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S^1-BOTT FUNCTIONS ON MANIFOLDS
S^1-ФУНКЦІЇ БОТТА НА МНОГОВИДАХ

We study S^1-Bott functions on compact smooth manifolds. In particular, we investigate S^1-invariant Bott functions on manifolds with circle action.

Вивчаються S^1-функції Ботта на компактних гладких многовидах. Зокрема, досліджуються S^1-інваріантні функції Ботта на гладких многовидах з дією кола.

1. Introduction. Let M^n be a compact closed manifold of dimension at least 3. We study the S^1-Bott functions on M^n. Separately we investigate S^1-invariant Bott functions on M^{2n} with a semi-free circle action which have finitely many fixed points. The aim of this paper is to find exact values of minimal numbers of singular circles of some indices of S^1-invariant Bott functions on M^{2n}.

Closely related to S^1-Bott function on a manifold M^n is a more flexible object, the decomposition of a round handle of M^n. In its turn, to study the round handles decomposition of M^n we use a diagram, i.e., a graph which carries the information about the handles.

2. S^1-Bott functions. Let M^n be a smooth manifold, $f\colon M^n \to [0,1]$ a smooth function, and $x \in M^n$ one of its critical points. Consider the Hessian $\Gamma_x(f)\colon T_x \times T_x \to \mathbf{R}$ at this point. Recall that the index of the Hessian is called the maximum dimension of T_x, where $\Gamma_x(f)$ is negative definite. The index of $\Gamma_x(f)$ is called the index of the critical point x, and the corank of $\Gamma_x(f)$ is called the corank of x. Suppose that the set of critical points of f forms a disjoint union of smooth submanifolds K_j^i whose dimensions do not exceed $n-1$. A connected critical submanifold $K_{j_0}^{i_0}$ is called **nondegenerate** if the Hessian is nondegenerate on subspaces orthogonal to $K_{j_0}^{i_0}$ (i.e., has corank equal to $n - i_0$) at each point $x \in K_{j_0}^{i_0}$.

Definition 2.1. *A mapping $f\colon M^n \to [0,1]$ is called a Bott function if all of its critical points form nondegenerate critical submanifolds which do not intersect the boundary of M^n.*

Consider the following important example of Bott functions:

Definition 2.2. *A mapping $f\colon M^n \to [0,1]$ is called an S^1-Bott function if all of its critical points form nondegenerate critical circles.*

Note that S^1-Bott functions do not exist on any smooth manifold [12]. S^1-Bott functions have been studied and used by many authors [1 – 7, 9, 11, 14].The following theorem can be found in [8, 11].

Theorem 2.1. *Let M^n be a smooth closed manifold, $f\colon M^n \to [0,1]$ be a S^1- Bott function, and $\gamma \subset M^n$ its critical circle. Then there is a system of coordinates in a neighborhood of γ of one of the following types*:

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1. *Trivial* $\nu\colon S^1\times D^{n-1}(\varepsilon)\to M^n$, *where* $D^{n-1}(\varepsilon)$ *is a disc of radius* ε, $\nu(S^1\times 0)=\gamma$, *and* $f\big(\nu(\theta,x)\big)=-x_1^2-\ldots-x_\lambda^2+x_{\lambda+1}^2+\ldots+x_{n-1}^2$, *for* $(\theta,x)\in S^1\times D^{n-1}(\varepsilon)$.

2. *Twisted* $\tau\colon([0,1]\times D^{n-1}(\varepsilon)/\sim)\to M^n$, *where* τ *is a smooth embedding such that* $(\tau([0,1])\times\times 0/\sim)=\gamma$ *and* $f(\tau(t,x))=-x_1^2-\ldots-x_\lambda^2+x_{\lambda+1}^2+\ldots+x_{n-1}^2$, *for* $(t,x)\in(\tau\colon[0,1]\times\times D^{n-1}(\varepsilon)/\sim)$. *Here* $([0,1]\times D^{n-1}(\varepsilon)/\sim)$ *is diffeomorphic to* $S^1\times D^{n-1}(\varepsilon)$ *by identifying* $0\times D^{n-1}(\varepsilon)$ *and* $1\times D^{n-1}(\varepsilon)$ *by the mapping*: $(0,x_1,\ldots,x_\lambda,x_{\lambda+1},\ldots,x_{n-1})\leftrightarrow(1,-x_1,\ldots,x_\lambda,-x_{\lambda+1},\ldots,x_{n-1})$.

The number λ *is called the index of the critical circle* γ.

Let M^n be a smooth manifold, and $f\colon M^n\to[0,n]$ an S^1-Bott function. We say that f is a **nice** S^1-Bott function if the submanifold $M_i(f)=f^{-1}\Big[0,i+\dfrac{1}{2}\Big]$ contains all closed orbits of index $\lambda\le i$. Each nice S^1-Bott function defines a filtration on the manifold M^n : $M_0(f)\subset M_1(f)\subset\ldots\ldots\subset M_{n-1}(f)\subset M^n$. It is well known [11] that the existence of a nice S^1-Bott function on a manifold is equivalent to existence of a decomposition of the manifold by round handles. We recall some necessary definitions.

