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ON IMPULSIVE STURM – LIOUVILLE OPERATORS WITH SINGULARITY AND SPECTRAL PARAMETER IN BOUNDARY CONDITIONS

ПРО ІМПУЛЬСНІ ОПЕРАТОРИ ШТУРМА – ЛІУВІЛЛЯ ІЗ СИНГУЛЯРНІСТЮ ТА СПЕКТРАЛЬНИМ ПАРАМЕТРОМ У ГРАНИЧНИХ УМОВАХ

We study properties and the asymptotic behavior of spectral characteristics for a class of singular Sturm – Liouville differential operators with discontinuity conditions and an eigenparameter in boundary conditions. We also determine the Weyl function for this problem and prove uniqueness theorems for a solution of the inverse problem corresponding to this function and spectral data.

Досліджено властивості та асимптотичну поведінку спектральних характеристик для класу сингулярних диференціальних операторів Штурма – Ліувілля з розривними умовами та власним параметром у граничних умовах. Визначено функцію Вейля для цієї задачі та доведено теореми про єдиність розв'язку оберненої задачі, що відповідає цій функції та спектральним даним.

1. Introduction. In spectral theory, the inverse problem is the usual name for any problem in which it is required to ascertain the spectral data that will determine a differential operator uniquely and a method of construction of this operator from the data. This kind of problem was first formulated and investigated by Ambartsumyan in 1929 [1]. Since 1946, various forms of the inverse problem have been considered by N. Levinson [2], B. M. Levitan [3], G. Borg [4], and now there exists an extensive literature on the [5]. Later, the inverse problems having specified singularities were considered in [6].

Spectral functions are important for determining the operators, that is, for solving the inverse problem for differential operators. However, in finite intervals, the integral representations for the solution of differential equations which generate the operator with initial conditions are more useful for investigating the spectral properties of the operator.

In case of $q(x) \equiv 0$, since this operator is the singular Sturm – Liouville operator, linearly independent solutions of this kind of differential equation could be given with hypergeometric functions and this integral representation is also a representation for hypergeometric functions. For this reason, obtaining this kind of integral representation is so important. Therefore, when it is obtained, these integral representations can be used for asymptotic behaviours of hypergeometric functions as $x \to +\infty$.

In interval (a, b), i.e., when the given interval is finite, Sturm – Liouville operator which is generated by the differential expression $\ell(y) := -y''(x) + q(x)y(x)$ satisfies the condition $q(x) \in L_1\,(a, b)$ in general. In singular case, i.e., when interval (a, b) is infinite or the function $q(x)$ has nonintegrable singularity in extremity points of interval, the condition of $q(x) \in L_{1,\mathrm{loc}}\,(a, b)$ is given.

When $q(x)$ is a first order singular generalized function, singular Sturm – Liouville operator which has a potential as $q = u'$ by using concept of generalized derivative such that $u \in L_2\,(0, 1)$ has been defined in [7, 8].

On the other hand, one-dimensional Schrödinger operators $S = -d^2/dx^2 + q$ with real-valued distributional potentials q in $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ are studied in [9]. The operator S can then be rigorously defined e.g. by the so-called regularization method that was used in [10] in the particular case $q(x) = 1/x$ and then developed for generic distributional potentials in $W^{-1}_{2,\mathrm{loc}}(\mathbb{R})$ by Savchuk and Shkalikov [8, 11]; see also recent extensions to more general differential expressions in [12, 13].

Moreover self-adjoint extensions of differential operators generated by differential expression $\ell(y)$ which has a potential $q(x) = u'(x)$ such that $u \in L_2\,(0,1)$ are studied in [8]. When $a \neq \neq 2, 4, 6, \ldots,$ generalized functions can be corresponded to the functions $|x|^{-a} \operatorname{sgn} x$ by using the method of canonical regularization [14]. When $a < \dfrac{3}{2}$, generalized functions which are obtained by this way can be shown as generalized derivative of functions from the space L_2 and therefore Sturm – Liouville operator which is given by the differential expression $\ell(y)$ is defined such that it has a potential like $q(x) = |x|^{-a} \operatorname{sgn} x$. In [15], when $q(x) = Cx^{-a}$ and $a < \dfrac{3}{2}$, $C \in R$, a regularization of constructing boundary-value problem for Sturm – Liouville equation which has this type of potential has been given.

As in this studies of [16] and [17], when $q(x) = Cx^{-a}$ and $a \in [1, 2)$, all self-adjoint extensions of operators generated by the differential expression $\ell(y)$ which has this type of potential according to boundary conditions have been given and therefore when $a \in [1, 2)$, regularization of constructing boundary-value problems for Sturm – Liouville equation which has this type of potential has been investigated. Regularization in the [8] and [16] coincides only when $a < \dfrac{3}{2}$.

Let's consider the differential expression

$$\ell(y) := -y''(x) + \frac{C}{x^a} y(x) + q(x)y(x), \quad 0 < x < \pi, \tag{1.1}$$

where C is a real number, $q(x)$ is a real valued bounded function.

We shall define an operator L_0': $L_0' y = \ell(y)$, on the q set of $D_0' = C_0^\infty(0, \pi)$. It is obvious that the operator L_0' is symmetric in the space of $L_2[0, \pi]$. We say that the operator L_0 which is the closure of L_0' is the minimal operator generated by the differential expression (1.1). The conjugate L_0^* of the operator L_0 is said to be the maximal operator generated by the differential expression (1.1).

In [16], all maximal dissipative and accumulative and also self-adjoint extensions of the operator L_0 have been studied according to the domain and boundary conditions of minimal and maximal operators generated by differential expression (1.1).

We define $\Gamma_\alpha y$ by $(\Gamma_\alpha y)\,(x) = y'(x) - u(x)y(x)$, where $u(x) = C\dfrac{x^{1-\alpha}}{1-\alpha}$.

It has been shown in [16] that if $y(x) \in D\,(L_0^*)$ then the function $(\Gamma_\alpha y)(x)$ has a limit as $x \to 0^+$, i.e.,

$$\lim_{x \to 0^+} (\Gamma_\alpha y)(x) = (\Gamma_\alpha y)(0).$$

Hence the domain $D\,(L_0)$ of minimal operator L_0 generated by differential expression (1.1) contains only functions $y(x) \in D\,(L_0^*)$ such that function $y(x)$ satisfies the conditions $y(0) = = y(\pi) = (\Gamma_\alpha y)\,(0) = y'(\pi) = 0$.

Let us consider the boundary-value problem L for the equation

$$\ell(y) := -y''(x) + \frac{C}{x^\alpha} y(x) + q(x)y(x) = \lambda y(x), \quad \lambda = k^2, \tag{1.2}$$

on the interval $0 < x < \pi$ with the boundary conditions

$$U(y) := y(0) = 0, \qquad V(y) := \left(\alpha_1 k^2 + \alpha_2\right) y(\pi) + \left(\beta_1 k^2 + \beta_2\right) y'(\pi) = 0 \tag{1.3}$$

and with the jump conditions

$$y\,(a+0) = \beta y\,(a-0),$$

$$y'\,(a+0) = \beta^{-1} y'\,(a-0),$$

where λ is spectral parameter; $\alpha \in (1, 3/2)$, $C, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2$ are real numbers, $\alpha_2\beta_1 - \beta_2\alpha_1 > 0$ and $a \in \left(\frac{\pi}{2}, \pi\right)$, $\beta \neq 1$, $\beta > 0, q(x)$ is a real valued bounded function and $q(x) \in L_2(0, \pi)$.

