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INTEGRAL MANIFOLDS FOR SEMILINEAR EVOLUTION EQUATIONS AND ADMISSIBILITY OF FUNCTION SPACES**

ІНТЕГРАЛЬНІ МНОГОВИДИ ДЛЯ НАПІВЛІНІЙНИХ ЕВОЛЮЦІЙНИХ РІВНЯНЬ ТА ДОПУСТИМІСТЬ ПРОСТОРІВ ФУНКЦІЙ

We prove the existence of integral (stable, unstable, center) manifolds for the solutions to the semilinear integral equation $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))d\xi$ in the case where the evolution family $(U(t,s))_{t\geq s}$ has an exponential trichotomy on a half-line or on the whole line, and the nonlinear forcing term f satisfies the φ-Lipschitz conditions, i.e., $\|f(t,x)-f(t,y)\| \leqslant \varphi(t)\|x-y\|$, where $\varphi(t)$ belongs to some classes of admissible function spaces. Our main method invokes the Lyapunov – Perron methods, rescaling procedures, and the techniques of using the admissibility of function spaces.

Доведено існування інтегральних (стійких, нестійких, центральних) многовидів для розв'язків напівлінійного інтегрального рівняння $u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))d\xi$ у випадку, коли сім'я еволюцій $(U(t,s))_{t\geq s}$ має експоненціальну трихотомію на півосі або на всій осі, а нелінійний збурюючий член f задовольняє φ-ліпшицеві умови, тобто $\|f(t,x)-f(t,y)\| \leqslant \varphi(t)\|x-y\|$, де $\varphi(t)$ належить до деяких класів допустимих просторів функцій. Наш основний метод базується на методах Ляпунова – Перрона, процедурах перемасштабування та техніці застосування допустимості просторів функцій.

1. Introduction and preliminaries. Consider the semilinear evolution equation of the form

$$\frac{dx}{dt} = A(t)x(t) + f(t,x(t)), \quad t \in \mathbb{J}, \tag{1.1}$$

where $\mathbb{J}$ is a subinterval of the real line $\mathbb{R}$; each $A(t)$ is a (possibly unbounded) linear operator acting in a Banach space X, $x(t) \in X$, and $f(\cdot,\cdot)\colon \mathbb{J}\times X \to X$ is a nonlinear operator. When the linear part (i.e., the equation $dx/dt = A(t)x(t)$) of the above equation has an exponential dichotomy (or trichotomy), one shall try to find conditions imposed on the nonlinear forcing term f such that the equation (1.1) has an integral manifold (e.g., a stable, unstable, or center manifold).

Such early results can be traced back to Hadamard [10], Perron [29, 30], Bogoliubov and Mitropolsky [4, 5] for the case of matrix coefficients $A(t)$, to Daleckii and Krein [8] for the case of bounded coefficients acting on Banach spaces, and to Henry [12] for the case of unbounded coefficients. At this point, we would like to quote the sentence by Anosov [1]:

„Every five years or so, if not more often, some one „discovers" the theorem of Hadamard and Perron, proving it by Hadamard's method of proof or by Perron's".

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The Hadamard's method is generalized to the so-called graph transform method which has been used, e.g., in the works [2, 13, 22] to prove the existence of invariant manifolds. This method is more far-reaching and related to complicated choices of the transforms between graphs representing the involved manifolds. Meanwhile, the Perron's method is now extended to the well-known Lyapunov – Perron method aimed at the construction of the so-called Lyapunov – Perron equations (or operators) involving the differential equations under consideration to show the existence of the integral manifolds. It seems to be more natural to use the Lyapunov – Perron method to handle with the flows or semiflows which are generated by semilinear evolution equations since in this case it is relatively simple to construct such Lyapunov – Perron equations or operators. We refer the reader to [3, 7, 8, 11, 12, 15, 16, 35] and reference therein for more information on the matter.

To our best knowledge, the most popular conditions for the existence of invariant manifolds are the exponential dichotomy (or trichotomy) of the linear part $\dfrac{dx}{dt} = A(t)x$ and the uniform Lipschitz continuity of the nonlinear part $f(t,x)$ with sufficiently small Lipschitz constants (i.e., $\|f(t,x) - - f(t,y)\| \leqslant q\|x-y\|$ for q small enough). The purpose of this paper is establishing the existence of stable, unstable, and center-stable manifolds when the linear part of equation (1.1) has an exponential trichotomy on the half-line or on the whole line under more general conditions on the nonlinear term $f(t,x)$, that is the non-uniform Lipschitz continuity of f, i.e., $\|f(t,x) - f(t,y)\| \leqslant \varphi(t)\|x-y\|$ for φ being a real and positive function which belongs to admissible function spaces defined in Definition 2.3 below. Under some conditions on φ, we will prove the existence of center manifolds for the equation (1.1) provided that the linear part $\dfrac{dx}{dt} = A(t)x$ has an exponential trichotomy. Our method is to transform to the case of exponential dichotomy by some rescaling procedures, and then applying our techniques and results in [15] where we have used the Lyapunov – Perron method and the characterization (obtained in [14]) of the exponential dichotomy of evolution equations in admissible spaces of functions defined on the half-line $\mathbb{R}_+$ to construct the structures of solutions of the equation (1.1) in a mild form, which belong to some certain classes of admissible spaces on which we could implement some well-known procedures in functional analysis such as: constructing of contraction mapping; using of Implicite Function Theorem, etc. The use of admissible spaces has helped us to construct the invariant manifolds for equation (1.1) in the case of dichotomic linear parts without using the smallness of Lipschitz constants of nonlinear forcing terms in classical sense. Instead, the "smallness" is understood as the sufficient smallness of $\sup_{t\geq 0}\displaystyle\int_t^{t+1}\varphi(\tau)d\tau$ (see the conditions in Theorem 4.7 in [15]). Consequently, we have obtained the existence of invariant-stable manifolds for the case of dichotomic linear parts under very general conditions on the nonlinear term $f(t,x)$ (see [15]). Using these results and rescaling procedures we shall prove, in the present paper, the existence of center manifolds for the mild solutions of the equation (1.1) in the case of trichotomic linear parts under the same conditions on the nonlinear term $f(t,x)$ as in [15]. Moreover, using the same method we can also obtain the existence of unstable and center-unstable manifolds in the case of dichotomic and trichotomic linear parts (respectively) for the evolution equations defined on the whole line. Our main results are contained in Theorems 3.1, 4.1, 4.2, and Corollaries 4.1, 4.2, 4.3. We also illustrate our results in the Examples 5.2, 5.3.

We now recall some notions.

Definition 1.1. *Let $\mathbb{J}$ be one of the following intervals*: $\mathbb{R}_+$ *or* $\mathbb{R}$. *A family of operators* $\{U(t,s)\}_{t\geq s,\ t,s\in\mathbb{J}}$ *acting on a Banach space* X *is a* (*strongly continuous, exponential bounded*) *evolution family on* $\mathbb{J}$ *if*

(i) $U(t,t) = Id$ *and* $U(t,r)U(r,s) = U(t,s)$ *for all* $t \geq r \geq s$ *and* $t,s,r \in \mathbb{J}$,

(ii) *the map* $(t,s) \mapsto U(t,s)x$ *is continuous on* $\mathbb{J}$ *for every* $x \in X$,

(iii) $\|U(t,s)x\| \leqslant Ke^{\omega(t-s)}\|x\|$ *for all* $t \geq s$, $t,s,r \in \mathbb{J}$, *and* $x \in X$, *for some constants* K, ω.

The notion of an evolution family arises naturally from the theory of evolution equation which are well-posed. Meanwhile, if the abstract Cauchy problem

$$\begin{aligned} \frac{du(t)}{dt} &= A(t)u(t), \quad t \geq s, \quad t, \quad s \in \mathbb{J}, \\ u(s) &= x_s \in X, \end{aligned} \tag{1.2}$$

is well-posed, there exists an evolution family $(U(t,s))_{t\geq s,\ t,s\in\mathbb{J}}$ such that the solution of the problem (1.2) is given by $u(t) = U(t,s)u(s)$.

For more details on the notion and some problems focus on properties and applications of evolution family we refer the reader to Pazy [28], Henry [12], and Nagel and Nickel [9]. For a given evolution family, we have the following concept of an exponential trichotomy of evolution families on $\mathbb{J}$ as follows.

Definition 1.2. *Let $\mathbb{J}$ be one of the following intervals*: $\mathbb{R}_+$ *or* $\mathbb{R}$. *A given evolution family* $(U(t,s))_{t\geq s,\ t,s\in\mathbb{J}}$ *on* $\mathbb{J}$ *is said to have an exponential trichotomy on* $\mathbb{J}$ *if there are three families of projections* $(P_j(t))_{t\in\mathbb{J}}$, $j = 1,2,3$, *positive constants* N, α, β *with* $\alpha < \beta$ *such that the following conditions are satisfied*:

(i) $\sup_{t\in\mathbb{J}} \|P_j(t)\| < \infty,\ j = 1,2,3,$

(ii) $P_1(t) + P_2(t) + P_3(t) = Id$ *for all* $t \in \mathbb{J}$, *and* $P_j(t)P_i(t) = 0$ *for all* $j \neq i$,

(iii) $P_j(t)U(t,s) = U(t,s)P_j(s)$, *for all* $t \geq s \geq 0,\ j = 1,2,3,$

(iv) $U(t,s)|_{ImP_j(s)}$ *are homeomorphisms from* $\operatorname{Im} P_j(s)$ *onto* $\operatorname{Im} P_j(t)$ *for all* $t \geq s$, $t,s \in \mathbb{J}$, *and* $j = 2,3$, *respectively; also we denote the inverse of* $U(t,s)|_{ImP_2(s)}$ *by* $U(s,t)_|$ (*here* $s \leqslant t$),

(v) *the following estimates hold*:

$$\|U(t,s)P_1(s)x\| \leqslant Ne^{-\beta(t-s)}\|P_1(s)x\|,$$

$$\|U(s,t)_|P_2(t)x\| \leqslant Ne^{-\beta(t-s)}\|P_2(t)x\|,$$

$$\|U(t,s)P_3(s)x\| \leqslant Ne^{\alpha(t-s)}\|P_3(s)x\|,$$

$$\textit{for all}\ \ t \geq s, \quad t,s \in \mathbb{J}, \quad x \in X.$$

The evolution family is said to have an *exponential dichotomy on* $\mathbb{J}$ if it has an exponential trichotomy for which the family of projections $P_3(t)$ is trivial, i.e., $P_3(t) = 0$ for all $t \in \mathbb{J}$. In this case, we remark that the property (i) is a consequence of other properties (see [21], Lemma 4.2), and we also denote $P(t) := P_1(t)$ called dichotomy projections.

2. Function spaces and admissibility. We recall some notions of function spaces and admissibility. We refer the readers to Massera and Schäffer [20] (Chapter 2) for wide classes of function spaces that play a fundamental role throughout the study of differential equations in the case of bounded coefficients $A(t)$ (see also Räbiger and Schnaubelt [31] (§ 1) for some classes of admissible Banach function spaces of functions defined on the whole line $\mathbb{R}$).

Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}_+$. As already known, the set of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) that are integrable on every compact subinterval $J \subset \mathbb{R}_+$ becomes, with the topology of convergence in the mean on every such J, a locally convex topological vector space, which we denote by $L_{1,\mathrm{loc}}(\mathbb{R}_+)$. A set of seminorms defining the topology of $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is given by $p_n(f) := \displaystyle\int_{J_n} |f(t)|dt$, $n \in \mathbb{N}$, where $\{J_n\}_{n\in\mathbb{N}} = \{[n, n+1]\}_{n\in\mathbb{N}}$ is a countable set of abutting compact intervals whose union is $\mathbb{R}_+$. With this set of seminorms one can see (see [20], Chapter 2, § 20) that $L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is a Fréchet space.