Definition 2.3. *We define an* n*-dimensional round handle* R_λ *of index* λ *by* $R_\lambda=S^1\times D^\lambda\times\times D^{n-\lambda-1}$, *where* D^i *is a disc of dimension* i.

Define a twisted n*-dimensional round handle* TR_λ *of index* λ, $0<\lambda<n-1$, *by* $TR_\lambda=[0,1]\times\times D^\lambda\times D^{n-\lambda-1}/\sim$, *where identification is given by the map*: $(0,x_1,\ldots,x_\lambda,x_{\lambda+1},\ldots,x_{n-1})\leftrightarrow\leftrightarrow(1,-x_1,\ldots,x_\lambda,-x_{\lambda+1},\ldots,x_{n-1})$.

Apparently, Thurston [15] was the first to note that the existence of a S^1-Bott function on manifold is equivalent to existence of a decomposition of the manifold by handles. We shall describe this fact in more details.

Definition 2.4. *We say that the manifold* M^n_λ *is obtained from a smooth manifold* M^n *by attaching a round handle of index* λ *if* $M^n_\lambda=M^n\bigcup_\varphi S^1\times D^\lambda\times D^{n-\lambda-1}$, *where* $\varphi\colon S^1\times\partial D^\lambda\times\times D^{n-\lambda-1}\longrightarrow\partial M^n$ *is a smooth embedding.*

Manifold M^n_λ *is obtained from a smooth manifold* M^n *by gluing a twisted round handles of index* λ, *if* $M^n_\lambda=M^n\bigcup_\varphi[0,1]\times D^\lambda\times D^{n-\lambda-1}/\sim$, *where* $\varphi\colon([0,1]\times\partial D^\lambda\times D^{n-\lambda-1}/\sim)\to M^n$ *is a smooth embedding.*

Definition 2.5. *Decomposition of a smooth manifold* M^n *by round handles is called a filtration* $\partial M^n\times[0,1]=M^n_0(R)\subset M^n_1(R)\subset\ldots\subset M^n_{n-1}(R)=M^n$, *where the manifold* $M^n_i(R)$ *obtained from the manifold* $M^n_{i-1}(R)$ *by gluing round and twisted round handles of index* i. *In the case when* M^n *is a closed manifold, filtration begins with round handles of index* 0.

In what follows we recall the relationship between S^1 and the decomposition by round handles [11].

Theorem 2.2. *Let* M^n *be a smooth closed manifold. The following two conditions are equivalent* :

1. *On the manifold* M^n *there is a nice* S^1*-Bott function with the critical circles* $\gamma_1,\ldots,\gamma_k$ *of index* $\lambda_1,\ldots,\lambda_k$ *with trivial coordinate systems and critical circles* $\tilde\gamma_1,\ldots,\tilde\gamma_l$ *of indices* $\mu_1,\ldots,\mu_l$ *with twisted coordinate systems.*

2. *Manifold M^n admits a decomposition by round handles consisting of round handles $R_{\lambda_1}, \ldots$ $\ldots, R_{\lambda_k}$ of index $\lambda_1, \ldots, \lambda_k$ and of twisted round handles $TR_{\mu_1}, \ldots, TR_{\mu_l}$ of indices $\mu_1, \ldots, \mu_l$ so that the critical circle γ_i corresponds to a round handle R_{λ_i}, $1 \leq i \leq k$, and the critical circle $\tilde{\gamma}_j$ corresponds to a twisted round handle TR_{μ_j}, $1 \leq j \leq l$.*

Thus each nice S^1-Bott function on manifold M^n generates a round handle decomposition of M^n and vice versa.

The following result belongs to D. Asimov [5].

Theorem 2.3. *Let M^n be a smooth closed manifold $(n > 3)$, and suppose that the Euler characteristic $\chi(M^n) = 0$. Then M^n admits a round handle decomposition.*

For three-dimensional manifolds the situation is much more complicated [1, 12], there are closed three-manifolds which do not admit a round handle decomposition. Of recent results in the three-dimensional Poincare conjecture implies that a simply connected three-dimensional manifold admit a round handle decomposition.

We are interested in conditions when an S^1-Bott function on M^n has the property that all of its critical circles have trivial coordinate system. We recall the necessary facts from [4].

By definition an n-dimensional **handle** H_λ of index λ is $H_\lambda = D^\lambda \times D^{n-\lambda}$. We say that a smooth manifold M^n_λ is obtained from a smooth manifold M^n by attaching handles of index λ if $M^n_\lambda = M^n \bigcup_\varphi D^\lambda \times D^{n-\lambda}$, where $\varphi\colon \partial D^\lambda \times D^{n-\lambda} \longrightarrow \partial M^n$ is a smooth embedding. $\partial D^\lambda \times 0$ $(D^\lambda \times 0)$ is called the core (disc), and $\partial D^{n-\lambda} \times 0$ $(D^{n-\lambda} \times 0)$ is called co-core sphere (disc) of the handle $D^\lambda \times D^{n-\lambda}$.