The boundary-value problems that contain the spectral parameter in boundary conditions linearly were investigated in [18 – 20]. In [18, 21], an operator-theoretic formulation of the problems of the form (1.2) – (1.4) has been given. Oscillation and comparison results have been obtained in [22 – 24]. In case of $\alpha_1 \neq 0$, problem (1.2) – (1.4) is associated with the physical problem of cooling a thin solid bar one end of which is placed in contact with a finite amount of liquid at time zero (see [18] and also [25] in it). Assuming that heat flows only into the liquid which has un-uniform density $\rho(x)$ and is convected only form the liquid into the surrounding medium, the initial boundary-value prolem for a bar of length one takes the form

$$u_t = \rho(x) u_{xx}, \tag{1*}$$

$$u_x(0, t) = 0, \tag{2*}$$

$$-kAu_x(\pi^-, t) = qM\left(\frac{dv}{dt}\right) + k_1 B v(t) \qquad \text{for all} \quad t, \tag{3*}$$

$$u(x, 0) = u_0(x) \qquad \text{for} \quad x \in [0, \pi], \tag{4*}$$

$$v(0) = v_0$$

after factoring out the steady-state solution where

$$\rho(x) = \begin{cases} 1, & 0 < x < a, \\ \alpha^2, & a < x < \pi. \end{cases}$$

Assuming that the rate of heat transfer across the liquid-solid interface is proportional to the difference in temperature between the end of the bar and the liquid with which it is in contract (Newton's law of cooling) and applying Fourier's law of heat conduction at $x = \pi$, we get

$$v(t) = u(\pi, t) + kc^{-1} u_x(\pi^{-1}, t) \qquad \text{for} \quad t > 0,$$

where $c > 0$ is the coefficient of heat transfer for the liquid. If we put $u(x, t) = y(x)\exp(-\lambda t)$ then the problem (1.2) – (1.4) will appear to be consequence of the above problem. Indeed, the condition (1.3) is obtained from (2*) and the condition (1.4) is obtained from (3*) easily. Here $\alpha_1 = \dfrac{c}{k}$, $\beta_2 = -\dfrac{cA + k_1 B}{qM}$ and $\alpha_2 = -\dfrac{k_1 Bc}{qMk}$. Finally, if we put

$$t = \begin{cases} x, & 0 < x < a, \\ \alpha x, & a < x < \pi, \end{cases}$$

then the discontinuity conditions (1.4) and a particular case of (1.2) will appear. This corresponds to the case of nonperfect thermal contact. Since the density is changed at one point in interval, both of the intensity and the instant velocity of heat change at this point. Hence, (1.2) – (1.4) will appear to be consequence of the above problem.

Boundary-value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is of central importance in disiplines ranging from engineering to the geo-sciences.

For example, discontinuous inverse problems appear in electronics for constructing parameters of heterogeneous electronic lines with desirable technical characteristics [26, 27]. After reducing corresponding mathematical model we come to boundary-value problem L where $q(x)$ must be constructed from the given spectral information which describes desirable amplitude and phase characteristics. Spectral information can be used to reconstruct the permittivity and conductivity profiles of one-dimensional discontinuous medium [28]. Boundary-value problems with discontinuties in an interior point also appear in geophysical models for oscillations of the Earth [29]. Here, the main discontinuity is cased by reflection of the shear waves at the base of the crust. Further, it is known that inverse spectral problems play an important role for investigating some nonlinear evolution equations of mathematical physics. Discontinuous inverse problems help to study the blow-up behaviour of solutions for such nonlinear equations. We also note that inverse problem considered here appears in mathematics for investigating spectral properties of some classes of differential, integrodifferential and integral operators.

It must be noted that some special cases of the considered problem (1.2) – (1.4) arise after an application of the method of seperation of variables to the varied assortment of physical problems. For example, some boundary-value problems with discontinuity condition arise in heat and mass transfer problems (see, for example, [31]), in vibrating string problems when the string loaded additionally with point masses (see, for example, [25]) and in diffraction problems (see, for example, [30]). Moreover, some of the problems with boundary conditions depend on the spectral parameter occur in the theory of small vibrations of a damped string and freezing of the liquid (see, for example, [32, 33, 25]).

Furthermore, representation with transformation operator was shown in [17], as in [34] and [35].

In this study, properties of characteristic function of L_0 and asymptotic behaviours of spectral characteristics of considering operator have been given such that the remaining parts are in the space ℓ_2 as in [35].

Moreover three statements of the inverse problem of the reconstruction of the boundary problem from the Weyl function, from the spectral data $\{\lambda_n, \alpha_n\}_{n\geq 0}$ and from two spectra $\{\lambda_n, \mu_n\}_{n\geq 0}$ have been studied. These inverse problems are generalizations of the well known inverse problems for the Sturm – Liouville operator (see [36, 37]).

2. Representation for the solution. We define

$$y_1(x) = y(x), \qquad y_2(x) = (\Gamma_\alpha y)(x) = y'(x) - u(x)\, y(x), \qquad u(x) = C\frac{x^{1-\alpha}}{1-\alpha}$$

and let's write the expression of left-hand side of equation (1.1) as follows:

$$\ell(y) = -\left[(\Gamma_\alpha y)(x)\right]' - u(x)\,(\Gamma_\alpha y)(x) - u^2(x)y + q(x)y = k^2 y. \tag{2.1}$$

Then equation (1.1) reduces to the system

$$\begin{aligned} &y_1' - y_2 = u(x)y_1, \\ &y_2' + k^2 y_1 = -u(x)y_2 - u^2(x)y_1 + q(x)y_1 \end{aligned} \tag{2.2}$$

with the boundary conditions

$$y_1(0) = 0, \qquad \left(\alpha_1 k^2 + \alpha_2\right) y_1(\pi) + \left(\beta_1 k^2 + \beta_2\right) y_2(\pi) = 0 \tag{2.3}$$

and with the jump conditions

$$\begin{aligned} &y_1(a+0) = \beta y_1(a-0), \\ &y_2(a+0) = \beta^{-1} y_2(a-0) - 2\beta^{-} u(a) y_1(a-0). \end{aligned} \tag{2.4}$$

Matrix form of system (2.2)

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} u & 1 \\ -k^2 - u^2 + q & -u \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.5}$$

or $y' = Ay$ such that $A = \begin{pmatrix} u(x) & 1 \\ -k^2 - u^2(x) + q(x) & -u(x) \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Since $x = 0$ is a regular singular end point for equation (2.5), Theorem 2 in [38] (see Remark 1.2, p. 56) extends to interval $[0, \pi]$. For this reason, by [38], there exists only one solution of the system (2.2) which satisfies the initial condititons $y_1(\xi) = \upsilon_1$, $y_2(\xi) = \upsilon_2$ for each $\xi \in [0, \pi]$, $\upsilon = (\upsilon_1, \upsilon_2)^T \in C^2$, especially the initial conditions $y_1(0) = 1$, $y_2(0) = h$.

Definition 2.1. *The first component of the solution of system* (2.2) *which satisfies the initial condititons* $y_1(\xi) = \upsilon_1$, $y_2(\xi) = (\Gamma_\alpha y)(\xi) = \upsilon_2$ *is called the solution of equation* (1.2) *which satisfies the same initial conditions.*

It was shown in [17] by the successive approximations method that (see [37]) the following theorem is true.

Theorem 2.1. *Each solution of system* (2.2) *which satisfying the initial conditions* $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ ik \end{pmatrix}$ *and the jump conditions* (2.4), *has the form*:

for $x < a$

$$y_1 = e^{ikx} + \int_{-x}^{x} K_{11}(x,t) e^{ikt} dt,$$

$$y_2 = ike^{ikx} + b(x)e^{ikx} + \int_{-x}^{x} K_{21}(x,t)\, e^{ikt} dt + ik \int_{-x}^{x} K_{22}(x,t) e^{ikt} dt,$$

for $x > a$

$$y_1 = \beta^+ e^{ikx} + \beta^- e^{ik(2a-x)} + \int\limits_{-x}^{x} K_{11}(x,t) e^{ikt} dt,$$

$$y_2 = ik\beta^+ e^{ikx} - ik\beta^- e^{ik(2a-x)} + b\,(x) \Big[\beta^+ e^{ikx} + \beta^- e^{ik(2a-x)}\Big] +$$

$$+ \int\limits_{-x}^{x} K_{21}(x,t) e^{ikt} dt + ik \int\limits_{-x}^{x} K_{22}(x,t) e^{ikt} dt,$$

where

$$b(x) = -\frac{1}{2} \int\limits_{0}^{x} \left[u^2(s) - q(s)\right] e^{-\frac{1}{2}\int_s^x u(t)dt} ds, \qquad K(x,t) = \begin{pmatrix} K_{11}(x,t) & 0 \\ K_{21}(x,t) & K_{22}(x,t) \end{pmatrix},$$

$$K_{11}\,(x,x) = \frac{\beta^+}{2} u(x),$$

$$K_{21}\,(x,x) = b'(x) - \frac{1}{2} \left\{ \beta^+ \int\limits_{0}^{x} \left[u^2\,(s) - q(s)\right] K_{11}\,(s,s)\, ds + \int\limits_{0}^{x} u\,(s)\, K_{11}\,(s,s)\, ds \right\},$$

$$K_{22}\,(x,x) = -\frac{\beta^+}{2} u(x) - \beta^+ b(x), \qquad \beta^{\pm} = \frac{1}{2}\left(\beta \pm \frac{1}{\beta}\right).$$

3. Properties of the spectrum. In this section, properties of the spectrum of problem L have been given.

Let us denote problem L as L_0 in the case of $C = 0$ and $q(x) \equiv 0$.