Let V be a normed space (with norm $\|\cdot\|_V$) and W be a locally convex Hausdorff topological vector space. Then, we say that V is *stronger than* W if $V \subseteq W$ and the indentity map from V into W is continuous. The latter condition is equivalent to the fact that for each continuous seminorm π of W there exists a number $\beta_\pi > 0$ such that $\pi(x) \leqslant \beta_\pi \|x\|_V$ for all $x \in V$. We write $V \hookrightarrow W$ to indicate that V is stronger than W. If, in particular, W is also a normed space (with norm $\|\cdot\|_W$) then the relation $V \hookrightarrow W$ is equivalent to the fact that $V \subseteq W$ and there is a number $\alpha > 0$ such that $\|x\|_W \leqslant \alpha\|x\|_V$ for all $x \in V$ (see [20], Chapter 2 for detailed discussions on this matter).

We can now define Banach function spaces as follows.

Definition 2.1. *A vector space E of real-valued Borel-measurable functions on $\mathbb{R}_+$ (modulo λ-nullfunctions) is called a Banach function space (over $(\mathbb{R}_+, \mathcal{B}, \lambda)$) if*

(1) *E is Banach lattice with respect to a norm $\|\cdot\|_E$, i.e., $(E, \|\cdot\|_E)$ is a Banach space, and if $\varphi \in E$ and ψ is a real-valued Borel-measurable function such that $|\psi(\cdot)| \leqslant |\varphi(\cdot)|$ λ-a.e., then $\psi \in E$ and $\|\psi\|_E \leqslant \|\varphi\|_E$,*

(2) *the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure, and $\sup_{t\geq 0} \|\chi_{[t,t+1]}\|_E < \infty$ and $\inf_{t\geq 0} \|\chi_{[t,t+1]}\|_E > 0$,*

(3) *$E \hookrightarrow L_{1,\mathrm{loc}}(\mathbb{R}_+)$.*

For a Banach function space E we remark that the condition (3) in the above definition means that for each compact interval $J \subset \mathbb{R}_+$ there exists a number $\beta_J \geq 0$ such that $\displaystyle\int_J |f(t)|dt \leqslant \beta_J \|f\|_E$ for all $f \in E$.

We state the following trivial lemma which will be frequently used in our strategy.

Lemma 2.1. *Let E be a Banach function space. Let φ and ψ be real-valued, measurable functions on $\mathbb{R}_+$ such that they coincide with each other outside a compact interval and they are essentially bounded (in particular, continuous) on this compact interval. Then $\varphi \in E$ if and only if $\psi \in E$.*

We then define Banach spaces of vector-valued functions corresponding to Banach function spaces as follows.

Definition 2.2. *Let E be a Banach function space and X be a Banach space endowed with the norm $\|\cdot\|$. We set*

$$\mathcal{E} := \mathcal{E}(\mathbb{R}_+, X) := \Big\{ f \colon \mathbb{R}_+ \to X \colon f \ \text{ is strongly measurable and } \ \|f(\cdot)\| \in E \Big\}$$

(*modullo* λ*-nullfunctions*) *endowed with the norm*

$$\|f\|_{\mathcal{E}} := \| \|f(\cdot)\| \|_E.$$

One can easily see that $\mathcal{E}$ *is a Banach space. We call it the Banach space corresponding to the Banach function space* E.

We now introduce the notion of admissibility in the following definition.

Definition 2.3. *The Banach function space* E *is called admissible if it satisfies*

(i) *there is a constant* $M \geq 1$ *such that for every compact interval* $[a, b] \in \mathbb{R}_+$ *we have*

$$\int\limits_a^b |\varphi(t)| dt \leqslant \frac{M(b-a)}{\|\chi_{[a,b]}\|_E} \|\varphi\|_E \quad \text{for all} \quad \varphi \in E, \tag{2.1}$$

(ii) *for* $\varphi \in E$ *the function* $\Lambda_1 \varphi$ *defined by* $\Lambda_1 \varphi(t) := \int_t^{t+1} \varphi(\tau) d\tau$ *belongs to* E,

(iii) E *is* T_τ^+*-invariant and* T_τ^-*-invariant, where* T_τ^+ *and* T_τ^- *are defined, for* $\tau \in \mathbb{R}_+$, *by*

$$T_\tau^+ \varphi(t) := \begin{cases} \varphi(t-\tau) & \text{for } \ t \geq \tau \geq 0, \\ 0 & \text{for } \ 0 \leqslant t \leqslant \tau, \end{cases} \tag{2.2}$$

$$T_\tau^- \varphi(t) := \varphi(t+\tau) \quad \text{for} \quad t \geq 0.$$

Moreover, there are constants N_1, N_2 *such that* $\|T_\tau^+\| \leqslant N_1$, $\|T_\tau^-\| \leqslant N_2$ *for all* $\tau \in \mathbb{R}_+$.

***Example* 2.1.** Besides the spaces $L_p(\mathbb{R}_+)$, $1 \leqslant p \leqslant \infty$, and the space

$$\mathbf{M}(\mathbb{R}_+) := \left\{ f \in L_{1,\mathrm{loc}}(\mathbb{R}_+) \colon \sup_{t \geq 0} \int\limits_t^{t+1} |f(\tau)| d\tau < \infty \right\}$$

endowed with the norm $\|f\|_{\mathbf{M}} := \sup_{t\geq 0} \int_t^{t+1} |f(\tau)| d\tau$, many other function spaces occuring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}$, $1 < p < \infty$, $1 \leqslant q < \infty$ (see [6, p. 284], Theorem 3, [36]) and, more general, the class of rearrangement invariant function spaces over $(\mathbb{R}_+, \mathcal{B}, \lambda)$ (see [17]) are admissible.

***Remark* 2.1.** If E is an admissible Banach function space then $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$. Indeed, put $\beta := \inf_{t \geq 0} \|\chi_{[t,t+1]}\|_E > 0$ (by Definition 2.1 (2)). Then, from Definition 2.3 (i) we derive

$$\int\limits_t^{t+1} |\varphi(\tau)| d\tau \leqslant \frac{M}{\beta} \|\varphi\|_E \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \varphi \in E. \tag{2.3}$$

Therefore, if $\varphi \in E$ then $\varphi \in \mathbf{M}(\mathbb{R}_+)$ and $\|\varphi\|_{\mathbf{M}} \leqslant \dfrac{M}{\beta} \|\varphi\|_E$. We thus obtain $E \hookrightarrow \mathbf{M}(\mathbb{R}_+)$.

We now collect some properties of admissible Banach function spaces in the following proposition (see [14], Proposition 2.6 and originally in [20]).

Proposition 2.1. *Let E be an admissible Banach function space. Then the following assertions hold.*

(a) *Let $\varphi \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ such that $\varphi \geq 0$ and $\Lambda_1\varphi \in E$, where, Λ_1 is defined as in Definition* 2.3(ii). *For $\sigma > 0$ we define functions $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ by*

$$\Lambda'_\sigma\varphi(t) := \int\limits_0^t e^{-\sigma(t-s)}\varphi(s)ds,$$

$$\Lambda''_\sigma\varphi(t) := \int\limits_t^\infty e^{-\sigma(s-t)}\varphi(s)ds.$$

Then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ belong to E. In particular, if $\sup_{t\geq 0}\int_t^{t+1}\varphi(\tau)d\tau < \infty$ (this will be satisfied if $\varphi \in E$ (see Remark 2.1)) *then $\Lambda'_\sigma\varphi$ and $\Lambda''_\sigma\varphi$ are bounded. Moreover, denoted by $\|\cdot\|_\infty$ for* ess sup-*norm, we have*

$$\|\Lambda'_\sigma\varphi\|_\infty \leqslant \frac{N_1}{1-e^{-\sigma}}\|\Lambda_1 T_1^+\varphi\|_\infty \qquad \textit{and} \qquad \|\Lambda''_\sigma\varphi\|_\infty \leqslant \frac{N_2}{1-e^{-\sigma}}\|\Lambda_1\varphi\|_\infty \tag{2.4}$$

for operator T_1^+ and constants N_1, N_2 defined as in Definition 2.3.

(b) *E contains exponentially decaying functions $\psi(t) = e^{-\alpha t}$ for $t \geq 0$ and any fixed constant $\alpha > 0$.*

(c) *E does not contain exponentially growing functions $f(t) := e^{bt}$ for $t \geq 0$ and any fixed constant $b > 0$.*

***Remark* 2.2.** If we replace the half-line $\mathbb{R}_+$ by any infinite (or half-infinite) interval $\mathbb{I}$ (e.g., $\mathbb{I} = \mathbb{R}_-, \mathbb{R}$, or any $(-\infty, t_0]$ for fixed $t_0 \in \mathbb{R}$, etc.), then we have the similar notions of admissible spaces on the interval $\mathbb{I}$ with slight changes as follow:

(1) In Definition 2.3, the translations semigroups T_τ^+ and T_τ^- for $\tau \in \mathbb{R}_+$ should be replaced by T_τ^+ and T_τ^- defined for $\tau \in \mathbb{I}$ as

$$T_\tau^+\varphi(t) := \begin{cases} \varphi(t-\tau) & \text{for } t \text{ and } t-\tau \text{ belonging to } \mathbb{I}, \\ 0 & \text{for } t \in \mathbb{I} \text{ but } t-\tau \notin \mathbb{I}, \end{cases} \tag{2.5}$$

$$T_\tau^-\varphi(t) := \varphi(t+\tau) \quad \text{for} \quad t \in \mathbb{I}.$$

(2) In Proposition 2.1 (a), the functions Λ'_σ and Λ''_σ should be replaced by

$$\Lambda'_\sigma\varphi(t) := \int\limits_t^{t_0} e^{-\sigma|t-s|}\varphi(s)ds, \quad \text{here} \quad t_0 = \infty \quad \text{if} \quad \mathbb{I} = \mathbb{R}, \quad \text{and} \quad t_0 = 0 \quad \text{if} \quad \mathbb{I} = \mathbb{R}_-,$$

$$\Lambda''_\sigma\varphi(t) := \int\limits_{-\infty}^t e^{-\sigma|s-t|}\varphi(s)ds.$$

(3) In Proposition 2.1 (b) and (c) the functions $\psi(t) = e^{-\alpha t}$ ($t \geq 0$, and fixed $\alpha > 0$) should be replaced by $\psi(t) = e^{-\alpha|t|}$, $t \in \mathbb{I}$ and fixed $\alpha > 0$; and the functions $f(t) := e^{bt}$ for $t \geq 0$ and any fixed constant $b > 0$ should be replaced by $f(t) := e^{b|t|}$, $t \in \mathbb{I}$ and fixed $b > 0$.

These notions will be used in Section 5. We denote the admissible function space of the functions defined on $\mathbb{I}$ by $E_{\mathbb{I}}$. If $\mathbb{I} = \mathbb{R}_+$, we denote simply $E := E_{\mathbb{R}_+}$. For a function φ defined on the whole line we denote the restriction of φ on $\mathbb{I}$ by $\varphi|_{\mathbb{I}}$. It is obvious that, if the function $\varphi \in E_{\mathbb{R}}$, then $\varphi|_{\mathbb{I}} \in E_{\mathbb{I}}$.

