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A STUDY ON TENSOR PRODUCT SURFACES IN LOW-DIMENSIONAL EUCLIDEAN SPACES

ДОСЛІДЖЕННЯ ПОВЕРХОНЬ ТЕНЗОРНОГО ДОБУТКУ В ЕВКЛІДОВИХ ПРОСТОРАХ МАЛОЇ РОЗМІРНОСТІ

We consider a special case for curves in two-, three-, and four-dimensional Euclidean spaces and obtain a necessary and sufficient condition for the tensor product surfaces of the planar unit circle centered at the origin and these curves to have a harmonic Gauss map.

Розглянуто спеціальний випадок для кривих у дво-, три- та чотиривимірних евклідових просторах і отримано необхідну та достатню умову, за якої поверхні тензорного добутку плоского одиничного кола з центром у початку координат та цих кривих мають гармонічне гауссове зображення.

1. Introduction. Tensor product of two immersions of a given Riemannian manifold is one of the intersting topics in differential geometry. This notion is a generalization of the quadratic representation of a submanifold. In spatial case, a tensor product surface is obtained by taking the tensor product of two curves. In [7], many properties such as minimality and totally reality are studied for tensor product of two planar curves.

We also know that the Gauss map is one of the important topics in study of surfaces. In the other hand harmonic functions have very properties in advanced analysis. Therefore we want search for tensor product surfaces that have harmonic Gauss map in special cases. In this paper, since the case of general dimension involves rather tedious calculations, we will restrict ourselves to low dimensions.

2. Preliminaries. In this section we recall some standard definitions and results from Riemannian geometry. Let M be an n-dimensional Riemannian manifold, $\mathbb{E}^m$ be an m-dimensional Euclidean space and $\varphi\colon M \to \mathbb{E}^m$ be an isometric immersion. Let $\overline{\bigtriangledown}$ be the Levi-Civita connection of $\mathbb{E}^m$ and $\bigtriangledown$ the induced connection on M. We denote the second fundamental form of M in $\mathbb{E}^m$ by II, normal connection in the normal bundle of M by $\bigtriangledown^{\perp}$ and the shape operator in the direction of normal vector field n by A_n. It is well known that the two later notions are related by

$$\langle II(X,Y), n\rangle = \langle A_n\, X, Y\rangle, \tag{1}$$

where X, Y are vector fields tangent to M. For an n-dimensional submanifold M in $\mathbb{E}^m$, M is said to be totally geodesic if $II \equiv 0$. Furthermore, the Gaussian and Weingarten formula are given respectively by

$$\overline{\bigtriangledown}_X Y = \bigtriangledown_X Y + II(X,Y), \tag{2}$$

$$\overline{\bigtriangledown}_X n = -A_n X + \bigtriangledown^{\perp}_X n. \tag{3}$$

Let $G(n,m)$ be the Grassmannian consisting of all oriented n-planes through the origin of $\mathbb{E}^m$ [1]. For an isometric immersion $\varphi\colon M \to \mathbb{E}^m$, the (generalized) Gauss map $\Gamma\colon M \to G(n,m)$ of φ is a smooth map which carries $p \in M$ into the oriented n-plane in $\mathbb{E}^m$ which obtained from the parallel translation of the tangent space of M at p in $\mathbb{E}^m$ [6]. It is well known that $G(n,m)$ is canonically imbedded in $\wedge^n\mathbb{E}^m$, the vector space obtained by the exterior product of n vectors in $\mathbb{E}^m$. It must be said that we can assume $\wedge^n\mathbb{E}^m$ as Euclidean space $\mathbb{E}^N$, where $N = \binom{m}{n}$. So the Gauss map at $p \in M$ can be write as $\Gamma(p) = (e_1 \wedge e_2 \wedge \ldots \wedge e_n)(p)$, where $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_m\}$ is an adapted local orthonormal frame field in $\mathbb{E}^m$ such that $e_1, e_2, \ldots, e_n$ are tangent to M and $e_{n+1}, \ldots, e_m$ are normal to M [3].

We denote by $C^\infty(TM)$ the space of all sections of TM. The (rough) Laplacian of f in $C^\infty(TM)$ is defined by

$$\Delta f = -\sum_i \big(\overline{\nabla}_{e_i}\overline{\nabla}_{e_i} f - \overline{\nabla}_{\nabla_{e_i} e_i} f\big). \tag{4}$$

Note that in this paper smooth can be replace by second differentiability. We also assume that an Euclidean smooth curve $c\colon \mathbb{R} \to \mathbb{E}^n$ with parametrization $c(t) = \big(\alpha_1(t), \alpha_2(t), \ldots, \alpha_n(t)\big)$ is called i-th suitable and denoted by i-s, $1 \le i \le n$, if $\alpha_i(t) \neq 0$ and $\alpha_i'(t) \neq 0$ for $t \in \mathbb{R}$, where from now on we use prime to denote the differential respect to t.

3. A tensor product surface in $\mathbb{E}^4$ with harmonic Gauss map. This section is similar to a part of Section 3 in [2], but because of deficiency in essential assumptions for i-s curve and for development of results to other dimensions in Sections 4 and 5, we state this section with another discipline on normal vector basis.