When $C = 0$ and $q(x) \equiv 0$, it is easily shown that solution $\varphi_0(x,k)$ satisfying the initial conditions $\varphi_0\,(0,k) = 0$, $(\Gamma_\alpha \varphi_0)\,(0,k) = k$ and the jump conditions (2.4), is shown as

$$\varphi_0(x,k) = \begin{cases} \sin kx & \text{for } \ x < a, \\ \beta^+ \sin kx + \beta^- \sin k\,(2a - x) & \text{for } \ x > a, \end{cases}$$
$$(\Gamma_\alpha \varphi_0)\,(x,k) = \begin{cases} k \cos kx & \text{for } \ x < a, \\ k\beta^+ \cos kx - k\beta^- \cos k\,(2a - x) & \text{for } \ x > a. \end{cases} \tag{3.1}$$

We denote characteristic function, eigenvalues sequence and normalizing constant sequence by $\Delta(k)$, $\{k_n\}$ and $\{\alpha_n\}$, respectively. Denote

$$\Delta(k) = \langle \psi(x,k), \varphi(x,k) \rangle\,, \tag{3.2}$$

where

$$\langle y(x), z(x) \rangle := y\,(x)\,(\Gamma_\alpha z)\,(x) - (\Gamma_\alpha y)\,(x) z(x).$$

We define normalizing constants by

$$\alpha_n = \int_0^\pi \varphi^2(x, k_n)dx + \frac{1}{\rho}\left[\alpha_1\varphi(\pi, k_n) + \beta_1(\Gamma_\alpha\varphi)(\pi, k_n)\right]^2,$$

where $\rho = \alpha_2\beta_1 - \alpha_1\beta_2$.

According to the Liouville formula, $\langle\psi(x,k), \varphi(x,k)\rangle$ does not depend on x.

We shall assume that $\varphi(x,k)$ and $\psi(x,k)$ are solutions of equation (1.2) under the following initial conditions:

$$\varphi(0,k) = 0, \qquad (\Gamma_\alpha\varphi)(0,k) = k, \qquad \psi(\pi,k) = \beta_1 k^2 + \beta_2, \qquad (\Gamma_\alpha\psi)(\pi,k) = -(\alpha_1 k^2 + \alpha_2).$$

Clearly, for each x, functions $\langle\psi(x,k), \varphi(x,k)\rangle$ are entire in k and

$$\Delta(k) = V(\varphi) = U(\psi) = (\alpha_1 k^2 + \alpha_2)\varphi(\pi,k) + (\beta_1 k^2 + \beta_2)(\Gamma_\alpha\varphi)(\pi,k) = \psi(0,k). \tag{3.3}$$

By using the representation of the function $y(x,k)$ for the solution $\varphi(x,k)$:

$$\varphi(x,k) = \varphi_0(x,k) + \int_0^\pi \widetilde{K}_{11}(\pi,t)\sin kt dt \tag{3.4}$$

is obtained.

Lemma 3.1 (Lagrange's formula). *Let $y, z \in D(L_0^*)$. Then*

$$(L_0^* y, z) = \int_0^\pi \ell(y)\overline{z}dx = (y, L_0^* z) + [y, \overline{z}]\left(\Big|_0^{a-0} + \Big|_{a+0}^{\pi}\right),$$

where $[y,\overline{z}]\left(\Big|_0^{a-0} + \Big|_{a+0}^{\pi}\right) = \left[(\Gamma_\alpha\overline{z})(x)y(x) - (\Gamma_\alpha y)(x)\overline{z(x)}\right]\left(\Big|_0^{a-0} + \Big|_{a+0}^{\pi}\right)$.

Proof. We have

$$(L_0^* y, z) = -\int_0^\pi (y' - u\,y)'\,\overline{z}dx - \int_0^\pi u\,(y' - u\,y)\,\overline{z}dx - \int_0^\pi (u^2 - q(x))\,y\overline{z}dx =$$

$$= \int_0^\pi (y' - u\,y)\left(\overline{z}' - u\overline{z}\right)dx - \int_0^\pi (u^2 - q(x))\,y\overline{z}dx - (\Gamma_\alpha y)(x)\overline{z(x)}\left(\Big|_0^{a-0} + \Big|_{a+0}^{\pi}\right) =$$

$$= \int_0^\pi y\ell(\overline{z})\,dx + [y,\overline{z}]\left(\Big|_0^{a-0} + \Big|_{a+0}^{\pi}\right) = (y, L_0^* z) + [y,\overline{z}]\left(\Big|_0^{a-0} + \Big|_{a+0}^{\pi}\right).$$

Lemma 3.2. *The zeros $\{k_n\}$ of the characteristic function coincide with the eigenvalues of the boundary-value problem L. The functions $\varphi(x,k_n)$ and $\psi(x,k_n)$ are eigenfunctions and there exists a sequence $\{\gamma_n\}$ such that*

$$\psi(x,k_n) = \gamma_n\varphi(x,k_n), \quad \gamma_n \neq 0. \tag{3.5}$$

Proof. 1. Let k_0 be a zero of the function $\Delta(k)$. Then by virtue of equation (3.2) and (3.3), $\psi(x,k_0) = \gamma_0\varphi(x,k_0)$ and the functions $\psi(x,k_0), \varphi(x,k_0)$ satisfy the boundary conditions (1.3). Hence, k_0 is an eigenvalue and $\psi(x,k_0), \varphi(x,k_0)$ are eigenfunctions related to k_0.

2. Let k_0 be an eigenvalue of L, and let y_0 be a corresponding eigenfunctions. Then $U(y_0) = V(y_0) = 0$. Clearly $y_0(0) = 0$. Without loss of generality, we put $(\Gamma_\alpha y_0)(0) = ik$. Hence $y_0(x) \equiv \varphi(x,k_0)$. Thus, from equation (3.3), $\Delta(k_0) = V(\varphi(x,k_0)) = V(y_0(x)) = 0$ is obtained.

Lemma 3.3. *Eigenvalues of the problem L are simple and separated.*

Proof. Since $\varphi(x,k)$ and $\psi(x,k)$ are solutions of equation (1.2), it is obtained that

$$-\psi''(x,k) + \left[u'(x) + q(x)\right]\psi(x,k) = k^2\psi(x,k),$$

$$-\varphi''(x,k_n) + \left[u'(x) + q(x)\right]\varphi(x,k_n) = k_n^2\varphi(x,k_n).$$

If first equation is multiplied by $\varphi(x,k_n)$, second equation is multiplied by $\psi(x,k)$ and substracting them side by side and finally integrating over the interval $[0,\pi]$, then the following equality is obtained:

$$\frac{d}{dx}\langle\psi(x,k), \varphi(x,k_n)\rangle = \left(k^2 - k_n^2\right)\psi(x,k)\varphi(x,k_n),$$

$$\langle\psi(x,k), \varphi(x,k_n)\rangle\left[\Big|_0^{a-0} + \Big|_{a+0}^{\pi}\right] = \left(k^2 - k_n^2\right)\int\limits_0^\pi \psi(x,k)\varphi(x,k_n)dx.$$

If jump conditions (1.4) and $\alpha_n = \int\limits_0^\pi \varphi^2(x,k_n)dx + \dfrac{1}{\rho}\left[\alpha_1\varphi(\pi,k_n) + \beta_1(\Gamma_\alpha\varphi)(\pi,k_n)\right]^2$ are considered, it is obtained that

$$\int\limits_0^\pi \psi(x,k_n)\varphi(x,k_n)\,dx + \frac{1}{\rho}\left[\alpha_1\varphi(\pi,k_n) + \beta_1(\Gamma_\alpha\varphi)(\pi,k_n)\right],$$

$$\left[\alpha_1\psi(\pi,k_n) + \beta_1(\Gamma_\alpha\psi)(\pi,k_n)\right] = -\dot{\Delta}(k_n) \qquad \text{as} \quad k \to k_n.$$

From Lemma 3.2, we get that

$$\alpha_n\gamma_n = -\dot{\Delta}(k_n). \tag{3.6}$$

It is obvious that $\dot{\Delta}(k_n) \neq 0$.

