К О Р О Т К І П О В І Д О М Л Е Н Н Я

UDC 512.5

T. Amouzegar Kalati (Quchan Inst. Engineering and Technology, Iran),
D. Keskin Tütüncü (Hacettepe Univ., Ankara, Turkey)

A NOTE ON NONCOSINGULAR LIFTING MODULES

ПРО НЕКОСИНГУЛЯРНІ МОДУЛІ ІЗ ВЛАСТИВІСТЮ ПІДНЯТТЯ

Let R be a right perfect ring. Let M be a noncosingular lifting module which does not have any relatively projective component. Then M has finite hollow dimension.

Нехай R — праве досконале кільце, а M — некосингулярний модуль із властивістю підняття, що не має жодної відносно проективної компоненти. Тоді M має скінченну дуальну розмірність Голді.

1. Introduction. Throughout this paper all rings are associative with identity and modules are unitary right modules. A module M is said to have *finite hollow dimension* if there exists an epimorphism from M to a finite direct sum of n hollow factor modules with small kernel. A module M is called *lifting* if for every $A \leq M$, there exists a direct summand B of M such that $B \subseteq A$ and $A/B \ll M/B$. A module M is amply supplemented and every coclosed submodule of M is a direct summand of M if and only if M is lifting by [1] (22.3(d)). In [5], Talebi and Vanaja defined $\overline{Z}(M)$ as follows:

$$\overline{Z}(M) = \mathrm{Re}\,(M, \mathcal{S}) = \bigcap \big\{ \mathrm{Ker}(g) \,|\, g \in \mathrm{Hom}\,(M, L), L \in \mathcal{S} \big\},$$

where $\mathcal{S}$ denotes the class of all small modules.

They called M a *cosingular* (*noncosingular*) module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$).

In this note, as we state in the abstract, we prove the following main theorem:

Let R be a right perfect ring. Let M be a noncosingular lifting module which does not have any relatively projective component. Then M has finite hollow dimension.

For all undefined notions we refer to [1].

2. Results. An R-module M is called *dual Rickart* if, for any element $\phi \in S = \mathrm{End}(M)$, $\mathrm{Im}\phi = eM$, where $e^2 = e \in S$.

Lemma 2.1. *Let $M = \oplus_{i\in\mathbb{N}} M_i$ be a dual Rickart module and let $(f_i\colon M_i \to M_{i+1})_{\mathbb{N}}$ be a sequence of homomorphisms. Then for any finitely many elements $a_1, a_2, \ldots, a_n \in M_1$, there exist some $r \in \mathbb{N}$ and a homomorphism $h\colon M_{r+1} \to M_r$ such that $f_{r-1}f_{r-2}\ldots f_1(a_k) = hf_rf_{r-1}\ldots f_1(a_k)$ for $k = 1, 2, \ldots, n$.*

In particular, if M_1 is finitely generated, then $f_{r-1}f_{r-2}\ldots f_1 = hf_rf_{r-1}\ldots f_1$.

Proof. It is easy to see by [6] (43.3(3)).

In [3], Keskin Tütüncü and Tribak introduced the concept of dual Baer modules. A module M is called a *dual Baer* module if for every right ideal I of S, $\sum_{\phi\in I} \mathrm{Im}\phi$ is a direct summand of M. It is clear that every dual Baer module is dual Rickart.

Lemma 2.2. *Let $M = \oplus_{i=1}^{\infty} M_i$, where each M_i is local noncosingular. If, for each i, there is an epimorphism $f_i\colon M_i \longrightarrow M_{i+1}$, which is not an isomorphism, then M is not lifting.*

Proof. Let $M = \oplus_{i=1}^{\infty} M_i$ be a lifting module and $(f_i \colon M_i \to M_{i+1})_{\mathbb{N}}$ be a sequence of epimorphisms, which are non-isomorphisms. By [3] (Theorem 2.14) and Lemma 2.1, there exist an $r \in \mathbb{N}$ and a homomorphism $h \colon M_{r+1} \to M_r$ such that $f_{r-1} f_{r-2} \ldots f_1 = h f_r f_{r-1} \ldots f_1$. Since all f_i are epimorphisms, we have $h f_r = 1_{M_r}$. Hence f_r is an isomorphism, a contradiction.