Let $c_1\colon \mathbb{R} \to \mathbb{E}^2$ be the unit planar circle centered at origin of $\mathbb{E}^2$ with parametrization $c_1(t) = (\cos s, \sin s)$ and $c_2\colon \mathbb{R} \to \mathbb{E}^2$ be a unit speed i-s smooth curve in $\mathbb{E}^2$. Without loss of generality let $c_2(t) = \big(\alpha(t), \beta(t)\big)$ with $\alpha(t) \neq 0, \alpha'(t) \neq 0$ for $t \in \mathbb{R}$, i.e., 1-s curve. The tensor product surface M of two curves c_1 and c_2 is given by

$$f = c_1 \otimes c_2\colon \mathbb{R}^2 \to \mathbb{E}^4,$$

$$f(s,t) = \big(\alpha(t)\cos s, \beta(t)\cos s, \alpha(t)\sin s, \beta(t)\sin s\big).$$

Assume that $f(s,t) = c_1(s) \otimes c_2(t)$ defines an isometric immersion of $\mathbb{R}^2$ into $\mathbb{R}^4$. It follows directly that

$$e_1 = \frac{1}{\|c_2\|}\big(- \alpha(t)\sin s, -\beta(t)\sin s, \alpha(t)\cos s, \beta(t)\cos s\big)$$

and

$$e_2 = \big(\alpha'(t)\cos s, \beta'(t)\cos s, \alpha'(t)\sin s, \beta'(t)\sin s\big)$$

are form an orthonormal basis for tangent space to M. Moreover, an orthonormal basis for normal space to M is given by

$$e_3 = \frac{1}{\|c_2\|}\big(\beta(t)\sin s, -\alpha(t)\sin s, -\beta(t)\cos s, \alpha(t)\cos s\big),$$

$$e_4 = \big(\beta'(t)\cos s, -\alpha'(t)\cos s, \beta'(t)\sin s, -\alpha'(t)\sin s\big).$$

In this section, for simplification of relations, we have to introduce following abbreviations:

$$A = -\frac{\alpha(t)\alpha'(t) + \beta(t)\beta'(t)}{\|c_2\|^2}, \qquad D = \frac{\alpha(t)\beta'(t) - \alpha'(t)\beta(t)}{\|c_2\|}, \qquad E = \alpha''(t)\beta'(t) - \alpha'(t)\beta''(t).$$

Now covariant differentiation with respect to e_1 and e_2 give us

$$\overline{\nabla}_{e_1} e_1 = (A)e_2 + \left(-\frac{D}{\|c_2\|}\right) e_4, \qquad \overline{\nabla}_{e_1} e_2 = (-A)e_1 + \left(\frac{D}{\|c_2\|}\right) e_3,$$
$$\overline{\nabla}_{e_2} e_1 = \left(\frac{D}{\|c_2\|}\right) e_3, \qquad \overline{\nabla}_{e_2} e_2 = (E)e_4 \tag{5}$$

and

$$\overline{\nabla}_{e_1} e_3 = \left(-\frac{D}{\|c_2\|}\right) e_2 + (-A)e_4, \qquad \overline{\nabla}_{e_1} e_4 = \left(\frac{D}{\|c_2\|}\right) e_1 + (A)e_3,$$
$$\overline{\nabla}_{e_2} e_3 = \left(-\frac{D}{\|c_2\|}\right) e_1, \qquad \overline{\nabla}_{e_2} e_4 = (-E)e_2. \tag{6}$$

The first result of above relations follows from (3) and (6) as follows.

Lemma 1. *Let M be the tensor product surface $f = c_1 \otimes c_2$ of unite circle c_1 centered at origin in Euclidean plane $\mathbb{E}^2$ and a unite speed 1-s smooth curve $c_2(t) = \big(\alpha(t), \beta(t)\big)$ in $\mathbb{E}^2$, then*

$$A_{e_3} = \begin{bmatrix} 0 & \dfrac{D}{\|c_2\|} \\ \dfrac{D}{\|c_2\|} & 0 \end{bmatrix}, \qquad A_{e_4} = \begin{bmatrix} \dfrac{-D}{\|c_2\|} & 0 \\ 0 & E \end{bmatrix}. \tag{7}$$

It must be said that in Lemma 1 the similar result is true, if we replace 1-s by 2-s. The same note is correct for all future subjects about 1-s.

Theorem 1. *Let M be a tensor product surface of Euclidean planar circle $c_1(s) = (\cos s, \sin s)$ and a unit speed, 1-s smooth curve $c_2(t) = \big(\alpha(t), \beta(t)\big)$ in $\mathbb{E}^2$. The Gauss map of M is harmonic, if and only if M is part of a plane.*

Proof. If we apply (4) for the Gauss map $\Gamma = e_1 \wedge e_2$, then by a direct calculation and use of (5) and (6), we get the following expression for Laplacian of the Gauss map:

$$-\triangle\Gamma = \sum_{i=1}^{2} \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}\Gamma) - \overline{\nabla}_{\nabla_{e_i} e_i}\Gamma =$$

$$= \sum_{i=1}^{2} \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}(e_1 \wedge e_2)) - \overline{\nabla}_{\nabla_{e_i} e_i}(e_1 \wedge e_2) =$$

$$= \sum_{i=1}^{2} \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i} e_1 \wedge e_2 + e_1 \wedge \overline{\nabla}_{e_i} e_2) - (\overline{\nabla}_{\nabla_{e_i} e_i} e_1 \wedge e_2) - (e_1 \wedge \overline{\nabla}_{\nabla_{e_i} e_i} e_2) =$$

$$= -\left[\frac{3D^2}{\|c_2\|^2} + E^2\right] (e_1 \wedge e_2) + [\ldots]\,(e_1 \wedge e_4) - [\ldots](e_2 \wedge e_3) + [\ldots]\,(e_3 \wedge e_4). \tag{8}$$

If the Gauss map of M is harmonic, i.e., $\triangle\Gamma = 0$, then (8) implies that

$$\frac{3D^2}{\|c_2\|^2} + E^2 = 0. \tag{9}$$

Since all terms in the right – hand side of (10) are non-negative, so we have

$$D \equiv E \equiv 0.$$

This implies that M is a totally geodesic surface in $\mathbb{E}^4$ by (7), therefore M is part of a plane. The converse is obvious.