Since the function $\Delta(k)$ is an entire function ok k, the zeros of $\Delta(k)$ are separated.

Lemma 3.3 is proved.

Now, let problems be

$$L\colon \begin{cases} -y'' + [u'(x) + q(x)]\,y = \lambda y, \\ (\Gamma_\alpha y)(0) - hy(0) = 0, \\ (\beta_1\lambda + \beta_2)(\Gamma_\alpha y)(\pi) + (\alpha_1\lambda + \alpha_2)y(\pi) = 0, \\ \begin{cases} y(a+0) = \beta y(a-0), \\ (\Gamma_\alpha y)(a+0) = \beta^{-1}(\Gamma_\alpha y)(a-0) - 2\beta^{-}u(a)y(a-0), \end{cases} \end{cases}$$

$$\widetilde{L}\colon \begin{cases} -y'' + [u'(x) + q(x)]\, y = \mu y, \\ (\Gamma_\alpha y)(0) - hy(0) = 0, \\ (\widetilde{\beta}_1 \mu + \widetilde{\beta}_2)(\Gamma_\alpha y)(\pi) + (\widetilde{\alpha}_1 \mu + \widetilde{\alpha}_2) y(\pi) = 0, \\ \begin{cases} y(a+0) = \beta y(a-0), \\ (\Gamma_\alpha y)(a+0) = \beta^{-1} (\Gamma_\alpha y)(a-0), \end{cases} \end{cases}$$

where $\alpha_1 \widetilde{\beta}_1 = \widetilde{\alpha}_2 \beta_2$, $\alpha_1 \widetilde{\beta}_2 = \widetilde{\alpha}_2 \beta_1$, $\alpha_2 \widetilde{\beta}_1 = \widetilde{\alpha}_1 \beta_2$. Let $\{\lambda_n\}_{n \geq 0}$ and $\{\mu_n\}_{n \geq 0}$ be the eigenvalues of the problems L and $\widetilde{L}$ respectively.

Lemma 3.4. *The eigenvalues of the problems L and $\widetilde{L}$ are interlace, i.e.,*

$$\lambda_n < \mu_n < \lambda_{n+1}, \quad \textit{if} \quad \alpha_2 \widetilde{\beta}_2 < \widetilde{\alpha}_2 \beta_2 \qquad \textit{and} \qquad \mu_n < \lambda_n < \mu_{n+1}, \quad \textit{if} \quad \alpha_2 \widetilde{\beta}_2 > \widetilde{\alpha}_2 \beta_2, \quad n \geq 0, \tag{3.7}$$

where $\alpha_1 \widetilde{\alpha}_2 > \widetilde{\alpha}_1 \alpha_2$ and $\beta_1 \widetilde{\beta}_2 > \widetilde{\beta}_1 \beta_2$.

Proof. As in the proof of Lemma 3.3, we get that

$$\frac{d}{dx} \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle = (\lambda - \mu)\, \varphi(x, \lambda)\, \varphi(x, \mu)$$

and from here

$$(\lambda - \mu) \int_0^\pi \varphi(x, \lambda)\, \varphi(x, \mu)\, dx = \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle \left[\Big|_0^{a-0} + \Big|_{a+0}^{\pi} \right] =$$

$$= \varphi(\pi, \lambda)(\Gamma_\alpha \varphi)(\pi, \mu) - (\Gamma_\alpha \varphi)(\pi, \lambda)\, \varphi(\pi, \mu) =$$

$$= \frac{\widetilde{\alpha}_1 \alpha_2 - \alpha_1 \widetilde{\alpha}_2}{\alpha_2 \widetilde{\beta}_2 - \widetilde{\alpha}_2 \beta_2} (\lambda - \mu) \varphi(\pi, \lambda) \varphi(\pi, \mu) +$$

$$+ \frac{\widetilde{\beta}_1 \beta_2 - \beta_1 \widetilde{\beta}_2}{\alpha_2 \widetilde{\beta}_2 - \widetilde{\alpha}_2 \beta_2} (\lambda - \mu)(\Gamma_\alpha \varphi)(\pi, \lambda)(\Gamma_\alpha \varphi)(\pi, \mu) +$$

$$+ \frac{1}{\alpha_2 \widetilde{\beta}_2 - \widetilde{\alpha}_2 \beta_2} \left[\widetilde{\Delta}(\lambda)\, \Delta(\mu) - \widetilde{\Delta}(\mu)\, \Delta(\lambda) \right].$$

Hence

$$(\lambda - \mu) \int_0^\pi \varphi(x, \lambda)\, \varphi(x, \mu)\, dx = \frac{\widetilde{\alpha}_1 \alpha_2 - \alpha_1 \widetilde{\alpha}_2}{\alpha_2 \widetilde{\beta}_2 - \widetilde{\alpha}_2 \beta_2} (\lambda - \mu) \varphi(\pi, \lambda) \varphi(\pi, \mu) +$$

$$+ \frac{\widetilde{\beta}_1 \beta_2 - \beta_1 \widetilde{\beta}_2}{\alpha_2 \widetilde{\beta}_2 - \widetilde{\alpha}_2 \beta_2} (\lambda - \mu)(\Gamma_\alpha \varphi)(\pi, \lambda)(\Gamma_\alpha \varphi)(\pi, \mu) +$$

$$+ \frac{1}{\alpha_2 \widetilde{\beta}_2 - \widetilde{\alpha}_2 \beta_2} \left[\frac{\widetilde{\Delta}(\lambda) - \widetilde{\Delta}(\mu)}{\lambda - \mu} \Delta(\mu) - \frac{\Delta(\lambda) - \Delta(\mu)}{\lambda - \mu} \widetilde{\Delta}(\mu) \right].$$

As $\mu \to \lambda$

$$\int\limits_0^\pi \varphi^2(x,\lambda)\,dx = \frac{1}{\alpha_2\widetilde{\beta}_2 - \widetilde{\alpha}_2\beta_2}\times$$

$$\times\Big[(\widetilde{\alpha}_1\alpha_2 - \alpha_1\widetilde{\alpha}_2)\,\varphi^2(\pi,\lambda) + \left(\widetilde{\beta}_1\beta_2 - \beta_1\widetilde{\beta}_2\right)(\Gamma\varphi)^2(\pi,\lambda) +$$

$$+\dot{\widetilde{\Delta}}(\lambda)\,\Delta(\lambda) - \dot{\Delta}(\lambda)\,\widetilde{\Delta}(\lambda)\Big], \tag{3.8}$$

where $\dot{\Delta}(\lambda) = \dfrac{d}{d\lambda}\Delta(\lambda)$, $\dot{\widetilde{\Delta}}(\lambda) = \dfrac{d}{d\lambda}\widetilde{\Delta}(\lambda)$. From equation (3.8), for $-\infty < \lambda < \infty$, if $\widetilde{\Delta}(\lambda) \neq 0$,

$$\frac{1}{\widetilde{\Delta}^2(\lambda)}\left[\int\limits_0^\pi \varphi^2(x,\lambda)\,dx - \frac{(\widetilde{\alpha}_1\alpha_2 - \alpha_1\widetilde{\alpha}_2)\varphi^2(\pi,\lambda) + (\widetilde{\beta}_1\beta_2 - \beta_1\widetilde{\beta}_2)(\Gamma_\alpha\varphi)^2(\pi,\lambda)}{\widetilde{\alpha}_2\beta_2 - \alpha_2\widetilde{\beta}_2}\right] =$$

$$= -\frac{1}{(\widetilde{\alpha}_2\beta_2 - \alpha_2\widetilde{\beta}_2)}\frac{d}{d\lambda}\left(\frac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}\right)$$

is obtained.

If $\alpha_2\widetilde{\beta}_2 < \widetilde{\alpha}_2\beta_2$ then $\dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)}$ is monotonically decreasing in the set of $R\setminus\{\mu_n, n\geq 0\}$. Thus it is obvious that $\lim\limits_{\lambda\to\mu_n^{\pm 0}}\dfrac{\Delta(\lambda)}{\widetilde{\Delta}(\lambda)} = \pm\infty$.