In the case of infinite-dimensional phase spaces, instead of equation (1.1), for an evolution family $(U(t,s))_{t\geq s,\ t,s\in\mathbb{J}}$ where $\mathbb{J} = \mathbb{R}_+$or $\mathbb{R}$, we consider the integral equation

$$u(t) = U(t,s)u(s) + \int_s^t U(t,\xi)f(\xi,u(\xi))d\xi \quad \text{for a.e.} \quad t \geq s, \quad t,s \in \mathbb{J}. \tag{2.6}$$

We note that, if the evolution family $(U(t,s))_{t\geq s,\ t,s\in\mathbb{J}}$ arises from the well-posed Cauchy problem (1.2) then the function u, which satisfies (2.6) for some given function f, is called a mild solution of the inhomogeneous problem

$$\frac{du(t)}{dt} = A(t)u(t) + f(t,u(t)), \quad t \geq s, \quad t,s \in \mathbb{J},$$

$$u(s) = x_s \in X.$$

We refer the reader to Pazy [28] for more detailed treatment on the relations between classical and mild solutions of evolution equations (see also [9, 18, 35]).

To obtain the existence of an integral manifold for equation (2.6), beside the exponential dichotomy (or trichotomy) of the evolution family, we also need the properties of (local) φ-Lipschitz of the nonlinear term f in the following definitions in which we suppose as above that $\mathbb{J}$ is one of the infinite intervals $\mathbb{R}_+$ or $\mathbb{R}$. Also, we let $E_{\mathbb{J}}$ be an admissible Banach function space on $\mathbb{J}$. When $\mathbb{J} = \mathbb{R}_+$, we simply write E instead of $E_{\mathbb{R}_+}$.

Definition 2.4 (Local φ-Lipschitz functions)**.** *Let φ be a positive function belonging to $E_{\mathbb{J}}$, and B_ρ be the ball with radius ρ in X, i.e., $B_\rho := \{x \in X\colon \|x\| \leqslant \rho\}$. A function $f\colon \mathbb{J} \times B_\rho \to X$ is said to be local φ-Lipschitz of the class (M,φ,ρ) for some positive constants M, ρ if f satisfies*

(i) $\|f(t,x)\| \leqslant M\varphi(t)$ *for a.e.* $t \in \mathbb{J}$ *and all* $x \in B_\rho$,

(ii) $\|f(t,x_1) - f(t,x_2)\| \leqslant \varphi(t)\|x_1 - x_2\|$ *for a.e.* $t \in \mathbb{J}$ *and all* $x_1, x_2 \in B_\rho$.

Remark **2.3.** If $f(t,0) = 0$ then, the condition (ii) in the above definition already implies that f belongs to class (ρ,φ,ρ).

We next recall the definition of φ-Lipschitz functions.

Definition 2.5 (φ-Lipschitz functions)**.** *Let φ be a positive function belongs to $E_{\mathbb{J}}$. A function $f\colon \mathbb{J} \times X \to X$ is said to be φ-Lipschitz if f satisfies*

(i) $f(t,0) = 0$ *for a.e.* $t \in \mathbb{J}$,

(ii) $\|f(t,x_1) - f(t,x_2)\| \leqslant \varphi(t)\|x_1 - x_2\|$ *for a.e.* $t \in \mathbb{J}$ *and all* $x_1, x_2 \in X$.

3. Exponential trichotomy and center-stable manifolds on $\mathbb{R}_+$. In this section, we will generalize Theorem 4.7 in [15] to the case that the evolution family $(U(t,s))_{t\geq s\geq 0}$ has an exponential trichotomy on $\mathbb{R}_+$ and the nonlinear forcing term f is φ-Lipschitz.

In this case, the interval $\mathbb{J} = \mathbb{R}_+$. For an evolution family $(U(t,s))_{t\geq s\geq 0}$ we rewrite the integral equation (2.6) for $\mathbb{J} = \mathbb{R}_+$ as

$$u(t) = U(t,s)u(s) + \int\limits_s^t U(t,\xi)f(\xi,u(\xi))d\xi \quad \text{for a.e.} \quad t \geq s, \quad t,s \in \mathbb{R}_+. \tag{3.1}$$

Precisely, we will prove that there exists a center-stable manifold for the solutions of equation (3.1).

Theorem 3.1. *Let the evolution family $(U(t,s))_{t\geq s\geq 0}$ have an exponential trichotomy with the corresponding constants N, α, β $(\alpha < \beta)$, and projections $(P_j(t))_{t\geq 0}$, $j = 1,2,3$, given in Definition* 1.2. *Suppose that $f\colon \mathbb{R}_+ \times X \to X$ be φ-Lipschitz, where φ is the positive function belonging to E such that $k < \min\left\{\dfrac{1}{N+1}, \dfrac{1-e^{\alpha-\beta}}{1-e^{-\beta}}\right\}$, here k is defined by*

$$k := \frac{(1+H)N}{1-e^{-\beta}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big). \tag{3.2}$$

Then there exists a center-stable manifold $\mathbf{C} = \big\{(t,\mathbf{C}_t) \mid t \in \mathbb{R}_+ \text{ and } \mathbf{C}_t \subset X\big\}$ for the solutions of equation (3.1), *with the family $(\mathbf{C}_t)_{t\geq 0}$ being the graphs of the family of Lipschitz continuous mappings $(g_t)_{t\geq 0}$ (i.e., $\mathbf{C}_t := \operatorname{graph}(g_t) = \big\{x + g_t x \mid x \in \operatorname{Im}(P_1(t)+P_3(t))\big\}$ for each $t \geq 0$) where $g_t\colon \operatorname{Im}(P_1(t)+P_3(t)) \to \operatorname{Im} P_2(t)$ has the Lipschitz constant $\dfrac{Nk}{1-k}$ independent of t, such that the following properties hold*:

(i) *to each $x_0 \in \mathbf{C}_{t_0}$ there corresponds one and only one solution $u(t)$ of equation* (3.1) *on $[t_0,\infty)$ and it satisfies $u(t_0) = x_0$ and $\operatorname{ess\,sup}_{t\geq t_0}\|e^{-\gamma t}u(t)\| < \infty$, where $\gamma := \dfrac{\alpha+\beta}{2}$,*

(ii) *$\mathbf{C}_t$ is homeomorphism to $X_1(t)\oplus X_3(t)$ for all $t \geq 0$, where $X_1(t) = P_1(t)X$, $X_3(t) = P_3(t)X$,*

(iii) *$\mathbf{C}$ is invariant under the equation* (3.1) *in the sense that, if $u(t)$ is the solution of equation* (3.1) *satisfying $u(t_0) = x_0 \in \mathbf{C}_{t_0}$ and $\operatorname{ess\,sup}_{t\geq t_0}\|e^{-\gamma t}u(t)\| < \infty$, then $u(s) \in \mathbf{C}_s$ for all $s \geq t_0$,*

(iv) *every two solutions $u_1(t)$, $u_2(t)$ on the center-stable manifold $\mathbf{C}$ satisfy the condition that there exist positive constants μ and C_μ independent of $t_0 \geq 0$ such that*

$$\|x(t) - y(t)\| \leqslant C_\mu e^{(\gamma-\mu)(t-t_0)}\big\|(P_1(t_0)+P_3(t_0))x(t_0) - (P_1(t_0)+P_3(t_0))y(t_0)\big\|$$

for all $t \geq t_0$.

Proof. Set $P(t) := P_1(t) + P_3(t)$ and $Q(t) := P_2(t) = I - P(t)$. We consider the following rescaling evolution family:

$$\tilde{U}(t,s)x := e^{-\gamma(t-s)}U(t,s)x \quad \text{for all} \quad t \geq s \geq 0, \quad x \in X,$$

where $\gamma := \dfrac{\alpha+\beta}{2}$.

It is easy to check that $(\tilde{U}(t,s))_{t\geq s\geq 0}$ is an evolution family on *X*.

We now claim that $(\tilde{U}(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with the projection $P(t)$ and $Q(t)$ on the half-line. Infact, it suffices to verify the estimates in Definition 1.2.

By the definition of exponential trichotomy we have

$$\|\tilde{U}(s,t)Q(t)x\| \leqslant Ne^{-(\beta-\gamma)(t-s)}\|Q(t)x\| = Ne^{-\frac{(\beta-\alpha)}{2}(t-s)}\|Q(t)x\|$$

$$\text{for all} \quad t \geq s \geq 0, \quad x \in X.$$

On the other hand,

$$\|\tilde{U}(t,s)P(s)x\| = e^{-\gamma(t-s)}\|U(t,s)[P_1(s)+P_3(s)]x\| \leqslant$$

$$\leqslant Ne^{-(\gamma+\alpha)(t-s)}\|P_1(s)x\| + e^{-(\gamma-\alpha)(t-s)}\|P_3(s)x\| \leqslant$$

$$\leqslant Ne^{-\frac{(\beta-\alpha)}{2}(t-s)}(\|P_1(s)x\| + \|P_3(s)x\|) =$$

$$= Ne^{-\frac{(\beta-\alpha)}{2}(t-s)}\big(\|P_1(s)(P_1(s)+P_3(s))x\| + \|P_3(s)(P_1(s)+P_3(s))x\|\big) \leqslant$$

$$\leqslant NHe^{-\frac{(\beta-\alpha)}{2}(t-s)}\big(\|(P_1(s)+P_3(s))x\| + \|(P_1(s)+P_3(s))x\|\big) =$$

$$= 2NHe^{-\frac{(\beta-\alpha)}{2}(t-s)}\|P(s)x\| \quad \text{for all} \quad t \geq s \geq 0, \quad x \in X$$

(here we use the fact that $H := \sup_{t\geq 0}\{\|P_1(t)\|, \|P_2(t)\|, \|P_3(t)\|\} < \infty$).

We finally obtain the following estimate:

$$\|\tilde{U}(t,s)P(s)x\| \leqslant 2NHe^{-\frac{(\beta-\alpha)}{2}(t-s)}\|P(s)x\| \quad \text{for all} \quad t \geq s \geq 0, \quad x \in X.$$

Therefore, $(\tilde{U}(t,s))_{t\geq s\geq 0}$ has an exponential dichotomy with the projections $(P(t))_{t\geq 0}$ and the dichotomy constants $N' := \max\{N, 2NH\}$, $\beta' = \dfrac{\beta-\alpha}{2} > 0$.

Put $\tilde{x}(t) := e^{-\gamma t}x(t)$, and define the mapping F as follows:

$$F\colon \mathbb{R}_+ \times X \to X,$$

$$F(t,x) = e^{-\gamma t}f(t, e^{\gamma t}x) \quad \text{for all} \quad t \geq 0, \quad x \in X.$$

We can easily verify that the operator F is also φ-Lipschitz. Thus, we can rewrite the equation (3.1) in the new form

$$\tilde{x}(t) = \tilde{U}(t,s)\tilde{x}(s) + \int_s^t \tilde{U}(t,\xi)F(\xi,\tilde{x}(\xi))d\xi \quad \text{for a.e.} \quad t \geq s \geq 0. \tag{3.3}$$

Hence, by [15] (Theorem 4.7), we obtain that, if

$$k = \frac{(1+H)N(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty)}{1-e^{-\beta}} < \frac{1}{1+N},$$

then there exists an invariant stable manifold $\mathbf{C}$ for the solutions of equation (3.3). Return to equation (3.1) by using the relation $x(t) := e^{\gamma t}\tilde{x}(t)$ we can easily verify the properties of $\mathbf{C}$ which are stated in (i), (ii), (iii), and (iv). Thus, $\mathbf{C}$ is an invariant center-stable manifold for the solutions of equation (3.1).