Theorem 1 is proved.

4. A tensor product surface in $\mathbb{E}^6$ with harmonic Gauss map. Let $c_1 \colon \mathbb{R} \to \mathbb{E}^2$ be the unit planar circle centered at origin of $\mathbb{E}^2$ with parametrization $c_1(t) = (\cos s, \sin s)$ and $c_2 \colon \mathbb{R} \to \mathbb{E}^3$ ba a unit speed, i-s smooth curves in $\mathbb{E}^3$. Without loss of generality let $c_2(t) = \big(\alpha(t), \beta(t), \gamma(t)\big)$ be 1-s, then the tensor product surface M of two curves c_1 and c_2 is given by

$$f = c_1 \otimes c_2 \colon \mathbb{R}^2 \to \mathbb{E}^6,$$

$$f(s,t) = \Big(\alpha(t)\cos s, \beta(t)\cos s, \gamma(t)\cos s, \alpha(t)\sin s, \beta(t)\sin s, \gamma(t)\sin s\Big).$$

Assume that $f(s,t) = c_1(s) \otimes c_2(t)$ defines an isometric immersion of $\mathbb{R}^2$ into $\mathbb{R}^6$. It follows directly that

$$e_1 = \frac{1}{\|c_2\|}\Big(-\alpha(t)\sin s, -\beta(t)\sin s, -\gamma(t)\sin s, \alpha(t)\cos s, \beta(t)\cos s, \gamma(t)\cos s\Big)$$

and

$$e_2 = \Big(\alpha'(t)\cos s, \beta'(t)\cos s, \gamma'(t)\cos s, \alpha'(t)\sin s, \beta'(t)\sin s, \gamma'(t)\sin s\Big)$$

are form an orthonormal frame for tangent space to M. Moreover, an orthonormal basis for normal space to M is given by

$$e_3 = \frac{1}{\sqrt{\alpha^2+\beta^2}}\Big(\beta(t)\sin s, -\alpha(t)\sin s, 0, -\beta(t)\cos s, \alpha(t)\cos s, 0\Big),$$

$$e_4 = \frac{1}{\sqrt{\alpha'^2+\beta'^2}}\Big(\beta'(t)\cos s, -\alpha'(t)\cos s, 0, \beta'(t)\sin s, -\alpha'(t)\sin s, 0\Big),$$

$$e_5 = \frac{1}{\sqrt{\alpha^2+\gamma^2}}\Big(\gamma(t)\sin s, 0, -\alpha(t)\sin s, -\gamma(t)\cos s, 0, \alpha(t)\cos\Big),$$

$$e_6 = \frac{1}{\sqrt{\alpha'^2+\gamma'^2}}\Big(\gamma'(t)\cos s, 0, -\alpha'(t)\cos s, \gamma'(t)\sin s, 0, -\alpha'(t)\sin s\Big).$$

In this section, for simplification of relations, we have to introduce following abbreviations:

$$A = -\frac{\alpha(t)\alpha'(t) + \beta(t)\beta'(t) + \gamma(t)\gamma'(t)}{\|c_2\|^2}, \qquad B_1 = -\frac{\alpha(t)\alpha'(t) + \beta(t)\beta'(t)}{\alpha^2(t) + \beta^2(t)}$$

$$B_2 = -\frac{\alpha(t)\alpha'(t) + \gamma(t)\gamma'(t)}{\alpha^2(t) + \gamma^2(t)}, \qquad C_1 = -\frac{\alpha'(t)\alpha''(t) + \beta'(t)\beta''(t)}{\alpha'^2(t) + \beta'^2(t)},$$

$$C_2 = -\frac{\alpha'(t)\alpha''(t) + \gamma'(t)\gamma''(t)}{\alpha'^2(t) + \gamma'^2(t)}, \qquad D_1 = \frac{\alpha(t)\beta'(t) - \alpha'(t)\beta(t)}{\sqrt{\alpha^2(t) + \beta^2(t)}},$$

$$D_2 = \frac{\alpha(t)\gamma'(t) - \alpha'(t)\gamma(t)}{\sqrt{\alpha^2(t) + \gamma^2(t)}}, \qquad E_1 = \frac{\alpha''(t)\beta'(t) - \alpha'(t)\beta''(t)}{\sqrt{\alpha'^2(t) + \beta'^2(t)}},$$

$$E_2 = \frac{\alpha''(t)\gamma'(t) - \alpha'(t)\gamma''(t)}{\sqrt{\alpha'^2(t) + \gamma'^2(t)}}, \qquad F_1 = \frac{\alpha'(t)\beta(t) - \alpha(t)\beta'(t)}{\sqrt{\alpha'^2(t) + \beta'^2(t)}},$$