When $\alpha_2\widetilde{\beta}_2 > \widetilde{\alpha}_2\beta_2$, if we write the equality (3.8) as

$$\frac{1}{\Delta^2(\lambda)}\left[\int\limits_0^\pi \varphi^2(x,\lambda)\,dx - \frac{(\widetilde{\alpha}_1\alpha_2 - \alpha_1\widetilde{\alpha}_2)\varphi^2(\pi,\lambda) + (\widetilde{\beta}_1\beta_2 - \beta_1\widetilde{\beta}_2)(\Gamma_\alpha\varphi)^2(\pi,\lambda)}{\alpha_2\widetilde{\beta}_2 - \widetilde{\alpha}_2\beta_2}\right] =$$

$$= -\frac{1}{\alpha_2\widetilde{\beta}_2 - \widetilde{\alpha}_2\beta_2}\frac{d}{d\lambda}\left(\frac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)}\right),$$

for $-\infty < \lambda < \infty$, $\Delta(\lambda) \neq 0$, we get that the function $\dfrac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)}$ is monotonically decreasing in $R\setminus\{\lambda_n, n\geq 0\}$ and it is clear that $\lim\limits_{\lambda\to\lambda_n^{\pm 0}}\dfrac{\widetilde{\Delta}(\lambda)}{\Delta(\lambda)} = \pm\infty$. From here, we obtain (3.7).

Theorem 3.1. *The eigenvalues k_n, eigenfunctions $\varphi(x,k_n)$ and the normalizing numbers α_n of problem L have the following asymptotic behaviour*:

$$\sqrt{\lambda_n} = k_n = k_n^0 + \frac{d_n}{k_n^0} + \frac{\delta_n}{k_n^0}, \tag{3.9}$$

$$\varphi(x,k_n) = \beta^+\sin k_n^0 x + \beta^-\sin k_n^0(2a - x) + \frac{s_n}{k_n^0} + \frac{b_n}{k_n^0}, \tag{3.10}$$

$$\alpha_n = \frac{a}{2} + \left[\left(\beta^+\right)^2 + \left(\beta^-\right)^2\right]\left(\frac{\pi - a}{2}\right) - \left(\beta^+\right)\left(\beta^-\right)\cos 2k_n^0 a + \frac{\gamma_n}{n} + \frac{\xi_n}{n}, \tag{3.11}$$

where $\delta_n,\, s_n,\, \xi_n \in \ell_2,\, b_n, d_n, \gamma_n \in \ell_\infty$ *and* k_n^0 *are roots of* $\Delta_0(k) := k^3\big[\beta^+ \cos k\pi - \beta^- \cos k(2a - \pi)\big]$ *and* $k_n^0 = n + h_n,\, h_n \in \ell_\infty$.

Proof. Using (3.1), (3.3) and (3.4), we get

$$\Delta(k) = (\alpha_1 k^2 + \alpha_2)\varphi_0(\pi, k) + (\beta_1 k^2 + \beta_2)(\Gamma_\alpha \varphi_0)(\pi, k) +$$

$$+(\alpha_1 k^2 + \alpha_2)\int_0^\pi \widetilde{K}_{11}(\pi, t)\sin kt dt +$$

$$+(\beta_1 k^2 + \beta_2)\left[\int_0^\pi \widetilde{K}_{21}(\pi, t)\sin kt dt + \int_0^\pi \widetilde{K}_{22}(\pi, t)\cos kt dt\right] =$$

$$= (\alpha_1 k^2 + \alpha_2)\big(\beta^+ \sin k\pi + \beta^- \sin k(2a - \pi)\big) +$$

$$+(\beta_1 k^2 + \beta_2)\big(k\beta^+ \cos k\pi - k\beta^- \cos k(2a - \pi)\big) + k^3 O\left(\frac{\exp|\operatorname{Im} k|\pi}{|k|}\right) =$$

$$= \beta_1 \Delta_0(k) + (\alpha_1 k^2 + \alpha_2)\big(\beta^+ \sin k\pi + \beta^- \sin k(2a - \pi)\big) +$$

$$+\beta_2 k\big(\beta^+ \cos k\pi - \beta^- \cos k(2a - \pi)\big) + k^3 O\left(\frac{\exp|\operatorname{Im} k|\pi}{|k|}\right).$$

Denote

$$G_n = \left\{k\colon\, |k| = \left|k_n^0\right| + \frac{\sigma}{2},\;\; n = 0, \pm 1, \pm 2, \ldots\right\},$$

$$G_\delta = \left\{k\colon\, \left|k - k_n^0\right| \ge \delta,\; n = 0, \pm 1, \pm 2, \ldots, \delta > 0\right\},$$

where δ is sufficiently small positive number $\left(\delta \ll \dfrac{\sigma}{2}\right)$.

Since $|\Delta_0(k)| \ge k^3 C_\delta e^{|\operatorname{Im} k|\pi}$ for $k \in \overline{G_\delta}$ and $|\Delta(k) - \Delta_0(k)| < \dfrac{C_\delta}{2}|k|^3 e^{|\operatorname{Im} k|\pi}$ for sufficiently large values of n and $k \in G_n$, we get

$$|\Delta_0(k)| > C_\delta k^3 e^{|\operatorname{Im} k|\pi} > |\Delta(k) - \Delta_0(k)|.$$

It follows from that for sufficiently large values of n, functions $\Delta_0(k)$ and $\Delta_0(k) + \big(\Delta(k) - \Delta_0(k)\big) = \Delta(k)$ have the same number of zeros counting multiplicities inside contour G_n, according to Rouche's theorem. That is, they have the $(n+1)$ number of zeros: $k_0, k_1, \ldots, k_n$.

Analogously, it is shown by Rouche's theorem that for sufficiently large values of n, function $\Delta(k)$ has a unique of zero inside each circle $\left|k - k_n^0\right| < \delta$.

Since δ is sufficiently small number, representing of $k_n = k_n^0 + \varepsilon_n$ is acquired, where $\lim_{n\to\infty} \varepsilon_n = 0$.

Since numbers k_n are zeros of characteristic function $\Delta(k)$,

$$\Delta(k_n) = (\alpha_1 k_n^2 + \alpha_2)\big(\beta^+ \sin k_n\pi + \beta^- \sin k_n(2a - \pi)\big) +$$

$$+(\beta_1 k^2+\beta_2)\left(k_n\beta^+\cos k_n\pi - k_n\beta^-\cos k_n\left(2a-\pi\right)\right)+O\left(k_n^2\right).$$

From the last equality, we get

$$\beta^+\cos k_n\pi-\beta^-\cos k_n\left(2a-\pi\right)+\frac{\alpha_1}{\beta_1 k_n^2}\left[\beta^+\sin k_n\pi+\beta^-\sin k_n\left(2a-\pi\right)\right]+$$

$$+\frac{\alpha_2}{\beta_1 k_n^3}\left[\beta^+\sin k_n\pi+\beta^-\sin k_n\left(2a-\pi\right)\right]+$$

$$+\frac{\beta_2}{\beta_1 k_n^2}[\beta^+\sin k_n\pi-\beta^-\cos k_n\left(2a-\pi\right)]+O\left(\frac{1}{k_n}\right)=0.$$

If we write $k_n^0+\varepsilon_n$ instead of k_n and use $\Delta_0\left(k_n^0+\varepsilon_n\right)=\dot{\Delta}_0\left(k_n^0\right)\varepsilon_n+o\left(\varepsilon_n\right)$ and also the study [39] (see also [40]) is used then we get that $k_n^0=n+h_n$, where $\sup_n|h_n|<M$. Therefore,

$$\varepsilon_n=\frac{d_n}{n}+\frac{\delta_n}{n},\qquad \delta_n\in\ell_2,\quad d_n\in\ell_\infty.$$