Theorem 3.1 is proved.

***Remark* 3.1.** In case the evolution family has an exponential trichotomy and the nonlinear term f satisfies the local φ-Lipschitz of the class (M,φ,ρ) with $f(t,0)=0$ and the positive function $\varphi\in E$ satisfying $k<\min\left\{\dfrac{\rho}{2M},\dfrac{1}{N+1},\dfrac{1-e^{\alpha-\beta}}{1-e^{-\beta}}\right\}$ (here k is defined as in (3.2)), then by the similar ways as above and using the results in [15] (Theorem 3.8) we can obtain the existence of a *local center-stable manifold* for the solutions of equation (3.1), that is a set $\mathbf{C}\subset\mathbb{R}_+\times X$ such that there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous mappings

$$g_t\colon\ B_{\rho_0}\cap\operatorname{Im}(P_1(t)+P_3(t))\to B_{\rho_1}\cap\operatorname{Im}P_2(t),\quad t\in\mathbb{R}_+,$$

with Lipschitz constants independent of t satisfying:

(i) $\mathbf{C}=\{(t,x+g_t(x))\in\mathbb{R}_+\times\big(\operatorname{Im}(P_1(t)+P_3(t))\oplus\operatorname{Im}P_2(t)\big)\mid t\in\mathbb{R}_+,\,x\in B_{\rho_0}\cap\operatorname{Im}(P_1(t)+$ $+\,P_3(t))\}$, and we denote by $\mathbf{C}_t:=\{x+g_t(x)\mid(t,x+g_t(x))\in\mathbf{C}\}$,

(ii) $\mathbf{C}_t$ is homeomorphic to $B_{\rho_0}\cap\operatorname{Im}(P_1(t)+P_3(t))=\{x\in\operatorname{Im}(P_1(t)+P_3(t))\mid\|x\|\leqslant\rho_0\}$ for all $t\geq 0$,

(iii) to each $x_0\in\mathbf{C}_{t_0}$ there corresponds one and only one solution $u(t)$ of equation (3.1) on $[t_0,\infty)$ and it satisfies $u(t_0)=x_0$ and $\operatorname{ess\,sup}_{t\geq t_0}\|e^{-\gamma t}u(t)\|<\infty$, where $\gamma:=\dfrac{\alpha+\beta}{2}$,

(iv) every two solutions $u_1(t)$, $u_2(t)$ on the local center-stable manifold $\mathbf{C}$ satisfy the condition that there exist positive constants μ and C_μ independent of $t_0\geq 0$ such that

$$\|x(t)-y(t)\|\leqslant C_\mu e^{(\gamma-\mu)(t-t_0)}\big\|(P_1(t_0)+P_3(t_0))x(t_0)-(P_1(t_0)+P_3(t_0))y(t_0)\big\| \tag{3.4}$$

for all $t\geq t_0$.

4. Unstable manifolds for equations defined on the whole line. We now consider the case that the evolution family $(U(t,s))_{t\geq s}$ and the nonlinear forcing term f are defined on the whole line (i.e., the case $\mathbb{J}=\mathbb{R}$). That is to say, we will consider the integral equation

$$x(t)=U(t,s)x(s)+\int_s^t U(t,\xi)f(\xi,x(\xi))d\xi\quad\text{for a.e.}\quad t\geq s,\quad t,s\in\mathbb{R}. \tag{4.1}$$

As in Section 1, the solutions of the equation (4.1) is called the mild solutions of the equation

$$\frac{dx}{dt}=A(t)x+f(t,x),\quad t\in\mathbb{R},\quad x\in X, \tag{4.2}$$

where $A(t)$, $t\in\mathbb{R}$ (in general case), are unbounded operators in X, which are coefficients of a well-posed Cauchy problem

$$\frac{du(t)}{dt}=A(t)u(t),\quad t\geq s,$$

$$u(s)=x_s\in X,$$

whose solutions are given by $x(t)=U(t,s)x(s)$ as mentioned in Section 1. In this case, the existences of (local- or invariant-) stable manifolds on $\mathbb{R}$ are defined and proved by the same way as in the case of equations defined on a half-line $\mathbb{R}_+$ (see [15], Theorem 4.7). Therefore, we will pay our attention to the case of the unstable manifolds which are defined below.

4.1. Local-unstable manifolds on $\mathbb{R}$. We shall prove the existence of the local-unstable manifold under the conditions that the evolution family $(U(t,s))_{t\geq s}$ has an exponential dichotomy and the nonlinear term f is local φ-Lipschitz of the class (M,φ,ρ) for a relevant positive function $\varphi \in E_{\mathbb{R}}$.

We now give the description of a local-unstable manifold for the solutions of the integral equation (4.1) in the following definition in which we remind that by B_r we denote the ball in X with radius r centered at 0, i.e., $B_r = \{x \in X \mid \|x\| \leqslant r\}$.

Definition 4.1. *A set* $\mathbf{U} \subset \mathbb{R}\times X$ *is said to be a local-unstable manifold for the solutions of equation* (4.1) *if for every* $t\in\mathbb{R}$ *the phase spaces* X *splits into a direct sum* $X = X_1(t)\oplus X_2(t)$ *such that*

$$\inf_{t\in\mathbb{R}_+} Sn(X_1(t),X_2(t)) := \inf_{t\in\mathbb{R}_+} \inf\left\{\|x_1+x_2\|\colon x_i\in X_i(t),\ \ \|x_i\|=1,\ \ i=1,2\right\}>0,$$

and if there exist positive constants ρ, ρ_0, ρ_1 *and a family of Lipschitz continuous mappings*

$$h_t\colon\ B_{\rho_0}\cap X_2(t)\to B_{\rho_1}\cap X_1(t),\quad t\in\mathbb{R},$$

with the Lipschitz constants independent of t *such that*

(i) $\mathbf{U} = \{(t, x+h_t(x)) \in \mathbb{R}\times (X_2(t)\oplus X_1(t)) \mid x\in B_{\rho_0}\cap X_2(t)\}$, *and we denote by* $\mathbf{U}_t :=$ $:= \{x + h_t(x) \mid (t, x+h_t(x)) \in \mathbf{U}\}$,

(ii) $\mathbf{U}_t$ *is homeomorphic to* $B_{\rho_0}\cap X_2(t)$ *for all* $t\in\mathbb{R}$,

(iii) *to each* $x_0 \in \mathbf{U}_{t_0}$ *there corresponds one and only one solution* $x(t)$ *of equation* (4.1) *satisfying the conditions* $x(t_0)=x_0$ *and* $\operatorname{ess\,sup}_{t\leqslant t_0}\|x(t)\|\leqslant\rho$.

Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projection $P(t)$, $t\in\mathbb{R}$, and the dichotomy constants N, $\beta>0$. Then, we can define the Green's function as follows:

$$G(t,\tau) := \begin{cases} P(t)U(t,\tau) & \text{for } \ t\geq\tau, \\ -U(t,\tau)_|[I-P(\tau)] & \text{for } \ t<\tau. \end{cases} \tag{4.3}$$

Thus, we have

$$\|G(t,\tau)\|\leqslant (1+H)Ne^{-\beta|t-\tau|} \quad \text{for all} \quad t\neq\tau, \quad \text{where} \quad H=\sup_{t\in\mathbb{R}}\|P(t)\|<\infty.$$

We now prove the existence of a local-unstable manifold. To do that, we first construct the form of the solutions of the equation (4.1) which are bounded on the half-line $(-\infty,t_0]$. We denote by $\|\cdot\|_\infty$ the sup-norm on the half-line $(-\infty,t_0]$.

Lemma 4.1. *Let the evolution family* $(U(t,s))_{t\geq s}$ *have an exponential dichotomy with the corresponding projections* $P(t)$, $t\in\mathbb{R}$, *and the dichotomy constants* N, $\beta>0$. *Suppose that* φ *is the positive function which belongs to* $E_{\mathbb{R}}$. *Let* $f\colon\mathbb{R}\times B_\rho\to X$ *be local* φ*-Lipschitz of the class* (M,φ,ρ) *for some positive constants* M, ρ. *Let* $x(t)$ *be a solution of* (4.1) *such that* $\operatorname{ess\,sup}_{t\leqslant t_0}\|x(t)\|\leqslant\rho$ *for some fixed* t_0. *Then, for* $t\leqslant t_0$, *we have that* $x(t)$ *can be rewritten in the form*

$$x(t) = U(t,t_0)_|v + \int_{-\infty}^{t_0} G(t,\tau)f(\tau,x(\tau))d\tau \quad \textit{for all} \quad t\leqslant t_0, \tag{4.4}$$

and some $v\in X_2(t_0) = (I-P(t_0))X$, *where* $G(t,\tau)$ *is the Green's function defined above.*

Proof. Let

$$y(t) := \int\limits_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau)) d\tau \quad \text{for all} \quad t \leqslant t_0. \tag{4.5}$$

Then the function $y(\cdot)$ is bounded. Indeed, by the estimates of the Green's function G and the function f we have

$$\|y(\cdot)\|_\infty \leqslant \int\limits_{-\infty}^{t_0} (1+H) N e^{-\beta|t-\tau|} \|f(\tau, x(\tau))\| d\tau \leqslant$$

$$\leqslant (1+H)NM \left[\int\limits_{-\infty}^{t} e^{-\beta(t-\tau)} \|\varphi(\tau)\| d\tau + \int\limits_{t}^{t_0} e^{\beta(t-\tau)} \|\varphi(\tau)\| d\tau \right] \leqslant$$

$$\leqslant (1+H)NM \left[\frac{N_1 \|\Lambda_1 \varphi\|_\infty + N_2 \|\Lambda_1 T_1^+ \varphi\|_\infty}{1 - e^{-\beta}} \right] < \infty.$$

Next, by computing directly we verify that $y(\cdot)$ satisfies the integral equation

$$y(t_0) = U(t_0, t) y(t) + \int\limits_{t}^{t_0} U(t_0, \tau) f(\tau, x(\tau)) d\tau \quad \text{for all} \quad t \leqslant t_0. \tag{4.6}$$

Indeed, subtituting y from (4.5) to the right-hand side of (4.6) we obtain

$$U(t_0, t) y(t) + \int\limits_{t}^{t_0} U(t_0, \tau) f(\tau, x(\tau)) d\tau =$$

$$= U(t_0, t) \int\limits_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau)) d\tau + \int\limits_{t}^{t_0} U(t_0, \tau) f(\tau, x(\tau)) d\tau =$$

$$= U(t_0, t) \int\limits_{-\infty}^{t} U(t,\tau) P(\tau) f(\tau, x(\tau)) d\tau -$$

$$- U(t_0, t) \int\limits_{t}^{t_0} U(t,\tau)_| (I - P(\tau)) f(\tau, x(\tau)) d\tau + \int\limits_{t}^{t_0} U(t_0, \tau) f(\tau, x(\tau)) d\tau =$$

$$= \int\limits_{-\infty}^{t} U(t_0, \tau) P(\tau) f(\tau, x(\tau)) d\tau -$$

$$-\int_t^{t_0} U(t_0,t)U(t,\tau)_|(I-P(\tau))f(\tau,x(\tau))d\tau + \int_t^{t_0} U(t_0,\tau)f(\tau,x(\tau))d\tau =$$

$$= \int_{-\infty}^{t_0} U(t_0,\tau)P(\tau)f(\tau,x(\tau))d\tau = \int_{-\infty}^{t_0} G(t_0,\tau)f(\tau,x(\tau)) = y(t_0),$$

here we use the fact $U(t_0,t)U(t,\tau)_|(I-P(\tau)) = U(t_0,\tau)(I-P(\tau))$ that for all $t \leqslant \tau \leqslant t_0$.