$$F_2 = \frac{\alpha'(t)\gamma(t) - \alpha(t)\gamma'(t)}{\sqrt{\alpha'^2(t) + \gamma'^2(t)}}, \qquad L_1 = \sqrt{\frac{\alpha^2(t) + \beta^2(t)}{\alpha'^2(t) + \beta'^2(t)}},$$

$$G_{23} = \frac{\beta(t)\gamma'(t)}{\sqrt{(\alpha^2(t) + \beta^2(t))(\alpha'^2(t) + \gamma'^2(t))}}, \qquad \overline{G}_{23} = \frac{-\beta'(t)\gamma(t)}{\sqrt{(\alpha'^2(t) + \beta'^2(t))(\alpha^2(t) + \gamma^2(t))}},$$

$$H_{23} = \frac{B_2\beta\gamma + \beta(t)\gamma'(t)}{\sqrt{(\alpha^2(t) + \beta^2(t))(\alpha^2(t) + \gamma^2(t))}}, \qquad \overline{H}_{23} = \frac{B_1\beta\gamma + \beta'(t)\gamma(t)}{\sqrt{(\alpha^2(t) + \beta^2(t))(\alpha^2(t) + \gamma^2(t))}},$$

$$K_{23} = \frac{C_2\beta'\gamma' + \beta'(t)\gamma''(t)}{\sqrt{(\alpha'^2(t) + \beta'^2(t))(\alpha'^2(t) + \gamma'^2(t))}}, \qquad \overline{K}_{23} = \frac{C_1\beta'\gamma' + \beta''(t)\gamma'(t)}{\sqrt{(\alpha'^2(t) + \beta'^2(t))(\alpha'^2(t) + \gamma'^2(t))}},$$

$$L_2 = \sqrt{\frac{\alpha^2(t) + \gamma^2(t)}{\alpha'^2(t) + \gamma'^2(t)}}.$$

So covariant differentiation with respect to e_1 and e_2 give us

$$\overline{\nabla}_{e_1} e_1 = \big(A\|c_2\|\big)e_2 + \left(\frac{F_1}{\|c_2\|}\right)e_4 + \left(\frac{F_2}{\|c_2\|}\right)e_6, \qquad \overline{\nabla}_{e_2} e_1 = \left(\frac{D_1}{\|c_2\|}\right)e_3 + \left(\frac{D_2}{\|c_2\|}\right)e_5,$$

$$\overline{\nabla}_{e_1} e_2 = \big(-A\|c_2\|\big)e_1 + (D_1)e_3 + (D_2)e_5, \qquad \overline{\nabla}_{e_2} e_2 = (E_1)e_4 + (E_2)e_6 \tag{10}$$

and

$$\overline{\nabla}_{e_1}e_3 = (-D_1)e_2 + (-L_1B_1)e_4 + (G_{23})e_6, \qquad \overline{\nabla}_{e_2}e_3 = \left(-\frac{D_1}{\|c_2\|}\right)e_1 + (\overline{H}_{23})e_5,$$

$$\overline{\nabla}_{e_1}e_4 = \left(\frac{-F_1}{\|c_2\|}\right)e_1 + (B_1L_1)e_3 + (\overline{G}_{23})e_5, \qquad \overline{\nabla}_{e_2}e_4 = (-E_1)e_2 + (\overline{K}_{23})e_6,$$

$$\overline{\nabla}_{e_1}e_5 = (-D_2)e_2 + (-\overline{G}_{23})e_4 + (-L_2B_2)e_6, \qquad \overline{\nabla}_{e_2}e_5 = \left(\frac{-D_2}{\|c_2\|}\right)e_1 + (H_{23})e_3, \tag{11}$$

$$\overline{\nabla}_{e_1}e_6 = \left(\frac{-F_2}{\|c_2\|}\right)e_1 + (-G_{23})e_3 + (B_2L_2)e_5, \qquad \overline{\nabla}_{e_2}e_6 = (-E_2)e_2 + (K_{23})e_4.$$

The first result of above relations follows from (3) and (11) as follows.

Lemma 2. *Let $f = c_1 \otimes c_2$ be the tensor product of unite circle with parametrization $c_1 =$ $= (\cos s, \sin s)$ in Euclidean plane $\mathbb{E}^2$ and a unite speed 1-s smooth curve $c_2(t) = \big(\alpha(t), \beta(t), \gamma(t)\big)$ in $\mathbb{E}^3$, then*

$$A_{e_3} = \begin{bmatrix} 0 & D_1 \\ \dfrac{D_1}{\|c_2\|} & 0 \end{bmatrix}, \qquad A_{e_4} = \begin{bmatrix} \dfrac{F_1}{\|c_2\|} & 0 \\ 0 & E_1 \end{bmatrix},$$

$$A_{e_5} = \begin{bmatrix} 0 & D_2 \\ \dfrac{D_2}{\|c_2\|} & 0 \end{bmatrix}, \qquad A_{e_6} = \begin{bmatrix} d\dfrac{F_2}{\|c_2\|} & 0 \\ 0 & E_2 \end{bmatrix}. \tag{12}$$

The following theorem states a necessary and sufficient condition for the Gauss map of a tensor product surface with same conditions in Lemma 2 to be a harmonic function.