Lemma 2.2 is proved.

Recall that a family of modules $\{M_i \mid i \in I\}$ is called *locally semi-T-nilpotent* if, for any countable set of non-isomorphisms $\{f_n \colon M_{i_n} \to M_{i_{n+1}}\}$ with all i_n distinct in I, and for any $x \in M_{i_1}$, there exists k (depending on x) such that $f_k \ldots f_1(x) = 0$ (see [4]).

***Corollary* 2.1.** *Let M be a noncosingular lifting module such that $M = \oplus_{i=1}^{\infty} M_i$, where each M_i is local for all $i \in \mathbb{N}$. Then the family $\{M_i \mid i \in \mathbb{N}\}$ is locally semi-T-nilpotent.*

Proof. Consider any infinite sequence of non-isomorphisms f_n

$$M_{i_1} \overset{f_1}{\to} M_{i_2} \overset{f_2}{\to} \ldots M_{i_n} \overset{f_n}{\to} \ldots .$$

It is obvious that f_n is an epimorphism for all $n \geq 1$. By Lemma 2.2, it is easy to see that the family $\{M_i \mid i \in \mathbb{N}\}$ is locally semi-T-nilpotent.

Lemma 2.3. *Let U and V be noncosingular hollow modules such that the module $U \oplus V$ is lifting. Then there exists an epimorphism from U to V or V is U-projective.*

Proof. Let $M = U \oplus V$, $M_1 = U \oplus 0$ and $M_2 = 0 \oplus V$. Hence $M = M_1 \oplus M_2$. Suppose that there does not exist any epimorphism from U to V, i.e., from M_1 to M_2. We will show that V is U-projective. Let N be any nonzero proper submodule of M such that $M = N + M_1$. Since M is lifting, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. Let $M = K \oplus K'$ for some submodule K' of M. Note that K and K' are hollow. Since $M = K + M_1$, we have an epimorphism from M/K' to M_2. If $K' + M_1 = M$, then we have an epimorphism from M_1 to M/K'. So we have an epimorphism from M_1 to M_2, a contradiction. Thus $K' + M_1 \neq M$. Hence $(K' + M_1)/K' \ll M/K'$. Since every small module is cosingular, $(K' + M_1)/K'$ is cosingular. On the other hand, $(K' + M_1)/K' \cong M_1/(K' \cap M_1)$ is noncosingular. Hence $K' = K' + M_1$ and so $M_1 \leq K'$. Thus $M = K \oplus M_1$. By [6] (41.14), M_2 is M_1-projective, i.e., V is U-projective.

Theorem 2.1. *Let R be a right perfect ring. Let M be a noncosingular lifting module which does note have any relatively projective component. Then M has finite hollow dimension.*

Proof. By [3] (Theorem 2.14 and Corollary 2.6(ii)), there exists an index set I and hollow submodules M_i, $i \in I$, such that $M = \oplus_{i \in I} M_i$. Suppose that I is infinite. For all distinct i, j in I, $M_i \oplus M_j$ is lifting and hence by Lemma 2.3, there exists an epimorphism from M_i to M_j or M_j is M_i-projective. By hypothesis, there exists an epimorphism from M_i to M_j. Now by Lemma 2.2, there exists an infinite subset J of I such that $M_i \cong M_j$ for all $i, j \in J$ since $\oplus_{i \in I} M_i$ is lifting.

Let $i \in J$. Suppose that $\phi \colon M_i \longrightarrow M_i$ is a nonzero homomorphism. Since M_i is noncosingular and hollow, ϕ is an epimorphism. Suppose ϕ is not an isomorphism. Then for each $i, j \in J$, ϕ induces an epimorphism $\phi_{ij} \colon M_i \longrightarrow M_j$ which is not an isomorphism, contradicting Lemma 2.2. Thus ϕ is an isomorphism. It follows that the ring $\mathrm{End}(M_i)$ of endomorphisms of M_i is a division ring, and by [2] (Lemma 1), M_i is $M_i \cong M_j$-projective, a contradiction. Therefore, M has finite hollow dimension.