Thus, we have

$$y(t_0) = U(t_0,t)y(t) + \int_t^{t_0} U(t_0,\tau)f(\tau,x(\tau))d\tau.$$

On the other hand,

$$x(t_0) = U(t_0,t)x(t) + \int_t^{t_0} U(t_0,\tau)f(\tau,x(\tau))d\tau.$$

Then $x(t_0) - y(t_0) = U(t_0,t)[x(t)-y(t)]$. We need to prove that $x(t_0)-y(t_0) \in (I-P(t_0))X$.

Applying the operator $P(t_0)$ to the expression $x(t_0)-y(t_0) = U(t_0,t)[x(t)-y(t)]$, we have

$$\|P(t_0)[x(t_0)-y(t_0)]\| = \|U(t_0,t)P(t)[x(t)-y(t)]\| \leqslant Ne^{-\beta(t_0-t)}\|P(t)\|.\|x(t)-y(t)\|.$$

Since $\sup_{t\in\mathbb{R}}\|P(t)\| < \infty$ and $\|x(t)-y(t)\| \leqslant \|x(\cdot)\|_\infty + \|y(\cdot)\|_\infty < \infty$, letting $t \to -\infty$ we obtain that

$$\|P(t_0)[x(t_0)-y(t_0)]\| = 0.$$

It means that, $v := x(t_0)-y(t_0) \in (I-P(t_0))X = X_2(t_0)$ finishing the proof.

***Remark* 4.1.** By computing directly, we can see that the converse of Lemma 4.1 is also true. It means, all solutions of equation (4.4) satisfied the equation (4.1) for $t \leqslant t_0$.

Lemma 4.2. *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\beta > 0$. Suppose that φ is the positive function which belongs to E. Put*

$$k := \frac{(1+H)N}{1-e^{-\beta}}\big[N_1\|\Lambda_1\varphi\|_\infty + N_2\|\Lambda_1 T_1^+\varphi\|_\infty\big]. \tag{4.7}$$

Let $f\colon \mathbb{R}\times B_\rho \to X$ be local φ-Lipschitz of the class (M,φ,ρ) such that $k < \min\left\{1, \dfrac{\rho}{2M}\right\}$. Then there corresponds to each $v \in B_{\rho/2N} \cap X_2(t_0)$ one and only one solution $x(t)$ of the equation (4.1) *on $(-\infty,t_0]$ satisfying the conditions that $(I-P(t_0))x(t_0) = v$ and* $\operatorname{ess\,sup}_{t\leqslant t_0}\|x(t)\| \leqslant \rho$.

Proof. We consider in the space $L_\infty((-\infty,t_0],X)$ the ball

$$\mathcal{B}_\rho := \left\{x(\cdot) \in L_\infty((-\infty,t_0],X)\colon \|x(\cdot)\|_\infty := \operatorname*{ess\,sup}_{t\leqslant t_0}\|x(t)\| \leqslant \rho\right\}.$$

For $v \in B_{\rho/2N} \cap X_2(t_0)$ we will prove the transformation T defined by

$$(Tx)(t) = U(t,t_0)_{|}v + \int\limits_{-\infty}^{t_0} G(t,\tau)f(\tau,x(\tau))d\tau \quad \text{for all} \quad t \leqslant t_0,$$

acts from $\mathcal{B}_\rho$ into $\mathcal{B}_\rho$ and is a contraction. In fact, for $x(\cdot) \in \mathcal{B}_\rho$ we have that $\|f(t,x(t))\| \leqslant M\varphi(t)$. Therefore, putting

$$y(t) = U(t,t_0)_{|}v + \int\limits_{-\infty}^{t_0} G(t,\tau)f(\tau,x(\tau))d\tau \quad \text{for all} \quad t \leqslant t_0,$$

we obtain that $\|y(t)\| \leqslant Ne^{-\beta(t_0-t)}\|v\| + (1+H)NM\int_{-\infty}^{t_0} e^{-\beta|t-\tau|}\varphi(\tau)d\tau$. It follows from the admissibility of L_∞ that, $y(\cdot) \in L_\infty$ and

$$\|y(\cdot)\|_\infty \leqslant N\|v\| + \frac{(1+H)NM}{1-e^{-\beta}}\big(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty\big).$$

Using now the fact that $\|v\| \leqslant \dfrac{\rho}{2N}$ and

$$\frac{(1+H)N}{1-e^{-\beta}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) < \frac{\rho}{2M},$$

we have that $\|y(\cdot)\|_\infty \leqslant \rho$. Therefore, the transformation T acts from $\mathcal{B}_\rho$ to $\mathcal{B}_\rho$.

It follows from the estimates of G and $U(t,s)$ that

$$\|T(x) - T(y)\|_\infty \leqslant$$

$$\leqslant \frac{(1+H)N\|x(\cdot)-y(\cdot)\|_\infty}{1-e^{-\beta}}\big[N_1\|\Lambda_1\varphi\|_\infty + N_2\|\Lambda_1 T_1^+\varphi\|_\infty\big] = k\|x(\cdot)-y(\cdot)\|_\infty.$$

Since $k < 1$, we obtain that T is a contraction. By the Banach contraction mapping theorem, the lemma follows.

From Lemmas 4.1, 4.2 and using the same arguments as in [15] (Theorem 3.8) we obtain the existence of an unstable manifold in the following theorem.

Theorem 4.1. *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\beta > 0$. Then, for any $\rho > 0$ and $M > 0$, we have that, if f is local φ-Lipschitz of the class (M,φ,ρ) with the positive function $\varphi \in E_\mathbb{R}$ such that $k < \min\left\{\dfrac{\rho}{2M}, \dfrac{1}{N+1}\right\}$, here k is defined as in* 4.7, *there exists a local unstable manifold for the solutions of equation* (4.1). *Moreover, for any two solution $x_1(\cdot)$ and $x_2(\cdot)$ belonging to this manifold we have*

$$\big\|x_1(t) - x_2(t)\big\| \leqslant Ce^{\mu(t-t_0)}\big\|(I-P(t_0))x_1(t_0) - (I-P(t_0))x_2(t_0)\big\| \quad \textit{for all} \quad t \leqslant t_0, \tag{4.8}$$

where C, μ be the positive constants independent of t_0, $x_1(\cdot)$ and $x_2(\cdot)$.

Proof. The proof of this theorem can be done by the same way as in [15] (Theorem 3.8) replacing $\mathbb{R}_+$ by $\mathbb{R}$ and using the structures of bounded solutions as in Lemmas 4.1, 4.2. We just note that the family of Lipschitz mappings $(h_t)_{t\in\mathbb{R}}$ determining the local-unstable manifold is define by

$$h_t\colon\ B_{\rho/2N}\cap X_2(t)\to B_{\rho/2}\cap X_1(),$$

$$h_t(y)=\int\limits_{-\infty}^{t} G(t,s)f(s,x(s))ds$$

for $y\in B_{\rho/2N}\cap X_2(t)$, where $x(\cdot)$ is the unique solution in $L_\infty((-\infty,t],X)$ of equation (4.1) on $(-\infty,t]$ satisfying $(I-P(t))x(t)=y$ (note that the existence and uniqueness of $x(\cdot)$ is obtained in Lemma 4.2). Furthermore, the Lipschitz constant of h_t is $\dfrac{kN}{1-k}<1$ which is the same as that of g_t determining the local-stable manifold (see [15], Theorem 3.8).

Theorem 4.1 is proved.

From the existence of the local-stable and local-unstable manifolds of equation (4.1) defined on the whole line we have the following important corollary which describes the geometric picture of solutions to equation (4.1).

***Corollary* 4.1.** *Let the evolution family $(U(t,s))_{t\ge s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t\in\mathbb{R}$, and the dichotomy constants N, $\beta>0$. Then, for any $\rho>0$ and $M>0$, we have that, if f is local φ-Lipschitz of the class (M,φ,ρ) with the positive function $\varphi\in E_{\mathbb{R}}$ such that $k<\min\left\{\dfrac{\rho}{2M},\dfrac{1}{N+1},\dfrac{\rho}{2MN}\right\}$, here k is defined as in* (4.7), *then there exist a local-stable manifold $\mathbf{S}$ and a local-unstable manifold $\mathbf{U}$ for the solutions of equation* (4.1) *having the following properties*:

(a) *for each t_0 the intersection $\mathbf{S}_{t_0}\cap\mathbf{U}_{t_0}$ contains the unique element z_{t_0}*,

(b) *the solution $u_0(t)$ of equation* (4.1) *with initial condition $u_0(t_0)=z_{t_0}$ is bounded on the whole line $\mathbb{R}$*,

(c) *the solutions $u(t)$ of equation* (4.1) *satisfying $u(t_0)\in\mathbf{S}_{t_0}$ exponentially approach $u_0(t)$ as $t\to\infty$*,

(d) *the solutions $u(t)$ of equation* (4.1) *satisfying $u_0(t)\in\mathbf{U}_{t_0}$ exponentially approach $u_0(t)$ as $t\to-\infty$.*

Proof. (a) The condition that $x\in\mathbf{S}_{t_0}\cap\mathbf{U}_{t_0}$ is equivalent to the fact that there are $w\in B_{\rho_0}\cap\cap X_1(t_0)$ and $y\in B_{\rho_0}\cap X_2(t_0)$ such that $x=w+g_{t_0}w=h_{t_0}y+y$ where g_{t_0} and h_{t_0} are members of the families of Lipschitz continuous mappings $(g_t)_{t\in\mathbb{R}}$ determining $\mathbf{S}$ and $(h_t)_{t\in\mathbb{R}}$ determining $\mathbf{U}$, respectively. Then $w-h_{t_0}y=y-g_{t_0}w\in X_1(t_0)\cap X_2(t_0)=\{0\}$. This follows that $w=h_{t_0}y$ and $y=g_{t_0}w$. Therefore, $w=h_{t_0}(g_{t_0}w)=(h_{t_0}\circ g_{t_0})w$. We now estimate $g_{t_0}w$ for $w\in B_{\rho_0}\cap X_1(t_0)$ by using the formula (see [15], equation (18))

$$g_{t_0}(w)=\int\limits_{t_0}^{\infty} G(t_0,s)f(s,x(s))ds, \tag{4.9}$$

where $w\in B_{\rho/2N}\cap X_1(t_0)$ and $x(\cdot)$ is the unique solution in $\mathcal{B}_\rho$ of equation (4.1) on $[t_0,\infty)$ satisfying $P(t_0)x(t_0)=w$ (note that the existence and uniqueness of $x(\cdot)$ is obtained in [15] (Theorem 3.7)).