Theorem 2. *Let M be a tensor product surface of Euclidean planar circle $c_1(s) = (\cos s, \sin s)$ and unit speed, 1-s smooth curve $c_2(t) = \big(\alpha(t), \beta(t), \gamma(t)\big)$ in $\mathbb{E}^3$. The Gauss map of M is harmonic, if and only if M is part of a plane.*

Proof. Proof is exactly the same as proof of Theorem 1. If we apply (4), (10) and (11), then a tedious computation implies that the $\triangle\Gamma$, (rough) Laplacian of the Gauss map, is given by

$$-\triangle\Gamma = \sum_{i=1}^{2} \overline{\nabla}_{e_i}\big(\overline{\nabla}_{e_i}\Gamma\big) - \overline{\nabla}_{\nabla_{e_i}e_i}\Gamma\big) =$$

$$= -\left[\frac{F_1^2}{\|c_2\|^2} + \frac{F_2^2}{\|c_2\|^2} + \big(D_1^2 + D_2^2\big)\left(\frac{1}{\|c_2\|^2} + 1\right) + E_1^2 + E_2^2\right](e_1 \wedge e_2) +$$

$$+ [\ldots](e_1 \wedge e_4) + [\ldots](e_1 \wedge e_6) + [\ldots](e_2 \wedge e_3) + [\ldots](e_2 \wedge e_5) + [\ldots](e_3 \wedge e_4) +$$

$$+ [\ldots](e_3 \wedge e_6) + [\ldots](e_4 \wedge e_5) + [\ldots](e_5 \wedge e_6). \tag{13}$$

If the Gauss map of M is harmonic, i.e., $\triangle\Gamma = 0$, then (13) implies that

$$\frac{F_1^2}{\|c_2\|^2} + \frac{F_2^2}{\|c_2\|^2} + \left(D_1^2 + D_2^2\right)\left(\frac{1}{\|c_2\|^2} + 1\right) + E_1^2 + E_2^2 = 0. \tag{14}$$

Since all terms in the right-hand side of (14) are non-negative, so we have,

$$D_1 \equiv D_2 \equiv E_1 \equiv E_2 \equiv F_1 \equiv F_2 \equiv 0.$$

This implies that M is a totally geodesic surface in $\mathbb{E}^6$ by (12), therefore M is part of plane. The converse is obvious.

Theorem 2 is proved.

5. A tensor product surface in $\mathbb{E}^8$ with harmonic Gauss map. Let $c_1 \colon \mathbb{R} \to \mathbb{E}^2$ be the unit planar circle centered at origin in E^2 with parametrization $c_1(t) = (\cos s, \sin s)$ and $c_2 \colon \mathbb{R} \to \mathbb{E}^4$ ba a unit speed, i-s smooth curves in $\mathbb{E}^4$. Without loss of generality let $c_2(t) = (\alpha(t), \beta(t), \gamma(t), \delta(t))$ with $\alpha(t) \neq 0, \alpha'(t) \neq 0$ for every $t \in \mathbb{R}$, then the tensor product surface M of two curves c_1 and c_2 is given by

$$f = c_1 \otimes c_2 \colon \mathbb{R}^2 \to \mathbb{E}^8,$$

$$f(s,t) = \big(\alpha(t)\cos s, \beta(t)\cos s, \gamma(t)\cos s, \delta(t)\cos s, \alpha(t)\sin s, \beta(t)\sin s, \gamma(t)\sin s, \delta(t)\sin s\big).$$

Assume that $f(s,t) = c_1(s) \otimes c_2(t)$ defines an isometric immersion of $\mathbb{R}^2$ into $\mathbb{R}^8$. It follows directly that

$$e_1 =$$

$$= \frac{1}{\|c_2\|}\Big(- \alpha(t)\sin s, -\beta(t)\sin s, -\gamma(t)\sin s, -\delta(t)\sin s, \alpha(t)\cos s, \beta(t)\cos s, \gamma(t)\cos s, \delta(t)\cos s\Big)$$

and

$$e_2 = \Big(\alpha'(t)\cos s, \beta'(t)\cos s, \gamma'(t)\cos s, \delta'(t)\cos s, \alpha'(t)\sin s, \beta'(t)\sin s, \gamma'(t)\sin s, \delta'(t)\sin s\Big)$$

are form an orthonormal basis for tangent space of M. Moreover, an orthonormal basis for normal space to M is given by

$$e_3 = \frac{1}{\sqrt{\alpha^2 + \beta^2}}\Big(\beta(t)\sin s, -\alpha(t)\sin s, 0, 0, -\beta(t)\cos s, \alpha(t)\cos s, 0, 0\Big),$$

$$e_4 = \frac{1}{\sqrt{\alpha'^2 + \beta'^2}}\Big(\beta'(t)\cos s, -\alpha'(t)\cos s, 0, 0, \beta'(t)\sin s, -\alpha'(t)\sin s, 0, 0\Big),$$

$$e_5 = \frac{1}{\sqrt{\alpha^2 + \gamma^2}}\Big(\gamma(t)\sin s, 0, -\alpha(t)\sin s, 0, -\gamma(t)\cos s, 0, \alpha(t)\cos\Big), 0,$$

$$e_6 = \frac{1}{\sqrt{\alpha'^2 + \gamma'^2}}\Big(\gamma'(t)\cos s, 0, -\alpha'(t)\cos s, 0, \gamma'(t)\sin s, 0, -\alpha'(t)\sin s, 0\Big),$$

$$e_7 = \frac{1}{\sqrt{\alpha^2+\delta^2}}\Big(\delta(t)\sin s, 0, 0, -\alpha(t)\sin s, -\delta(t)\cos s, 0, 0, \alpha(t)\cos\Big),$$

$$e_8 = \frac{1}{\sqrt{\alpha'^2+\delta'^2}}\Big(\delta'(t)\cos s, 0, 0, -\alpha'(t)\cos s, \gamma'(t)\sin s, 0, 0, -\alpha'(t)\sin s\Big).$$