Thus, asymptotic formula (3.9) is true for the eigenvalues k_n of the problem L. Now, let's try to find the asymptotic formula for the eigenfunction

$$\varphi(x,k_n)=\beta^+\sin k_n x+\beta^-\sin k_n\left(2a-x\right)+\int\limits_0^x\widetilde{K}_{11}\left(x,t\right)\sin k_n t\,dt=$$

$$=\beta^+\sin\left(k_n^0+\varepsilon_n\right)x+\beta^-\sin\left(k_n^0+\varepsilon_n\right)\left(2a-x\right)-$$

$$-\frac{1}{k_n^0+\varepsilon_n}\int\limits_0^x\widetilde{K}_{11}(x,t)d\left[\cos\left(k_n^0+\varepsilon_n\right)t\right]dt=$$

$$=\beta^+\sin k_n^0 x+\beta^-\sin k_n^0\left(2a-x\right)-$$

$$-\frac{1}{k_n^0+\varepsilon_n}\left[\widetilde{K}_{11}(x,t)\cos k_n^0 t\right]\left(\Big|_0^{2a-x-0}+\Big|_{2a-x+0}^{x}\right)+$$

$$+\frac{1}{k_n^0+\varepsilon_n}\int\limits_0^x\widetilde{K}'_{11_t}(x,t)\cos k_n^0 t\,dt.$$

Since

$$\widetilde{K}_{11}\left(x,x\right)=\frac{\beta^+}{2}u(x),\qquad \widetilde{K}_{11}\left(x,2a-x+0\right)-\widetilde{K}_{11}\left(x,2a-x-0\right)=\frac{\beta^-}{2}u(x),$$

$$\int\limits_0^x\widetilde{K}'_{11_t}\left(x,t\right)\cos k_n^0 tdt\in\ell_2.$$

It is obtained that

$$\varphi(x,k_n)=\beta^+\sin k_n^0 x+\beta^-\sin k_n^0\left(2a-x\right)+$$

$$+\frac{\beta^{-}\cos k_n^0(2a-x)-\beta^{+}\cos k_n^0 x}{2k_n^0}u(x)+\frac{b_n}{n}+\frac{s_n}{n}, \qquad s_n\in\ell_2 \quad \text{and} \quad b_n\in\ell_\infty.$$

Then we get the asymptotic formula (3.10). Finally, in order to show that (3.11) is true, using (3.1) and (3.4), we obtain that

$$\alpha_n=\int\limits_0^\pi \varphi^2(x,k_n)dx+\frac{1}{\rho}\left[\alpha_1\varphi(\pi,k_n)+\beta_1(\Gamma_\alpha\varphi)(\pi,k_n)\right]^2=$$

$$=\int\limits_0^a\left[\sin^2 k_n x dx+\left(\int\limits_0^x \widetilde{K}_{11}(x,t)\sin k_n t dt\right)^2\right]+$$

$$+2\int\limits_0^a \sin k_n x\int\limits_0^x \widetilde{K}_{11}(x,t)\sin k_n t dt dx+$$

$$+\int\limits_a^\pi\left[\left(\beta^{+}\right)^2\sin^2 k_n x+\left(\beta^{-}\right)^2\sin^2 k_n(2a-x)+\int\limits_a^\pi\left(\int\limits_a^x \widetilde{K}_{11}(x,t)\sin k_n t dt\right)^2\right]dx+$$

$$+2\beta^{+}\beta^{-}\int\limits_a^\pi \sin k_n x\sin k_n(2a-x)dx+2\beta^{+}\int\limits_a^\pi \sin k_n x\int\limits_a^x \widetilde{K}_{11}(x,t)\sin k_n t dt dx+$$

$$+2\beta^{-}\int\limits_a^\pi \sin k_n(2a-x)\int\limits_a^x \widetilde{K}_{11}(x,t)\sin k_n t dt dx+\frac{1}{\rho}\left[\alpha_1\varphi(\pi,k_n)+\beta_1(\Gamma_\alpha\varphi)(\pi,k_n)\right]^2=$$

$$=\left[\left(\beta^{+}\right)^2+\left(\beta^{-}\right)^2\right]\left(\frac{\pi-a}{2}\right)+\frac{a}{2}-\beta^{+}\beta^{-}\cos 2k_n a+\frac{\gamma_n}{n}+\frac{\xi_n}{n}, \qquad \gamma_n\in\ell_\infty, \quad \xi_n\in\ell_2.$$

4. Inverse problem. Let $\Phi(x,k)$ be solution of (1.3) under the conditions

$$U(\Phi)=\Phi(0,k)=1, \qquad V(\Phi)=\left(\alpha_1 k^2+\alpha_2\right)\Phi(\pi,k)+\left(\beta_1 k^2+\beta_2\right)(\Gamma_\alpha\Phi)(\pi,k)=0$$

and the jump conditions (1.5). $C(x,k)$ be solution of (2.2) with the conditions $C(0,k)=1$, $(\Gamma_\alpha C)(0,k)=0$ and the jump conditions (2.4). It is clear that the functions $\psi(x,k)$ and $C(x,k)$ are entire in k. Then the function $\psi(x,k)$ can be represented as follows:

$$\psi(x,k)=\frac{1}{k}(\Gamma_\alpha\psi)(0,k)\varphi(x,k)+\Delta(k)C(x,k)$$

or

$$\frac{1}{\Delta(k)}\psi(x,k)=\frac{(\Gamma_\alpha\psi)(0,k)}{k\Delta(k)}\varphi(x,k)+C(x,k). \tag{4.1}$$

Denote

$$M(k):=\frac{(\Gamma_\alpha\psi)(0,k)}{k\Delta(k)}. \tag{4.2}$$

It is clear that

$$\Phi(x,k) = M(k)\varphi(x,k) + C\,(x,k). \tag{4.3}$$

The function $\Phi(x,k)$ is called the Weyl solution and the function $M(k)$ is called the Weyl function for the boundary-value problem L.

The Weyl solution and Weyl function are meromorphic functions with respect to k having poles in the spectrum of the problem L.

It follows from (4.1) and (4.2) that

$$\Phi(x,k) = \frac{\psi(x,k)}{\Delta\,(k)} \qquad \text{and} \qquad (\Gamma_\alpha\Phi)\,(0,k) = \frac{(\Gamma_\alpha\psi)\,(0,k)}{k\Delta(k)} = M(k). \tag{4.4}$$

Note that by virtue of equalities $\langle C(x,k), \varphi(x,k)\rangle \equiv 1$, (4.2) and (4.3) we have

$$\langle \Phi(x,k), \varphi(x,k)\rangle \equiv k, \qquad \langle \psi(x,k), \varphi\,(x,k)\rangle \equiv k\Delta(k). \tag{4.5}$$

Theorem 4.1. *The following representation holds*:

$$M(k) = \frac{1}{\alpha_0\,(k-k_0)} + \sum_{n=1}^{\infty}\left\{\frac{1}{\alpha_n\,(k-k_n)} + \frac{1}{\alpha_n^0 k_n^0}\right\}. \tag{4.6}$$

Proof. Let's write a representation solution $\psi(x,k) = -(\beta_1 k^2+\beta_2)C(x,k)+(\alpha_1 k^2+\alpha_2)S(x,k)$ as $\varphi\,(x,k)$:

for $x > a$

$$\psi(x,k) = -\left(\beta_1 k^2+\beta_2\right)\cos k\,(\pi - x) + \left(\alpha_1 k^2+\alpha_2\right)\sin k\,(\pi - x) +$$

$$+\int\limits_0^{\pi-x} \widetilde{N}_{11}(x,t)\left[-\left(\beta_1 k^2+\beta_2\right)\cos kt + \left(\alpha_1 k^2+\alpha_2\right)\sin kt\right]dt,$$

$$(\Gamma_\alpha\psi)\,(x,k) = -k\left[\left(\beta_1 k^2+\beta_2\right)\sin k\,(\pi - x) + \left(\alpha_1 k^2+\alpha_2\right)\cos k\,(\pi - x)\right] -$$

$$-b(x)\left[\left(\beta_1 k^2+\beta_2\right)\cos k\,(\pi - x) - \left(\alpha_1 k^2+\alpha_2\right)\sin k\,(\pi - x)\right] +$$

$$+\int\limits_0^{\pi-x} \widetilde{N}_{21}(x,t)\left[-\left(\beta_1 k^2+\beta_2\right)\cos kt + \left(\alpha_1 k^2+\alpha_2\right)\sin kt\right]dt +$$

$$+\int\limits_0^{\pi-x} k\widetilde{N}_{22}(x,t)\left[\left(\beta_1 k^2+\beta_2\right)\sin kt + \left(\alpha_1 k^2+\alpha_2\right)\cos kt\right]dt,$$

for $x < a$

$$\psi(x,k) = \beta^+\left[-\left(\beta_1 k^2+\beta_2\right)\cos k\,(\pi - x) + \left(\alpha_1 k^2+\alpha_2\right)\sin k\,(\pi - x)\right] +$$