***Corollary* 2.2.** *Let R be a right perfect ring. Let M be a noncosingular lifting module which does not have any relatively projective component. Then M satisfies ACC equivalently, DCC on supplements.*

Proof. By Theorem 2.1 and [1] (20.34).

Finally, we give the following:

Proposition 2.1. *Let R be a right perfect ring and let $M = \prod_{i=1}^{\infty} M_i$, where each M_i is hollow noncosingular. If, for each i, there is an epimorphism $f_i\colon M_{i+1} \longrightarrow M_i$, which is not an isomorphism, then M is not lifting.*

Proof. Assume that $g_1\colon P_1 \longrightarrow M_1$ is a projective cover of M_1. Since P_1 is projective, there exists a homomorphism $g_2\colon P_1 \longrightarrow M_2$ such that $f_1 g_2 = g_1$. Clearly, g_2 is epic. Then for each i, we may define inductively, $g_i\colon P_1 \longrightarrow M_i$ so that $f_i g_{i+1} = g_i$ and all g_i are epic. Note that P_1 and all M_i are local and so cyclic. Now we have the strictly descending sequence since each f_i is not monic for each i:

$$P_1 \supset \operatorname{Ker} g_1 \supset \operatorname{Ker} g_2 \supset \ldots .$$

Define the homomorphism $\chi\colon P_1 \longrightarrow M$ by $\chi(y) = (g_i(y))_{i \in I}$ $(y \in P_1)$. Let $\operatorname{Im}\chi = K$. Then K is local and nonzero. Assume that $K = xR$ for some nonzero element $x \in K$. We can suppose without loss of generality that $x = (0, 0, \ldots, 0, x_{n+2}, x_{n+3}, \ldots)$ for some positive integer n. Then $x \in N = \prod_{n+2}^{\infty} M_i$. So $K \subseteq N$. Note that K is coclosed in M by [5] (Lemma 2.3(2)).

Now, let $M = K \oplus K'$ for some submodule K' of M and let $y \in \operatorname{Ker} g_n$. Consider $t = (0, 0, \ldots, 0, g_{n+1}(y), g_{n+2}(y), \ldots) \in M = \prod_{i=1}^{\infty} M_i$. Then $t = t_1 + t_2$ for some $t_1 \in K$ and $t_2 \in K'$. Then $t_2 = t - t_1 \in K \cap K' = 0$. So $t = t_1 \in K \subseteq N$. Thus $g_{n+1}(y) = 0$ and so $y \in \operatorname{Ker} g_{n+1}$. It follows that $\operatorname{Ker} g_n = \operatorname{Ker} g_{n+1}$, a contradiction. Therefore K is not a direct summand of M and M is not a lifting module.

Acknowledgments. This work has been done during a visit of the first author to the second author in the Department of Mathematics, Hacettepe University in 2011, and she wishes to thank the department for their kind hospitality. The first author also wishes to thank the Ministry of Science of Iran for the support.

1. *Clark J., Lomp C., Vanaja N., Wisbauer R.* Lifting modules // Frontiers in Math. – Birkhäuser Verlag, 2006.
2. *Keskin D.* Finite direct sums of $(D1)$-modules // Tr. J. Math. – 1998. – **22**, № 1. – P. 85 – 91.
3. *Keskin Tütüncü D., Tribak R.* On dual Baer modules // Glasgow Math. J. – 2010. – **52**. – P. 261 – 269.
4. *Mohamed S. H., Müller B. J.* Continuous and discrete modules // London Math. Soc. Lect. Notes. Ser. 147. – Cambridge: Cambridge Univ. Press, 1990.
5. *Talebi Y., Vanaja N.* The torsion theory cogenerated by M-small modules // Communs Algebra. – 2002. – **30**, № 3. – P. 1449 – 1460.
6. *Wisbauer R.* Foundations of module and ring theory. – Reading: Gordon and Breach, 1991.

Received 26.12.11