In this section, for simplification of relations, we have to introduce following abbreviations:

$$A = -\frac{\alpha(t)\alpha'(t)+\beta(t)\beta'(t)+\gamma(t)\gamma'(t)+\delta(t)\delta'(t)}{\|c_2\|^2}, \qquad B_1 = -\frac{\alpha(t)\alpha'(t)+\beta(t)\beta'(t)}{\alpha^2(t)+\beta^2(t)},$$

$$B_2 = -\frac{\alpha(t)\alpha'(t)+\gamma(t)\gamma'(t)}{\alpha^2(t)+\gamma^2(t)}, \qquad B_3 = -\frac{\alpha(t)\alpha'(t)+\delta(t)\delta'(t)}{\alpha^2(t)+\delta^2(t)},$$

$$C_1 = -\frac{\alpha'(t)\alpha''(t)+\beta'(t)\beta''(t)}{\alpha'^2(t)+\beta'^2(t)}, \qquad C_2 = -\frac{\alpha'(t)\alpha''(t)+\gamma'(t)\gamma''(t)}{\alpha'^2(t)+\gamma'^2(t)},$$

$$C_3 = -\frac{\alpha'(t)\alpha''(t)+\delta'(t)\delta''(t)}{\alpha'^2(t)+\delta'^2(t)}, \qquad D_1 = \frac{\alpha(t)\beta'(t)-\alpha'(t)\beta(t)}{\sqrt{\alpha^2(t)+\beta^2(t)}},$$

$$D_2 = \frac{\alpha(t)\gamma'(t)-\alpha'(t)\gamma(t)}{\sqrt{\alpha^2(t)+\gamma^2(t)}}, \qquad D_3 = \frac{\alpha(t)\delta'(t)-\alpha'(t)\delta(t)}{\sqrt{\alpha^2(t)+\delta^2(t)}},$$

$$E_1 = \frac{\alpha''(t)\beta'(t)-\alpha'(t)\beta''(t)}{\sqrt{\alpha'^2(t)+\beta'^2(t)}}, \qquad E_2 = \frac{\alpha''(t)\gamma'(t)-\alpha'(t)\gamma''(t)}{\sqrt{\alpha'^2(t)+\gamma'^2(t)}},$$

$$E_3 = \frac{\alpha''(t)\delta'(t)-\alpha'(t)\delta''(t)}{\sqrt{\alpha'^2(t)+\delta'^2(t)}}, \qquad F_1 = \frac{\alpha'(t)\beta(t)-\alpha(t)\beta'(t)}{\sqrt{\alpha'^2(t)+\beta'^2(t)}},$$

$$F_2 = \frac{\alpha'(t)\gamma(t)-\alpha(t)\gamma'(t)}{\sqrt{\alpha'^2(t)+\gamma'^2(t)}}, \qquad F_3 = \frac{\alpha'(t)\delta(t)-\alpha(t)\delta'(t)}{\sqrt{\alpha'^2(t)+\delta'^2(t)}},$$

$$G_{23} = \frac{\beta(t)\gamma'(t)}{\sqrt{(\alpha^2(t)+\beta^2(t))(\alpha'^2(t)+\gamma'^2(t))}}, \qquad \overline{G}_{23} = \frac{-\beta'(t)\gamma(t)}{\sqrt{(\alpha'^2(t)+\beta'^2(t))(\alpha^2(t)+\gamma^2(t))}},$$

$$G_{24} = \frac{\beta(t)\delta'(t)}{\sqrt{(\alpha^2(t)+\delta^2(t))(\alpha'^2(t)+\delta'^2(t))}}, \qquad \overline{G}_{24} = \frac{-\beta'(t)\delta(t)}{\sqrt{(\alpha'^2(t)+\delta'^2(t))(\alpha^2(t)+\delta^2(t))}},$$

$$G_{34} = \frac{\gamma(t)\delta'(t)}{\sqrt{(\alpha^2(t)+\gamma^2(t))(\alpha'^2(t)+\delta'^2(t))}}, \qquad \overline{G}_{34} = \frac{-\gamma'(t)\delta(t)}{\sqrt{(\alpha'^2(t)+\gamma'^2(t))(\alpha^2(t)+\delta^2(t))}},$$

$$H_{23} = \frac{B_2\beta(t)\gamma(t)+\beta(t)\gamma'(t)}{\sqrt{(\alpha^2(t)+\beta^2(t))(\alpha^2(t)+\gamma^2(t))}}, \qquad \overline{H}_{23} = \frac{B_1\beta(t)\gamma(t)+\beta'(t)\gamma(t)}{\sqrt{(\alpha^2(t)+\beta^2(t))(\alpha^2(t)+\gamma^2(t))}},$$