$$+\beta^-\left[-\left(\beta_1 k^2+\beta_2\right)\cos k\,(\pi - 2a + x) + \left(\alpha_1 k^2+\alpha_2\right)\sin k\,(\pi - 2a + x)\right] +$$

$$+\int_0^{\pi-x}\widetilde{N}_{11}(x,t)\left[-\left(\beta_1k^2+\beta_2\right)\cos kt+\left(\alpha_1k^2+\alpha_2\right)\sin kt\right]dt,$$

$$(\Gamma_\alpha\psi)(x,k)=-k\beta^+\left[\left(\beta_1k^2+\beta_2\right)\sin k(\pi-x)+\left(\alpha_1k^2+\alpha_2\right)\cos k(\pi-x)\right]+$$

$$+k\beta^-\left[\left(\beta_1k^2+\beta_2\right)\cos k(\pi-2a+x)-\left(\alpha_1k^2+\alpha_2\right)\sin k(\pi-2a+x)\right]+$$

$$+b(x)\beta^+\left[-\left(\beta_1k^2+\beta_2\right)\cos k(\pi-x)+\left(\alpha_1k^2+\alpha_2\right)\sin k(\pi-x)\right]+$$

$$+b(x)\beta^-\left[\left(\beta_1k^2+\beta_2\right)\cos k(\pi-2a+x)-\left(\alpha_1k^2+\alpha_2\right)\sin k(\pi-2a+x)\right]+$$

$$+\int_0^{\pi-x}\widetilde{N}_{21}(x,t)\left[-\left(\beta_1k^2+\beta_2\right)\cos kt+\left(\alpha_1k^2+\alpha_2\right)\sin kt\right]dt+$$

$$+\int_0^{\pi-x}k\widetilde{N}_{22}(x,t)\left[\left(\beta_1k^2+\beta_2\right)\sin kt+\left(\alpha_1k^2+\alpha_2\right)\cos kt\right]dt,$$

where $\widetilde{N}_{ij}(x,t)=N_{ij}(x,t)-N_{ij}(x,-t)$, $i,j=1,2$. In the case of $C=0$ and $q(x)\equiv 0$, denote the solutions with $\psi_{01}(x,k)$ and $\psi_{02}(x,k)$, so we have

$$\psi(x,k)=\Psi_{01}(x,k)+f_1,$$

$$(\Gamma_\alpha\psi)(x,k)=(\Gamma_\alpha\Psi_{02})(x,k)+f_2,$$

where

$$f_1=\int_0^{\pi-x}\widetilde{N}_{11}(x,t)\left[-\left(\beta_1k^2+\beta_2\right)\cos kt+\left(\alpha_1k^2+\alpha_2\right)\sin kt\right]dt,$$

$$f_2=b(x)\begin{bmatrix}\beta^+\left[-\left(\beta_1k^2+\beta_2\right)\cos k(\pi-x)+\left(\alpha_1k^2+\alpha_2\right)\sin k(\pi-x)\right]+\\ +\beta^-\left[-\left(\beta_1k^2+\beta_2\right)\cos k(\pi-2a+x)+\left(\alpha_1k^2+\alpha_2\right)\sin k(\pi-2a+x)\right]\end{bmatrix}+$$

$$+\int_0^{\pi-x}\widetilde{N}_{21}(x,t)\left[-\left(\beta_1k^2+\beta_2\right)\cos kt+\left(\alpha_1k^2+\alpha_2\right)\sin kt\right]dt+$$

$$+\int_0^{\pi-x}k\widetilde{N}_{22}(x,t)\left[\left(\beta_1k^2+\beta_2\right)\sin kt+\left(\alpha_1k^2+\alpha_2\right)\cos kt\right]dt.$$

On the other hand, we can write

$$M(k)-M_0(k)=\frac{(\Gamma_\alpha\psi)(0,k)}{k\psi(0,k)}-\frac{(\Gamma_\alpha\psi_0)(0,k)}{k\psi_0(0,k)}=\frac{f_2}{k\Delta(k)}-\frac{f_1}{\Delta(k)}M_0(k).$$

Since $\lim_{|k|\to\infty} e^{-|\operatorname{Im} k|\pi} |f_i(k)| = 0$ and $\Delta(k) > C_\delta e^{|\operatorname{Im} k|\pi}$ for $k \in G_\delta$, the equality

$$\frac{f_2}{k\Delta(k)} - \frac{f_1}{\Delta(k)} M_0(k)$$

yields

$$\limsup_{|k|\to\infty \; k\in G_\delta} |M(k) - M_0(k)| = 0. \tag{4.7}$$

Weyl function $M(k)$ is meromorphic with respect to poles k_n. Using (3.3), (4.1) and Lemma 3.2, we calculate that

$$\begin{aligned} \operatorname*{Re}_{k=k_n} s M(k) &= \frac{(\Gamma_\alpha \psi)(0, k_n)}{k_n \dot{\Delta}(k_n)} = -\frac{1}{\alpha_n}, \\ \operatorname*{Re}_{k=k_n^0} s M_0(k) &= \frac{(\Gamma_\alpha \psi_0)\left(0, k_n^0\right)}{k_n^0 \dot{\Delta}(k_n^0)} = -\frac{1}{\alpha_n^0}. \end{aligned} \tag{4.8}$$

Consider the contour integral

$$I_n(k) = \frac{1}{2\pi i} \int\limits_{\Gamma_n} \frac{M(\mu) - M_0(\mu)}{k - \mu} d\mu, \qquad k \in \operatorname{int} \Gamma_n.$$

By virtue of (4.7) , we have $\lim_{n\to\infty} I_n(k) = 0$. On the other hand, the residue theorem and (4.8) yield

$$I_n(k) = -M(k) + M_0(k) + \sum_{k_n \in int\Gamma_n} \frac{1}{\alpha_n (k - k_n)} - \sum_{k_n^0 \in int\Gamma_n} \frac{1}{\alpha_n^0 (k - k_n^0)}.$$

Therefore, as $n \to \infty$, we get

$$M(k) = M_0(k) + \sum_{n=-\infty}^{n=+\infty} \frac{1}{\alpha_n (k - k_n)} + \sum_{n=-\infty}^{n=+\infty} \frac{1}{\alpha_n^0 (k - k_n^0)}.$$

It follows from the form of the function $M_0(k)$ that

$$M_0(k) = \frac{1}{\alpha_n^0 k} + \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^0} \left(\frac{1}{k - k_n^0} + \frac{1}{k_n^0} \right).$$

The composition of the last two equalities yields (4.6).

Theorem 4.1 is proved.

Let us formulate a theorem on the uniqueness of a solution of the inverse problem with the Weyl function. For this purpose, parallel with L, we consider the boundary-value problem $\widetilde{L}$ of the same form but with different potential $\widetilde{q}(x)$. It is asumed in what follows that if a certain symbol α denotes an object related to the problem L, then $\widetilde{\alpha}$ denotes the corresponding object related to the problem $\widetilde{L}$.

Theorem 4.2. *If $M(k) = \widetilde{M}(k)$ then $L = \widetilde{L}$. Thus the specification of the Weyl function uniquely determines the operator.*

Proof. Let us define the matrix $P(x,k)=[P_{jk}(x,k)]_{j,k=1,2}$ by the formula

$$P(x,k)\begin{pmatrix}\widetilde{\varphi} & \widetilde{\Phi}\\ \Gamma_\alpha\widetilde{\varphi} & \Gamma_\alpha\widetilde{\Phi}\end{pmatrix}=\begin{pmatrix}\varphi & \Phi\\ \Gamma_\alpha\varphi & \Gamma_\alpha\Phi\end{pmatrix}. \tag{4.9}$$