By (4.9) we have that

$$\|g_{t_0}(w)\| \leqslant \int\limits_0^\infty \|\mathcal{G}(t_0,s)\|\|f(s,x(s))\|ds \leqslant (1+H)NM\int\limits_0^\infty e^{-|t_0-s|}\varphi(s)ds \leqslant$$

$$\leqslant \frac{(1+H)NM}{1-e^{-\beta}}(N_1\|\Lambda_1 T_1^+\varphi\|_\infty + N_2\|\Lambda_1\varphi\|_\infty) = kM < \frac{\rho}{2N} \quad \left(\text{since } \ k < \frac{\rho}{2MN}\right).$$

Therefore, we obtain that $g_{t_0}\colon B_{\rho/2N}\cap X_1(t_0)\to B_{\rho/2N}\cap X_2(t_0)$. Similarly, we have $h_{t_0}\colon B_{\rho/2N}\cap \cap X_2(t_0)\to B_{\rho/2N}\cap X_1(t_0)$. This follows that

$$h_{t_0}\circ g_{t_0}\colon\ B_{\rho/2N}\cap X_1(t_0)\to B_{\rho/2N}\cap X_1(t_0).$$

Since the mappings g_{t_0} and h_{t_0} are both Lipschitz continuous with the same Lipschitz constant $\frac{kN}{1-k} < 1$ (see the proof of [15] (Theorem 3.8)), we obtain that $h_{t_0}\circ g_{t_0}$ is a contraction. Thus, there exists a unique w_0 such that $w_0 = (h_{t_0}\circ g_{t_0})w_0$. Putting $z_{t_0} = w_0 + g_{t_0}w_0$ we obtain that z_{t_0} is the unique element of the intersection $\mathbf{S}_{t_0}\cap \mathbf{U}_{t_0}$.

The property (b) follows from the definitions of the local-stable and local-unstable manifolds, respectively.

The properties (c) and (d) are consequences of the inequalities in [15] (Theorem 3.7, ineq. (13)) and (4.8), respectively.

4.2. Invariant unstable manifolds on $\mathbb{R}$. In this subsection we consider the existence of the invariant unstable manifold under the conditions that the evolution family has an exponential dichotomy, and the nonlinear term f is φ-Lipschitz continuous.

We now give the definition of an invariant unstable manifold for the solutions of the integral equation (4.1).

Definition 4.2. *A set $\mathbf{S}\subset\mathbb{R}\times X$ is said to be an invariant unstable manifold for the solutions of equation* (4.1) *if for every $t\in\mathbb{R}$ the phase spaces X splits into a direct sum $X = X_1(t)\oplus X_2(t)$ such that*

$$\inf_{t\in\mathbb{R}_+} Sn(X_1(t),X_2(t)) := \inf_{t\in\mathbb{R}_+}\inf\big\{\|x_1+x_2\|\colon x_i\in X_i(t),\ \ \|x_i\| = 1,\ \ i=1,2\big\} > 0,$$

and if there exists a family of Lipschitz continuous mappings

$$g_t\colon\ X_2(t)\to X_1(t),\quad t\in\mathbb{R},$$

with the Lipschitz constants independent of t such that

(i) $\mathbf{S} = \{(t, x+g_t(x))\in\mathbb{R}\times(X_2(t)\oplus X_1(t)) \mid x\in X_2(t)\}$, *and we denote by* $\mathbf{S}_t := \{x+g_t(x) \mid (t,x+g_t(x))\in\mathbf{S}\}$,

(ii) $\mathbf{S}_t$ *is homeomorphic to $X_2(t)$ for all $t\in\mathbb{R}$,*

(iii) *to each $x_0\in\mathbf{S}_{t_0}$ there corresponds one and only one solution $x(t)$ of equation* (4.1) *satisfying the conditions $x(t_0) = x_0$ and* $\operatorname{ess\,sup}_{t\leqslant t_0}\|x(t)\| < \infty$,

(iv) $\mathbf{S}$ *is invariant under the equation* (4.1) *in the sense that, if $x(\cdot)$ is a solution of equation* (4.1) *satisfying $x(t_0)\in\mathbf{S}_{t_0}$ and* $\operatorname{ess\,sup}_{t\leqslant t_0}\|x(t)\| < \infty$, *then $x(t)\in\mathbf{S}_t$ for all $t\leqslant t_0$.*

As in the previous subsection, we can construct the form of the solutions of equation (4.1) which are bounded on the half-line $(-\infty, t_0]$ in the following lemma whose proof can be done by the same way as in Lemma 4.1.

Lemma 4.3. *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\beta > 0$. Suppose that φ is the positive function which belongs to $E_{\mathbb{R}}$. Let $f\colon \mathbb{R}\times X \to X$ be φ-Lipschitz. Let $x(t)$ be a solution of* (4.1) *such that* $\operatorname{ess\,sup}_{t\leqslant t_0} \|x(t)\| < \infty$ *for some fixed t_0. Then, for $t \leqslant t_0$, we have that $x(t)$ can be rewritten in the form*

$$x(t) = U(t,t_0)_{|}v + \int_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau))d\tau \quad \textit{for all} \quad t \leqslant t_0, \tag{4.10}$$

and some $v \in X_2(t_0) = (I - P(t_0))X$, where $G(t,\tau)$ is the Green's function defined above.

***Remark* 4.2.** By computing directly, we can see that the converse of Lemma 4.3 is also true. It means, all solutions of equation (4.10) satisfied the equation (4.1) for $t \leqslant t_0$.

Similarly to Lemma 4.2 we have the following lemma which describes the existence and uniqueness of certain bounded solutions.

Lemma 4.4. *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\beta > 0$. Suppose that φ is the positive function which belongs to E. Let $f\colon \mathbb{R}\times X \to X$ be φ-Lipschitz satisfying $k < 1$, where k is defined as in* (4.7). *Then there corresponds to each $v \in X_2(t_0)$ one and only one solution $x(t)$ of the equation* (4.1) *on $(-\infty, t_0]$ satisfying the condition $(I - P(t_0))x(t_0) = v$ and* $\operatorname{ess\,sup}_{t\leqslant t_0} \|x(t)\| < \infty$.

Proof. For each $t_0 \in \mathbb{R}, v \in X_2(t_0)$ we consider the operator

$$T\colon\ L_\infty((-\infty, t_0], X) \to L_\infty((-\infty, t_0], X),$$

$$x \mapsto (Tx)(t) = U(t,t_0)_{|}v + \int_{-\infty}^{t_0} G(t,\tau) f(\tau, x(\tau))d\tau \quad \text{for all} \quad t \leqslant t_0.$$

It follows from the estimates of G and $U(t,s)$ that

$$\|T(x) - T(y)\|_\infty \leqslant$$

$$\leqslant \frac{(1+H)N\|x(\cdot) - y(\cdot)\|_\infty}{1 - e^{-\beta}} \left[N_1\|\Lambda_1\varphi\|_\infty + N_2\|\Lambda_1 T_1^+\varphi\|_\infty\right] = k\|x(\cdot) - y(\cdot)\|_\infty.$$

Since $k < 1$, we obtain that T is a contraction. By the Banach contraction mapping theorem, the lemma follows.

From Lemmas 4.3, 4.4 and using the same arguments as in [15] (Theorem 4.7) we obtain the existence of an invariant unstable manifold in the following theorem.

Theorem 4.2. *Let the evolution family $(U(t,s))_{t\geq s}$ have an exponential dichotomy with the corresponding projections $P(t)$, $t \in \mathbb{R}$, and the dichotomy constants N, $\beta > 0$. Suppose that $f\colon \mathbb{R}\times X \to X$ be φ-Lipschitz, where φ is the positive function which belongs to $E_{\mathbb{R}}$ such that*

$k < 1$, *here* k *defined as in* (4.7). *Then there exists an invariant unstable manifold for the solutions of equation* (4.1). *Moreover, for any two solution* $x_1(\cdot)$ *and* $x_2(\cdot)$ *belonging to this unstable manifold we have*

$$\|x_1(t) - x_2(t)\| \leqslant Ce^{\mu(t-t_0)}\|(I - P(t_0))x_1(t_0) - (I - P(t_0))x_2(t_0)\| \quad \textit{for all} \quad t \leqslant t_0,$$

where C, μ *be the positive constants independent of* t_0, $x_1(\cdot)$ *and* $x_2(\cdot)$.

Proof. The proof of this theorem can be done by the same way as in [15] (Theorem 4.7) replacing $\mathbb{R}_+$ by $\mathbb{R}$ and using the structures of bounded solutions as in Lemmas 4.3, 4.4. We just note that the family of Lipschitz mappings $(g_t)_{t\in\mathbb{R}}$ determining the unstable manifold is define by

$$g_t\colon\ X_2(t) \to X_1(t),$$

$$g_t(y) = \int\limits_{-\infty}^{t} G(t,s)f(s,x(s))ds$$

for $y \in X_2(t)$, where $x(\cdot)$ is the unique solution in $L_\infty((-\infty,t],X)$ of equation (4.1) on $(-\infty,t]$ satisfying $(I-P(t))x(t) = y$ (note that the existence and uniqueness of $x(\cdot)$ is obtained in Lemma 4.4).

Theorem 4.2 is proved.

Using now the similar arguments as in Corollary 4.1, we easily obtain the following corollary which describes the relations of solutions of equation (4.1) with initial values lying on the invariant stable or unstable manifolds and the solution lying on the intersection of the two manifolds.

***Corollary* 4.2.** *Let the evolution family* $(U(t,s))_{t\geq s}$ *have an exponential dichotomy with the corresponding projections* $P(t)$, $t \in \mathbb{R}$, *and the dichotomy constants* N, $\beta > 0$. *Suppose that* f *is* φ*-Lipschitz with the positive function* $\varphi \in E_\mathbb{R}$ *such that* $k < \dfrac{1}{N+1}$, *here* k *defined as in* (4.7). *Then there exist an invariant stable manifold* $\mathbf{S}$ *and an invariant unstable manifold* $\mathbf{U}$ *for the solutions of equation* (4.1) *having the following properties*:

(a) *for each* t_0 *the intersection* $\mathbf{S}_{t_0} \cap \mathbf{U}_{t_0}$ *contains the unique element* z_{t_0},

(b) *the solution* $u_0(t)$ *of equation* (4.1) *with initial condition* $u_0(t_0) = z_{t_0}$ *is bounded on the whole line* $\mathbb{R}$,

(c) *the solutions* $u(t)$ *of equation* (4.1) *satisfying* $u(t_0) \in \mathbf{S}_{t_0}$ *exponentially approach* $u_0(t)$ *as* $t \to \infty$,

(d) *the solutions* $u(t)$ *of equation* (4.1) *satisfying* $u(t_0) \in \mathbf{U}_{t_0}$ *exponentially approach* $u_0(t)$ *as* $t \to -\infty$.

4.3. Invariant center-unstable manifolds on $\mathbb{R}$**.** Using Theorem 4.2 and rescaling procedures similar to Theorem 3.1 to transform the trichotomy case to the dichotomy case, we can easily obtain the exsitence of an invariant center-unstable manifolds in the following theorem.

