P is (strongly) $n+1$-totally projective.*

(b) *If G/P is (strongly) n-totally projective, then G is (strongly) $n+1$-totally projective.*

Proof. We shall prove the statement only for n-totally projective groups since the situation with strongly n-totally projective groups is quite similar.

(a) If λ is an ordinal and $G_\lambda = G/p^\lambda G$, then there is an exact sequence

$$0 \to (P + p^\lambda G)/p^\lambda G \to G_\lambda \to G/(P + p^\lambda G) \to 0.$$

Since $p((P + p^\lambda G)/p^\lambda G) = \{0\}$ and G_λ is $p^{\lambda+n}$-projective, it follows that $H = G/(P + p^\lambda G)$ is $p^{\lambda+n+1}$-projective. However, if $Q = (P + p^\lambda G)/P \subseteq A = G/P$, then $Q \subseteq p^\lambda A$. In addition,

$$H \cong (G/P)/((P + p^\lambda G)/P) = A/Q$$

is $p^{\lambda+n+1}$-projective. Moreover, it follows also that

$$H_\lambda = H/p^\lambda H \cong A/Q/p^\lambda(A/Q) = A/Q/p^\lambda A/Q \cong A/p^\lambda A = A_\lambda$$

is $p^{\lambda+n+1}$-projective. Note that this implies that A is $n+1$-totally projective, as required.

(b) Suppose now that $A = G/P$ is n-totally projective. If $P' = G[p]/P \subseteq A[p]$, then by what we have already shown above $pG \cong G/G[p] \cong (G/P)/(G[p]/P) = A/P'$ is $n+1$-totally projective. However, this easily forces by [8] that G itself is $n+1$-totally projective, as claimed.

We will now establish some affirmations for n-simply presented groups and their direct summands called n-balanced projective groups. So, the next few results show that an n-balanced projective group must be pretty close to being n-simply presented, since they illustrate that the complementary summand can be chosen in special ways. Recall that a group B will be said to be a *BT-group* if it is isomorphic to a balanced subgroup of a totally projective group. It plainly follows that a *BT-group* is also an *IT-group* (i.e., one that is isomorphic to an isotype subgroup of a totally projective group).

Proposition 3. *Suppose G is a group of length λ. Then the following hold*:

(a) *If G is n-balanced projective, then there is a BT-group X with $p^\lambda X = \{0\}$ such that $G \oplus X$ is n-simply presented.*

(b) *If G is strongly n-balanced projective, then there is an IT-group K with $p^\lambda K = \{0\}$ such that $G \oplus K$ is strongly n-simply presented.*

Proof. (a) Using the notation of Theorem 1.2 from [7], we start with a balanced projective resolution

$$0 \to X \to Y \to G \to 0,$$

so that X is a BT-group. Knowing this, we can construct an n-balanced projective resolution

$$0 \to X \to Z \to G \to 0$$

of G. Since G is n-balanced projective, we can conclude that $G \oplus X \to Z$ is n-simply presented, as required.

(b) Using the notations of Lemma 1.4 and Theorem 1.5 of [7], there is a strongly n-balanced projective resolution of G given by

$$0 \to K(G) \to H(G) \to G \to 0$$

where $H(G) = \mathcal{K}(G[p^n])$ is strongly n-simply presented. Note that $H(G)[p^n]$ is isometric to the valuated direct sum $G[p^n] \oplus K(G)[p^n]$. It follows that $K(G)[p^n]$ embeds isometrically in $H(G)/G[p^n]$. Therefore $K(G)$ embeds as an isotype subgroup of $H(G)/G[p^n]$, which is obviously totally projective.

As immediate consequences, we derive the following corollaries.

***Corollary* 2.** *Let G be a (strongly) n-balanced projective group of countable length. Then there exists a direct sum of countable groups X of countable length such that $G \oplus X$ is (strongly) n-simply presented.*

Proof. Since IT-groups of countable length are direct sums of countable groups, we may directly apply Proposition 3.

***Corollary* 3.** *Let G be an n-balanced projective group. If the balanced projective dimension of G is at most 1, then there is a totally projective group X such that $G \oplus X$ is n-simply presented.*

Proof. Again, if

$$0 \to X \to Y \to G \to 0$$

is a balanced projective resolution of G, then X will be totally projective, and $G \oplus X$ will be n-simply presented.

***Corollary* 4.** *Let G and G' be strongly n-balanced projective groups. If $G[p^n]$ is isometric to $G'[p^n]$, so that they have the same length λ, then there are IT-groups K and K' of length at most λ such that $G \oplus K$ is isomorphic to $G' \oplus K'$.*

Proof. An isometry $G[p^n] \to G'[p^n]$ leads to an isomorphism $H(G) \to H(G')$, and thus the result follows from Proposition 3 (b).

We close the work with the following three problems:

Problem 1. Find an ω-totally $p^{\omega+n}$-projective group which is not n-totally projective, and an n-totally projective group with a uncountable first Ulm subgroup that is not ω-totally $p^{\omega+n}$-projective.

Problem 2. Does it follow that n-simply presented A-groups are strongly n-simply presented?

Problem 3. Does there exist a p^{ω_1+1}-projective N-group of length ω_1 which is not totally projective, i.e., is not a direct sum of countable groups?

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