Using (4.9) and (4.5) we calculate

$$\begin{gathered}P_{11}(x,k)=-\frac{1}{k}\Big[\varphi(x,k)\left(\Gamma_\alpha\widetilde{\Phi}\right)(x,k)-\Phi(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k)\Big],\\ P_{12}(x,k)=-\frac{1}{k}\Big[\Phi(x,k)\widetilde{\varphi}(x,k)-\varphi(x,k)\widetilde{\Phi}(x,k)\Big],\\ P_{21}(x,k)=-\frac{1}{k}\Big[\left(\Gamma_\alpha\varphi\right)(x,k)\left(\Gamma_\alpha\widetilde{\Phi}\right)(x,k)-\left(\Gamma_\alpha\Phi\right)(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k)\Big],\\ P_{22}(x,k)=-\frac{1}{k}\Big[\left(\Gamma_\alpha\Phi\right)(x,k)\widetilde{\varphi}(x,k)-\left(\Gamma_\alpha\varphi\right)(x,k)\widetilde{\Phi}(x,k)\Big]\end{gathered} \tag{4.10}$$

and

$$\begin{gathered}\varphi(x,k)=P_{11}(x,k)\widetilde{\varphi}(x,k)+P_{12}(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k),\\ \left(\Gamma_\alpha\varphi\right)(x,k)=P_{21}(x,k)\widetilde{\varphi}(x,k)+P_{22}(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k),\\ \Phi(x,k)=P_{11}(x,k)\widetilde{\Phi}(x,k)+P_{12}(x,k)\left(\Gamma_\alpha\widetilde{\Phi}\right)(x,k),\\ \left(\Gamma_\alpha\Phi\right)(x,k)=P_{21}(x,k)\widetilde{\Phi}(x,k)+P_{22}(x,k)\left(\Gamma_\alpha\widetilde{\Phi}\right)(x,k).\end{gathered} \tag{4.11}$$

It follows from (4.10), (4.2) and (4.5)

$$P_{11}(x,k)=1+\frac{1}{\Delta(k)}\Big[\varphi(x,k)\left(\left(\Gamma_\alpha\widetilde{\Psi}\right)(x,k)-\left(\Gamma_\alpha\Psi\right)(x,k)\right)-$$

$$-\Psi(x,k)\left(\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k)-\left(\Gamma_\alpha\varphi\right)(x,k)\right)\Big],$$

$$P_{12}(x,k)=\frac{1}{k\Delta(k)}\Big[\Psi(x,k)\widetilde{\varphi}(x,k)-\varphi(x,k)\widetilde{\Psi}(x,k)\Big],$$

$$P_{21}(x,k)=\frac{1}{k\Delta(k)}\Big[\left(\Gamma_\alpha\varphi\right)(x,k)\left(\Gamma_\alpha\widetilde{\Psi}\right)(x,k)-\left(\Gamma_\alpha\Psi\right)(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k)\Big],$$

$$P_{22}(x,k)=1+\frac{1}{k\Delta(k)}\Big[\left(\Gamma_\alpha\Psi\right)(x,k)\left(\widetilde{\varphi}(x,k)-\varphi(x,k)\right)-\left(\Gamma_\alpha\varphi\right)(x,k)\left(\widetilde{\Psi}(x,k)-\Psi(x,k)\right)\Big].$$

With respect to (4.10) and (4.2), for each fixed x, the functions $P_{jk}(x,k)$ are meromorphic in k with poles in the points k_n and $\widetilde{k}_n$. It follows from the representations of the solutions $\Psi(x,k)$ and $\varphi(x,k)$ that

$$\lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{11}(x,k)-1| = \lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{12}(x,k)| =$$

$$= \lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{22}(x,k)-1| = \lim_{\substack{k\to\infty\\ k\in G_\delta}} \max_{0\le x\le\pi} |P_{21}(x,k)| = 0. \tag{4.12}$$

According to (4.2), (4.3) we have

$$P_{11}(x,k) = -\frac{1}{k}\Big[\varphi(x,k)\left(\Gamma_\alpha\widetilde{C}\right)(x,k) - C(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k) +$$

$$+\left(\widetilde{M}(k)-M(k)\right)\varphi(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k)\Big],$$

$$P_{12}(x,k) = -\frac{1}{k}\Big[\widetilde{\varphi}(x,k)C(x,k) - \widetilde{C}(x,k)\varphi(x,k) + \left(M(k)-\widetilde{M}(k)\right)\varphi(x,k)\widetilde{\varphi}(x,k)\Big],$$

(4.13)

$$P_{21}(x,k) = -\frac{1}{k}\Big[\left(\Gamma_\alpha\varphi\right)(x,k)\left(\Gamma_\alpha\widetilde{C}\right)(x,k) - \left(\Gamma_\alpha C\right)(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k)\Big] -$$

$$-\frac{1}{k}\left[\left(\widetilde{M}(k)-M(k)\right)\left(\Gamma_\alpha\varphi\right)(x,k)\left(\Gamma_\alpha\widetilde{\varphi}\right)(x,k)\right],$$

$$P_{22}(x,k) = -\frac{1}{k}\Big[\widetilde{\varphi}(x,k)\left(\Gamma_\alpha C\right)(x,k) - \widetilde{C}(x,k)\left(\Gamma_\alpha\varphi\right)(x,k) +$$

$$+\left(M(k)-\widetilde{M}(k)\right)\left(\Gamma_\alpha\varphi\right)(x,k)\widetilde{\varphi}(x,k)\Big].$$

Thus if $M(k)=\widetilde{M}(k)$ then the functions $P_{jk}(x,k)$ are entire in k for each fixed x. Together with (4.12) we get that

$$P_{11}(x,k)\equiv 1, \qquad P_{12}(x,k)\equiv 0, \qquad P_{21}(x,k)\equiv 0, \qquad P_{22}(x,k)\equiv 1.$$

Substituting into (4.11), we get

$$\varphi(x,k)\equiv\widetilde{\varphi}(x,k), \qquad (\Gamma_\alpha\varphi)(x,k)\equiv(\Gamma_\alpha\widetilde{\varphi})(x,k),$$

$$\Phi(x,k)\equiv\widetilde{\Phi}(x,k), \qquad (\Gamma_\alpha\Phi)(x,k)\equiv\left(\Gamma_\alpha\widetilde{\Phi}\right)(x,k)$$

for all x and k. Consequently $L=\widetilde{L}$.

Theorem 4.2 is proved.

Theorem 4.3. *If $k_n=\widetilde{k}_n$, $\alpha_n=\widetilde{\alpha}_n$, $n\ge 0$, then $L=\widetilde{L}$. Thus, the specification of the spectral data $\{k_n,\alpha_n\}_{n\ge 0}$ uniquely determines the operator.*

Proof. Since

$$M(k) = \frac{1}{\alpha_0\,(k-k_0)} + {\sum_{n=1}^{\infty}}'\left\{\frac{1}{\alpha_n\,(k-k_n)} + \frac{1}{\alpha_n^0 k_n^0}\right\},$$

$$\widetilde{M}(k) = \frac{\widetilde{h}}{\widetilde{\alpha}_0\left(\widetilde{k}-\widetilde{k}_0\right)} + {\sum_{n=1}^{\infty}}'\left\{\frac{1}{\widetilde{\alpha}_n\left(\widetilde{k}-\widetilde{k}_n\right)} + \frac{1}{\widetilde{\alpha}_n^0\widetilde{k}_n^0}\right\} \tag{4.14}$$

under the hypothesis of the theorem and in view of (4.13), we get that $M(k) = \widetilde{M}(k)$ and consequently by Theorem 4.2, $L = \widetilde{L}$.

Theorem 4.4. *If* $k_n = \widetilde{k}_n$, $\mu_n = \widetilde{\mu}_n$, $n \geq 0$, *then* $L = \widetilde{L}$.

Proof. In view of properties of functions$\Delta(k)$ and $\widetilde{\Delta}(k)$, it is clear that $\lim_{k\to\infty} \dfrac{\Delta(k)}{\widetilde{\Delta}(k)} = 1$. Under the hypothesis $k_n = \widetilde{k}_n$, $\Delta(k)$ and $\widetilde{\Delta}(k)$ functions are entire we get that $\Delta(k) = \widetilde{\Delta}(k)$. From Lemma 3.2, we have $\widetilde{\psi}\left(x, \widetilde{k}_n\right) = \widetilde{\gamma}_n \widetilde{\varphi}\left(x, \widetilde{k}_n\right) = \widetilde{\gamma}_n \widetilde{\varphi}(x, k_n)$ and $\widetilde{\Psi}\left(x, \widetilde{k}_n\right) = \widetilde{\Psi}(x, k_n) = = \gamma_n \widetilde{\varphi}(x, k_n)$. It follows that $\gamma_n = \widetilde{\gamma}_n$ and so $\alpha_n = \widetilde{\alpha}_n$. Consequently by Theorem 4.3, $L = \widetilde{L}$.

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