Theorem 4.3. *Let the evolution family* $(U(t,s))_{t\geq s}$ *have an exponential trichotomy with the corresponding constants* K, α, β $(\alpha < \beta)$, *and projections* $(P_j(t))_{t\in\mathbb{R}}$, $j = 1,2,3$, *given in Definition* 1.2. *Suppose that* $f\colon \mathbb{R}_+ \times X \to X$ *be* φ*-Lipschitz, where* φ *is the positive function which belongs to* $E_\mathbb{R}$ *such that* $k < \min\left\{\dfrac{1}{N+1}, \dfrac{1-e^{\alpha-\beta}}{1-e^{-\beta}}\right\}$, *here* k *is defined by* (4.7). *Then there exists* *a center-unstable manifold* $\mathbf{C^u} = \{(t, \mathbf{C}^\mathbf{u}_t) \mid t \in \mathbb{R}_+$ and $\mathbf{C}^\mathbf{u}_t \subset X\}$ *for the solutions of equation* (3.1), *with the family* $(\mathbf{C}^\mathbf{u}_t)_{t\in\mathbb{R}}$ *being the graphs of the family of Lipschitz continuous mappings* $(h_t)_{t\in\mathbb{R}}$ $\big($*i.e.,* $\mathbf{C}^\mathbf{u}_t := \mathrm{graph}(h_t) = \big\{x + h_t x \mid x \in \mathrm{Im}(P_2(t) + P_3(t))\big\}$ *for each* $t \in \mathbb{R}\big)$ *where* $h_t \colon \mathrm{Im}(P_2(t) + P_3(t)) \to \mathrm{Im}\, P_1(t)$ *has the Lipschitz constant* $\dfrac{Nk}{1-k}$ *independent of* t, *such that the following properties hold*:

(i) *to each* $x_0 \in \mathbf{C}^\mathbf{u}_{t_0}$ *there corresponds one and only one solution* $u(t)$ *of equation* (3.1) *on* $(-\infty, t_0]$ *satisfying* $u(t_0) = x_0$ *and* $\mathrm{ess\,sup}_{t\leqslant t_0} \|e^{\gamma t}u(t)\| < \infty$, *where* $\gamma := \dfrac{\alpha+\beta}{2}$,

(ii) $\mathbf{C}^\mathbf{u}_t$ *is homeomorphism to* $X_2(t) \oplus X_3(t)$ for all $t \in \mathbb{R}$, *where* $X_2(t) = \mathrm{Im}\, P_2(t)$ *and* $X_3(t) = = \mathrm{Im}\, P_3(t)$,

(iii) $\mathbf{C^u}$ *is invariant under the equation* (3.1) *in the sense that, if* $u(t)$ *is the solution of equation* (3.1) *satisfying* $u(t_0) = x_0 \in \mathbf{C}^\mathbf{u}_{t_0}$ *and* $\mathrm{ess\,sup}_{t\leqslant t_0} \|e^{\gamma t}u(t)\| < \infty$, *then* $u(s) \in \mathbf{C}^\mathbf{u}_s$ *for all* $s \leqslant t_0$,

(iv) *every two solutions* $u_1(t)$, $u_2(t)$ *on the center-unstable manifold* $\mathbf{C^u}$ *satisfy the condition that there exist positive constants* μ *and* C_μ *independent of* $t_0 \geq 0$ *such that*

$$\|x(t) - y(t)\| \leqslant C_\mu e^{(\mu-\gamma)(t-t_0)}\big\|(P_1(t_0) + P_3(t_0))x(t_0) - (P_1(t_0) + P_3(t_0))y(t_0)\big\| \tag{4.11}$$

for all $t \leqslant t_0$.

Note that the existence of an invariant center-stable manifold on $\mathbb{R}$ is defined and proved by the same ways as in the case of half-line $\mathbb{R}_+$ (see Theorem 3.1).

From the existence of the invariant center-stable and center-unstable manifolds of equation (4.1) defined on the whole line we have the following important corollary describing the behavior of solutions to equation (4.1).

***Corollary* 4.3.** *Let the evolution family* $(U(t,s))_{t\geq s}$ *have an exponential dichotomy with the corresponding projections* $P(t)$, $t \in \mathbb{R}$, *and the dichotomy constants* N, α, $\beta > 0$. *Suppose that* f *is* φ*-Lipschitz with the positive function* $\varphi \in E_\mathbb{R}$ *such that*

$$k < \min\left\{\frac{1}{N+1}, \frac{1-e^{\alpha-\beta}}{1-e^{-\beta}}, \frac{\sqrt{2}-1}{N+\sqrt{2}-1}\right\},$$

here k *defined as in* (4.7). *Then there exist an invariant center-stable manifold* $\mathbf{C}$ *and an invariant center-unstable manifold* $\mathbf{C^u}$ *for the solutions of equation* (4.1) *having the following properties*:

(a) *for each* $t_0 \in \mathbb{R}$ *the intersection* $\mathbf{C}_{t_0} \cap \mathbf{C}^\mathbf{u}_{t_0}$ *is homeomorphism to* $X_3(t_0) = P_3(t_0)X$,

(b) *the solution* $u_0(t)$ *of equation* (4.1) *with initial condition* $u_0(t_0) \in \mathbf{C}_{t_0} \cap \mathbf{C}^\mathbf{u}_{t_0}$ *satisfies that* $\mathrm{ess\,sup}_{t\in\mathbb{R}} \|e^{-\gamma|t|}u(t)\| < \infty$, *where* $\gamma := \dfrac{\alpha+\beta}{2}$,

(c) *for the solution* $u(t)$ *of equation* (4.1) *satisfying* $u(t_0) \in \mathbf{C}_{t_0}$ *we have that* $e^{-\gamma t}u(t)$ *exponentially approaches* $e^{-\gamma t}u_0(t)$ *as* $t \to \infty$,

(d) *for the solution* $u(t)$ *of equation* (4.1) *satisfying* $u(t_0) \in \mathbf{C}^\mathbf{u}_{t_0}$ *we have that* $e^{\gamma t}u(t)$ *exponentially approaches* $e^{\gamma t}u_0(t)$ *as* $t \to -\infty$.

Proof. (a) Let us first prove that for each $z \in X_3(t)$ there exists a unique $w \in X_1(t) \oplus X_3(t)$ such that $w = h_t(z + g_t(w)) + z$, where g_t and h_t are the members of the Lipschitz mapping families $(g_t)_{t\in\mathbb{R}}$ and $(h_t)_{t\in\mathbb{R}}$ determining the invariant center-stable and center-unstable manifolds, respectively. Indeed, the mapping

$$\mathbf{L}\colon\ X_1(t)\oplus X_3(t)\to X_1(t)\oplus X_3(t),$$

$$y\mapsto h_t(z+g_t(y))+z$$

satisfies that

$$\|\mathbf{L}y_1-\mathbf{L}y_2\|=\|h_t(z+g_t(y_1))-h_t(z+g_t(y_2))\|\leqslant\frac{Nk}{1-k}\|g_t(y_1)-g_t(y_2)\|\leqslant$$

$$\leqslant\left(\frac{Nk}{1-k}\right)^2\|y_1-y_2\|.$$

Since $\dfrac{Nk}{1-k}<1$ we obtain that $\mathbf{L}$ is a contraction. Let w be its unique fixed point. Then w is the unique element in $X_1(t)\oplus X_3(t)$ such that $w=h_t(z+g_t(w))+z$.

Define now the mapping $\mathbf{D}\colon X_3(t)\to\mathbf{C}_t\cap\mathbf{C}_t^{\mathbf{u}}$ by $\mathbf{D}(z)=w+g_t(w)$, where w is the unique element in $X_1(t)\oplus X_3(t)$ such that $w=h_t(z+g_t(w))+z$. Then we have $w+g_t(w)=z+g_t(w)+$ $+\,h_t(z+g_t(w))\in\mathbf{C}_t\cap\mathbf{C}_t^{\mathbf{u}}$. The uniqueness of w yields that $\mathbf{D}$ is a well-defined mapping.

We next prove the sujectiveness of $\mathbf{D}$. For $x\in\mathbf{C}_t\cap\mathbf{C}_t^{\mathbf{u}}$ we have that there is $u\in X_1(t)\oplus X_3(t)$ and $v\in X_2(t)\oplus X_3(t)$ such that $x=u+g_t(u)=v+h_t(v)$. Then we have $u-h_t(v)=v-$ $-\,g_t(u)\in\big(X_1(t)\oplus X_3(t)\big)\cap\big(X_2(t)\oplus X_3(t)\big)=X_3(t)$. Therefore, there is a $z\in X_3(t)$ such that $u-h_t(v)=v-g_t(u)=z$. This follows that $u-h_t(z+g_t(u))=z$. As shown above, this relation means that $\mathbf{D}z=u+g_t(u)=x$. Therefore, $\mathbf{D}$ is surjevtive.

We now prove that $\mathbf{D}$ is a Lipschitz mapping. In fact, by the definition of $\mathbf{D}$ we have $\mathbf{D}(z_1)=$ $=w_1+g_t(w_1)$ and $\mathbf{D}(z_2)=w_2+g_t(w_2)$ for w_1 and w_2 being the unique solutions in $X_1(t)\oplus X_3(t)$ of equations $w_1=h_t(z_1+g_t(w_1))+z_1$ and $w_2=h_t(z_2+g_t(w_2))+z_2$, respectively. Then putting $l=\dfrac{Nk}{1-k}$ (the Lipschitz constant of g_t and h_t), we have

$$(1-l)\|w_1-w_2\|\leqslant\|\mathbf{D}(z_1)-\mathbf{D}(z_2)\|=$$

$$=\|z_1+h_t(z_1+g_t(w_1))+g_t(w_1)-\big(z_2+h_t(z_2+g_t(w_2))+g_t(w_2)\big)\|\leqslant$$

$$\leqslant\|z_1-z_2\|+l\|z_1-z_2\|+l\|g_t(w_1)-g_t(w_2)\|+\|g_t(w_2)-g_t(w_2)\|\leqslant$$

$$\leqslant(1+l)\|z_1-z_2\|+l(l+1)\|w_2-w_2\|.$$

Therefore, we obtain that $\|\mathbf{D}(z_1)-\mathbf{D}(z_2)\|\leqslant(1+l)\|z_1-z_2\|+\dfrac{l(l+1)}{1-l}\|\mathbf{D}(z_1)-\mathbf{D}(z_2)\|$. Thus,

$$\|\mathbf{D}(z_1)-\mathbf{D}(z_2)\|\leqslant\frac{1-l^2}{2-(1+l)^2}\|z_1-z_2\|,$$

here we note that $2-(1+l)^2>0$ since $k<\dfrac{\sqrt{2}-1}{N+\sqrt{2}-1}$. Hence, we obtain that $\mathbf{D}$ is a Lipschitz mapping with Lischitz constant $\dfrac{1-l^2}{2-(1+l)^2}$. This follows that $\mathbf{D}$ is continuous and injective. As shown above $\mathbf{D}$ is already surjective, therefore, $\mathbf{D}$ is bijetive. The inverse $\mathbf{D}^{-1}$ of $\mathbf{D}$ is defined as $\mathbf{D}^{-1}\colon \mathbf{C}_t \cap \mathbf{C}_t^{\mathbf{u}} \to X_3(t)$ with $\mathbf{D}^{-1}(w + g_t(w)) = z$ if $z = w - h_t(z + g_t(w))$ (note that, by the contraction-mapping arguments we can easily show that for each $w \in X_1(t) \oplus X_3(t)$ there exists a unique $z \in X_3(t)$ such that $z = w - h_t(z + g_t(w))$). We then prove that $\mathbf{D}^{-1}$ is also a Lipschitz mapping. Indeed, for $x_1 = u + g_t(u)$ and $x_2 = v + g_t(v)$ belonging to $\mathbf{C}_t \cap \mathbf{C}_t^{\mathbf{u}}$ we have that

$$\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\| = \|z_1 - z_2\| \leqslant$$

$$\leqslant \|w_1 - h_t(z_1 + g_t(w_1)) - \big(w_2 - h_t(z_2 + g_t(w_2))\big)\| \leqslant$$

$$\leqslant \|w_1 - w_2\| + l\|z_1 - z_2\| + l^2|\|w_1 - w_2\| =$$

$$= (1 + l^2)\|w_1 - w_2)\| + l\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\| \leqslant$$

$$\leqslant \frac{1 + l^2}{1 - l}\|w_1 + g_t(w_1) - w_2 - g_t(w_2)\| + l\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\| =$$

$$= \frac{1 + l^2}{1 - l}\|x_1 - x_2\| + l\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\|.$$

Therefore, we obtain that

$$\left\|\mathbf{D}^{-1}x_1 - \mathbf{D}^{-1}x_2\right\| \leqslant \frac{1 + l^2}{(1 - l)^2}\|x_1 - x_2\|.$$

Hence, $\mathbf{D}^{-1}$ is also Lipschitz mapping. This follows that $\mathbf{D}$ is a homeomorphism, and we obtain that $\mathbf{C}_t \cap \mathbf{C}_t^{\mathbf{u}}$ is homeomorphism to $X_3(t)$ for all $t \in \mathbb{R}$.