$$H_{24} = \frac{B_3\beta(t)\delta(t)+\beta(t)\delta'(t)}{\sqrt{(\alpha^2(t)+\beta^2(t))(\alpha^2(t)+\delta^2(t))}}, \qquad \overline{H}_{24} = \frac{B_1\beta(t)\delta(t)+\beta'(t)\delta(t)}{\sqrt{(\alpha^2(t)+\beta^2(t))(\alpha^2(t)+\delta^2(t))}},$$

$$H_{34} = \frac{B_3\gamma(t)\delta(t) + \gamma(t)\delta'(t)}{\sqrt{(\alpha^2(t) + \gamma^2(t))(\alpha^2(t) + \delta^2(t))}}, \qquad \overline{H}_{34} = \frac{B_2\gamma(t)\delta(t) + \gamma'(t)\delta(t)}{\sqrt{(\alpha^2(t) + \gamma^2(t))(\alpha^2(t) + \delta^2(t))}},$$

$$K_{23} = \frac{C_2\beta'(t)\gamma'(t) + \beta'(t)\gamma''(t)}{\sqrt{(\alpha'^2(t) + \beta'^2(t))(\alpha'^2(t) + \gamma'^2(t))}}, \qquad \overline{K}_{23} = \frac{C_1\beta'(t)\gamma'(t) + \beta''(t)\gamma'(t)}{\sqrt{(\alpha'^2(t) + \beta'^2(t))(\alpha'^2(t) + \gamma'^2(t))}},$$

$$K_{24} = \frac{C_3\beta'(t)\delta'(t) + \beta'(t)\delta''(t)}{\sqrt{(\alpha'^2(t) + \beta'^2(t))(\alpha'^2(t) + \delta'^2(t))}}, \qquad \overline{K}_{24} = \frac{C_1\beta'(t)\delta'(t) + \beta''(t)\delta'(t)}{\sqrt{(\alpha'^2(t) + \beta'^2(t))(\alpha'^2(t) + \delta'^2(t))}},$$

$$K_{34} = \frac{C_3\gamma'(t)\delta'(t) + \gamma'(t)\delta''(t)}{\sqrt{(\alpha'^2(t) + \gamma'^2(t))(\alpha'^2(t) + \delta'^2(t))}}, \qquad K_{34} = \frac{C_2\gamma'(t)\delta'(t) + \gamma''(t)\delta'(t)}{\sqrt{(\alpha'^2(t) + \gamma'^2(t))(\alpha'^2(t) + \delta'^2(t))}},$$

$$L_1 = \sqrt{\frac{\alpha^2(t) + \beta^2(t)}{\alpha'^2(t) + \beta'^2(t)}}, \qquad L_2 = \sqrt{\frac{\alpha^2(t) + \gamma^2(t)}{\alpha'^2(t) + \gamma'^2(t)}}, \qquad L_3 = \sqrt{\frac{\alpha^2(t) + \delta^2(t)}{\alpha'^2(t) + \delta'^2(t)}}.$$

Now covariant differentiation with respect to e_1 and e_2 give us,

$$\begin{aligned}
\overline{\nabla}_{e_1} e_1 &= \big(A\|c_2\|\big)e_2 + \left(\frac{F_1}{\|c_2\|}\right)e_4 + \left(\frac{F_2}{\|c_2\|}\right)e_6 + \left(\frac{F_3}{\|c_2\|}\right)e_8,\\
\overline{\nabla}_{e_2} e_1 &= \left(\frac{D_1}{\|c_2\|}\right)e_3 + \left(\frac{D_2}{\|c_2\|}\right)e_5 + \left(\frac{D_3}{\|c_2\|}\right)e_7,\\
\overline{\nabla}_{e_1} e_2 &= \big(-A\|c_2\|\big)e_1 + (D_1)e_3 + (D_2)e_5 + (D_3)e_7,\\
\overline{\nabla}_{e_2} e_2 &= (E_1)e_4 + (E_2)e_6 + (E_3)e_8,
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
\overline{\nabla}_{e_1} e_3 &= (-D_1)e_2 + (-B_1L_1)e_4 + (G_{23})e_6 + (G_{24})e_8,\\
\overline{\nabla}_{e_2} e_3 &= \left(-\frac{D_1}{\|c_2\|}\right)e_1 + (\overline{H}_{23})e_5 + (\overline{H}_{24})e_7,\\
\overline{\nabla}_{e_1} e_4 &= \left(\frac{-F_1}{\|c_2\|}\right)e_1 + (B_1L_1)e_3 + (\overline{G}_{23})e_5 + (\overline{G}_{24})e_7,\\
\overline{\nabla}_{e_2} e_4 &= (-E_1)e_2 + (\overline{K}_{23})e_6 + (\overline{K}_{24})e_8,\\
\overline{\nabla}_{e_1} e_5 &= (-D_2)e_2 + (-\overline{G}_{23})e_4 + (-L_2B_2)e_6 + (G_{34})e_8,\\
\overline{\nabla}_{e_2} e_5 &= \left(\frac{-D_2}{\|c_2\|}\right)e_1 + (H_{23})e_3 + (\overline{H}_{34})e_7,\\
\overline{\nabla}_{e_1} e_6 &= \left(\frac{-F_2}{\|c_2\|}\right)e_1 + (-G_{23})e_3 + (B_2L_2)e_5 + (\overline{G}_{34})e_7,
\end{aligned} \tag{16}$$