The property (b) follows from the definitions of the invariant center-stable and center-ustable manifolds, respectively.

The properties (c) and (d) are consequences of the inequalities (3.4) and (4.11), respectively.

Corollary 4.3 is proved.

5. Examples. In this section, we give some concrete examples of reaction-diffusion equations to illustrate our abstract results.

The reaction-diffusion processes are modeled by the following equation:

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t, x),$$

where $x(t)$ is the density of material, the partial differential operators $A(t)$ represent the diffusion, and f represents the source of material which, in many contexts, depends on time in diversified manners (see [23] (Chapter 11), [24, 37]). Therefore, sometimes one may not hope to have the uniformly Lipschitz continuity of f. Our theoretical results hence give a chance to consider the above reaction-diffusion equation in general cases. Let us start by the following equation.

***Example* 5.1.** Consider the reaction-diffusion equation of the form

$$\frac{dx(t)}{dt} = Ax(t) + f(t, x),$$

where A is a sectorial operator satisfying that the spectrum $\sigma(A)$ of A is decomposed in three disjoint sets that are $\{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda < 0\}$, $\{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda > 0\}$, and $\{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda = 0\}$ such that $\sigma(A) \cap i\mathbb{R}$ is of finitely many points. Then, A is a generator of an analytic semigroup $(T(t))_{t\geq 0}$. We define the evolution family $U(t,s) := T(t-s)$ for all $t \geq s \geq 0$. We now claim that it has an exponential trichotomy with an appropriate choice of projections. By the spectral mapping theorem for analytic semigroups we have that, for fixed t_0, the spectrum of the operator $T(t_0)$ splits into three disjoint sets σ_1, σ_2, σ_3, where $\sigma_1 \subset \{|z| < 1\}, \sigma_2 \subset \{|z| > 1\}, \sigma_3 \subset \{|z| = 1\}$ with σ_3 consisting of finitely many points.

Next, we choose $P_1 = P_1(t_0)$, $P_2 = P_2(t_0)$, $P_3 = P_3(t_0)$ be the Riesz projections corresponding to the spectral sets σ_1, σ_2, σ_3, respectively. Clearly, P_1, P_2 and P_3 commute with $T(t)$ for all $t \geq 0$.

Obviously, $P_1 + P_2 + P_3 = I$ and $P_iP_j = 0$ for $i \neq j$, and there are positive constants M, δ such that $\|T(t)P_1\| \leqslant Me^{-\delta t}$ for all $t \geq 0$. Furthermore, let $Q := P_2 + P_3 = I - P_1$ and consider the strongly continuous semigoup $(T_Q(t))_{t\geq 0}$ on the space $\mathrm{Im}\, Q$, where $T_Q(t) := T(t)Q$. Since $\sigma_2 \cup \sigma_3 = \sigma(T_Q(t_0))$, $(T_Q(t))_{t\geq 0}$ can be extended to a group $(T_Q(t))_{t\in\mathbb{R}}$ in $\mathrm{Im}\, Q$. As well-known in the semigroup theory, there are positive constants K, α, γ such that α can be chosen as small as required (we may let $\alpha < \gamma$), and the following estimates hold:

$$\|T_Q(-t)P_2\| \leqslant Ke^{-\gamma t} \quad \text{for all} \quad t \geq 0,$$

$$\|T_Q(t)P_3\| \leqslant Ke^{\alpha|t|} \quad \text{for all} \quad t \in \mathbb{R}.$$

Summing up the above discussions, we conclude that the evolution family $(U(t,s))_{t\geq s}$ has an exponential trichotomy with projections P_j, $j = 1, 2, 3$, and positive constants N, α, β, where

$$\beta := \min\{\delta, \gamma\},$$

$$N := \max\{K, M\}.$$

Thus, if f is φ-Lipschitz for some positive function φ satisfying that $\sup_{t\in\mathbb{R}} \int_t^{t+1} \varphi(\tau)d\tau$ is small enough, then the integral equation

$$x(t) = U(t,s)x(s) + \int_s^t U(t,\xi)f(\xi, x(\xi))d\xi \quad \text{for all} \quad t \geq s,$$

has a center manifold.

***Example* 5.2.** For fixed $n \in \mathbb{N}^*$, consider the equation

$$\begin{aligned} &w_t(x,t) = w_{xx}(x,t) + n^2 w(x,t) + \varphi(t)\sin(w(x,t)), \quad 0 \leqslant x \leqslant \pi, \quad t \in \mathbb{R}, \\ &w(0,t) = w(\pi,t) = 0, \quad t \in \mathbb{R}, \end{aligned} \tag{5.1}$$

where the step function $\varphi(t)$ is defined as in formula (5.2).

We define $X := L_2[0,\pi]$, and let $A : X \to X$ be defined by $A(y) = y'' + n^2 y$, with

$$D(A) = \big\{y \in X\colon\ y \ \text{ and } \ y'' \ \text{ are absolutely continuous}, \ \ y'' \in X, \ \ y(0) = y(\pi) = 0\big\}.$$

The equation (5.1) can now be rewritten as

$$\frac{du}{dt} = Au + f(t,u) \quad \text{for} \quad u(t) = w(\cdot,t),$$

where $f\colon \mathbb{R}\times X \to X$, with $f(t,u) = \varphi(t)\sin(u)$ for φ being defined for a constant $c > 1$ by

$$\varphi(t) = \begin{cases} |k| & \text{if } t \in \left[\frac{2k+1}{2} - \frac{1}{2^{|k|+c}},\ \frac{2k+1}{2} + \frac{1}{2^{|k|+c}}\right] \text{ for } k = 0, \pm1, \pm2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2}$$

Here, we note that φ can take any arbitrarily large value but we still have that

$$\sup_{t\in\mathbb{R}} \int\limits_t^{t+1} |\varphi(\tau)|d\tau \leqslant 2\sup_{k\in\mathbb{Z}} \int\limits_{\frac{2k+1}{2}-\frac{1}{2^{|k|+c}}}^{\frac{2k+1}{2}+\frac{1}{2^{|k|+c}}} |k|dt = 2\sup_{k\in\mathbb{Z}} \frac{|k|}{2^{|k|+c-2}} \leqslant \frac{1}{2^{c-1}}.$$

Therefore, $\varphi \in \mathbf{M}(\mathbb{R})$ which is an admissible space.

It can be seen that (see [9]) that A is the generator of an analytic semigroup $(T(t))_{t\geq 0}$. Since $\sigma(A) = \{-1+n^2, -4+n^2, \ldots, 0, -(1+n)^2+n^2, \ldots\}$, applying the spectral mapping theorem for analytic semigroups we get

$$\sigma(T(t)) = e^{t\sigma(A)} =$$

$$= \{e^{t(n^2-1)}, e^{t(n^2-4)}, \ldots, e^{t((n-1)^2-n^2)}\} \cup \{1\} \cup \{e^{-t((1+n)^2-n^2)}, e^{-t((2+n)^2-n^2)}, \ldots\}.$$

One can see easily that the nonlinear forcing term f is φ-Lipschitz. Using Example 5.1 we obtain that, if $\sup_{t\in\mathbb{R}} \int_t^{t+1} \varphi(\tau)d\tau$, which is less than $\frac{1}{2^{c-1}}$, is sufficient small (or c is sufficiently large), then there exists a center manifold for mild solutions of equation (5.1).

***Example* 5.3.** For fixed $n \in \mathbb{N}^*$, consider the equation

$$\begin{aligned} &w_t(x,t) = a(t)[w_{xx}(x,t) + n^2 w(x,t)] + \varphi(t)\sin(w(x,t)), \quad 0 \leqslant x \leqslant \pi, \quad t \in \mathbb{R}, \\ &w(0,t) = w(\pi,t) = 0, \quad t \in \mathbb{R}, \end{aligned} \tag{5.3}$$

where φ is defined as in (5.2); the function $a(\cdot) \in L_{1,\mathrm{loc}}(\mathbb{R})$ and satisfies the condition $\gamma_1 \geq a(t) \geq \gamma_0 > 0$ for fixed γ_0, γ_1 and a.e. $t \in \mathbb{R}$.

We put $X := L_2[0,\pi]$, and let $A\colon X \to X$ be defined by $A(y) = y'' + n^2 y$, with

$$D(A) = \big\{y \in X\colon\ y \text{ and } y'' \text{ are absolutely continuous}, \ y'' \in X,\ y(0) = y(\pi) = 0\big\}.$$

Putting $A(t) := a(t)A$, the equation (5.3) can now be rewritten as

$$\frac{du}{dt} = A(t)u + f(t,u) \quad \text{for} \quad u(t) = w(\cdot,t),$$

where $f\colon \mathbb{R}\times X \to X$, with $f(t,u) = \varphi(t)\sin(u)$.

Thus, as the above examples, A is a sectorial operator and generates an analytic semigroup $(T(t))_{t\geq 0}$, and $\sigma(A)$ satisfies the conditions as in Examples 5.1 and 5.2. Therefore, $A(t)$ “generates” the evolution family $(U(t,s))_{t\geq s}$ which is defined by the formula

$$U(t,s) = T\left(\int\limits_s^t a(\tau)d\tau\right).$$

Using the above arguments as in Examples 5.1 and 5.2 we have that the analytic semigroup $(T(t))_{t\geq 0}$ has an exponential trichotomy with the projections P_k, $k = 1,2,3$, and the trichotomy constants N, α, β where α is as small as required. Also, the following estimates hold:

(i) $\|T(t)|_{P_1X}\| \leqslant Ne^{-\beta t}$,

(ii) $\|T_2(-t)\| = \|(T(t)|_{P_2X})^{-1}\| \leqslant Ne^{-\beta t}$,

(iii) $\|T(t)|_{P_3X}\| \leqslant Ne^{\alpha t}$ for all $t \geq 0$.

From this, it is straightforward to check that the evolution family $(U(t,s))_{t\geq s}$ has an exponential trichotomy with the trichotomy projection P_k, $k = 1,2,3$, and the trichotomy constants N, β, α by the following estimates:

$$\|U(t,s)|_{P_1X}\| = \left\|T\left(\int\limits_s^t a(\tau)d\tau\right)\right|_{P_1X}\right\| \leqslant Ne^{-\beta(t-s)},$$

$$\|U(s,t)_|\| = \|(U(t,s)|_{P_2X})^{-1}\| = \left\|T\left(-\int\limits_s^t a(\tau)d\tau\right)\right|_{P_2X}\right\| \leqslant Ne^{-\beta(t-s)},$$

$$\|U(t,s)|_{P_3X}\| = \left\|T\left(\int\limits_s^t a(\tau)d\tau\right)\right|_{P_3X}\right\| \leqslant Ne^{\alpha(t-s)}$$

for all $t \geq s \geq 0$. Therefore, we obtain that, if $\sup_{t\in\mathbb{R}} \displaystyle\int_t^{t+1} \varphi(\tau)d\tau = \frac{1}{2^{c-1}}$ is sufficient small, then there exists a center manifold for mild solutions of equation (5.3).

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