$$\overline{\nabla}_{e_2} e_6 = (-E_2)e_2 + (K_{23})e_4 + (\overline{K}_{34})e_8,$$

$$\overline{\nabla}_{e_1} e_7 = (-D_3)e_2 + (-\overline{G}_{24})e_4 + (-\overline{G}_{34})e_6 + (-B_3L_3)e_8,$$

$$\overline{\nabla}_{e_2} e_7 = \left(\frac{-D_3}{\|c_2\|}\right) e_1 + (H_{24})e_3 + (H_{34})e_5,$$

$$\overline{\nabla}_{e_1} e_8 = \left(\frac{-F_3}{\|c_2\|}\right) e_1 + (-G_{24})e_3 + (G_{34})e_5 + (B_3L_3)e_7,$$

$$\overline{\nabla}_{e_2} e_8 = (-E_3)e_2 + (K_{24})e_4 + (K_{34})e_6.$$

The first result of above relations yields from (3) and (16) as follows.

Lemma 3. *Let $f = c_1 \otimes c_2$ be the tensor product surface of unite circle $c_1(s) = (\cos s, \sin s)$ in Euclidean plane $\mathbb{E}^2$ and a unite speed, 1-s smooth curve $c_2(t) = \big(\alpha(t), \beta(t), \gamma(t)\delta(t)\big)$ in $\mathbb{E}^4$, then*

$$A_{e_3} = \begin{bmatrix} 0 & D_1 \\ \dfrac{D_1}{\|c_2\|} & 0 \end{bmatrix}, \qquad A_{e_4} = \begin{bmatrix} \dfrac{F_1}{\|c_2\|} & 0 \\ 0 & E_1 \end{bmatrix},$$

$$A_{e_5} = \begin{bmatrix} 0 & D_2 \\ \dfrac{D_2}{\|c_2\|} & 0 \end{bmatrix}, \qquad A_{e_6} = \begin{bmatrix} \dfrac{F_2}{\|c_2\|} & 0 \\ 0 & E_2 \end{bmatrix}, \tag{17}$$

$$A_{e_7} = \begin{bmatrix} 0 & D_3 \\ \dfrac{D_3}{\|c_2\|} & 0 \end{bmatrix}, \qquad A_{e_8} = \begin{bmatrix} \dfrac{F_3}{\|c_2\|} & 0 \\ 0 & E_3 \end{bmatrix}.$$

Let $c_1(s) = (\cos s, \sin s)$ and $c_2(t) = \big(\alpha(t), \beta(t), \gamma(t), \delta(t)\big)$ be the unit circle in $\mathbb{E}^2$ and a unit speed, 1-s smooth curve in $\mathbb{E}^4$ respectively, then following theorem, gives us similar result to the Theorems 1 and 2 in $\mathbb{E}^8$.

Theorem 3. *Let M be the tensor product surface $f = c_1 \otimes c_2$, The Gauss map of M is harmonic, if and only if M is part of a plane.*

Proof. Similar to proof of Theorems 1 and 2, we apply (4), (15) and (16), then a tedious computation implies that the $\triangle\Gamma$, (rough) Laplacian of the Gauss map, is given by

$$-\triangle\Gamma = \sum_{i=1}^{2} \overline{\nabla}_{e_i}(\overline{\nabla}_{e_i}\Gamma) - \overline{\nabla}_{\nabla_{e_i} e_i}\Gamma) =$$

$$= -\left[\frac{F_1^2}{\|c_2\|^2} + \frac{F_2^2}{\|c_2\|^2} + \frac{F_3^2}{\|c_2\|^2} + (D_1^2 + D_2^2 + D_3^2)\left(\frac{1}{\|c_2\|^2} + 1\right) + E_1^2 + E_2^2 + E_3^2\right](e_1 \wedge e_2) +$$

$$+[\ldots](e_1 \wedge e_4) + [\ldots](e_1 \wedge e_6) + [\ldots](e_1 \wedge e_8) + [\ldots](e_2 \wedge e_3) + [\ldots](e_2 \wedge e_5) + [\ldots](e_2 \wedge e_7) +$$

$$+[\ldots](e_3 \wedge e_4) + [\ldots](e_3 \wedge e_6) + [\ldots](e_3 \wedge e_8) + [\ldots](e_4 \wedge e_5) + [\ldots](e_4 \wedge e_7) + [\ldots](e_5 \wedge e_6)+$$

$$+[\ldots](e_5 \wedge e_8) + [\ldots](e_6 \wedge e_7) + [\ldots](e_7 \wedge e_8). \tag{18}$$

If the Gauss map of M is harmonic, then (18) implies that

$$\frac{F_1^2}{\|c_2\|^2} + \frac{F_2^2}{\|c_2\|^2} + \frac{F_3^2}{\|c_2\|^2} + \left(D_1^2 + D_2^2 + D_3^2\right)\left(\frac{1}{\|c_2\|^2} + 1\right) + E_1^2 + E_2^2 + E_3^2 = 0. \tag{19}$$

Since all terms in the right-hand side of (19) are non-negative, so we have

$$D_1 \equiv D_2 \equiv D_3 \equiv E_1 \equiv E_2 \equiv E_3 \equiv F_1 \equiv F_2 \equiv F_3 \equiv 0.$$

This implies that M is a totally geodesic surface in $\mathbb{E}^8$ by (17), therefore M is part of a plane. The converse is obvious.

Theorem 3 is proved.

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