A decomposition of a smooth manifold M^n by handles is a filtration $\partial M^n \times [0,1] = M^n_0 \subset$ $\subset M^n_1 \subset \ldots \subset M^n_n = M^n$, where the manifold M^n_i is obtained from the manifold M^n_{i-1} by attaching handles of index i.

In the case where M^n is a closed manifold, filtration begins by handles of index 0. There is a close relationship between the expansion of manifold by round handles and handles, in [5] the following lemma was proved.

Lemma 2.1. *Let $M^n = M^n_1 + H_\lambda + H_{\lambda+1}$ be a smooth manifold obtained from manifold with boundary M^n_1 by attaching handles of indices λ and $\lambda + 1$, which do not intersect $(n > 2)$. Then if $\lambda > 0$, the manifold M^n can be represented as $M^n = M^n_1 + R_\lambda$, where R_λ denotes the round handle of index λ.*

Lemma 2.2. *Let M^n be a smooth manifold $(n > 2)$ obtained from manifolds with boundary M^n_1 by attaching a round (or twisted round) handles of index $\lambda > 0$. Then the manifold M^n can be represented as $M^n = M^n_1 + H_\lambda + H_{\lambda+1}$. If the round handle R_λ was glued, then the intersection index of H_λ and $H_{\lambda+1}$ is equal to 0.*

If we glue to the twisted handle TR_λ, then the intersection index H_λ and $H_{\lambda+1}$ is equal to ± 2.

Proof. The case when the handle is attached was proved in [4] (Lemma VIII.2). If glue twisted handle TR_λ to M^n_1 , then the argument is the same. Let $\varphi\colon ([0,1] \times \partial D^\lambda \times D^{n-\lambda-1}/\sim) \longrightarrow$ $\longrightarrow \partial M^n_1$ be a gluing map. Represent $\varphi([0,1] \times 0 \times 0/\sim)$ as the sum of two segments I_1 and I_2 such that $I_1 \cap I_2 = \partial I_1 = \partial I_2$ and $I_1 \cup I_2 = \big(\varphi\big([0,1] \times 0 \times 0/\sim\big)$. Consider the submanifold $H_\lambda = I_1 \times D^\lambda \times D^{n-\lambda-1}$. Obviously it can be regarded as a handle of index λ, which is attached to ∂M^n_1 along the set $\partial D^\lambda \times D^{n-\lambda-1} \times I_1$ with the restriction of φ. It is clear that the manifold

$H_{\lambda+1} = \overline{TR_\lambda \setminus (I_1 \times D^\lambda \times D^{n-\lambda-1})} = I_2 \times D^\lambda \times D^{n-\lambda-1})$ is the handle of the index $\lambda+1$, which is attached to $\partial(M_1^n \cup H_\lambda)$ along the set $(\partial I_2 \times D^\lambda \cup I_2 \times \partial D^\lambda) \times D^{n-\lambda-1}$.

By construction, the intersection index of these two handles is ± 2.

Lemma 2.2 is proved.

Lemma 2.3. *Let M^n be a smooth closed manifold, $f\colon M^n \to [0,1]$ an S^1-Bott function, and c its critical value. Suppose $\varepsilon > 0$, and that on the interval $[c-\varepsilon, c+\varepsilon]$ there are no other critical values. Assume that on the surface level $f^{-1}(c)$ there are critical circles $\gamma_1, \ldots, \gamma_k$ of indices $\lambda_1, \ldots, \lambda_k$ with trivial coordinate systems and there are critical circles $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$ of indices $\mu_1, \ldots, \mu_l$ with twisted coordinate systems. Then the homology groups $H_*\big(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$ is generated exactly by the handles which correspond to the critical circles $\gamma_1, \ldots, \gamma_k$, $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_l$. Each circle γ_i generates two subgroups that are isomorphic to $\mathbf{Z}$, a direct product of the homology group $H_{\lambda_i}\big(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$, and the other in the homology group $H_{\lambda_{i+1}}\big(f^{-1}[c-\varepsilon, c+ + \varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$. Each circle $\tilde{\gamma}_j$ generates a subgroup $\mathbf{Z}_2$ which is direct product in a group $H_{\mu_j}\big(f^{-1}[c-\varepsilon, c+\varepsilon], f^{-1}(c-\varepsilon), \mathbf{Z}\big)$.*

Proof. Consider a function f associated to the decomposition of the manifold $f^{-1}[c-\varepsilon, c+\varepsilon]$ by the round and twisted handles. Thus the critical circles lie on the same level of the decomposition of the round and twisted handles We can choose the handles so that they do not intersect each other. If we replace the round handles by handles from the previous lemma it follows that each twisted round handle of index λ generates the homology of a subgroup isomorphic to $\mathbf{Z}_2$ in dimension λ, and each round handle of index λ generates the homology of two subgroups isomorphic to $\mathbf{Z}$ in dimensions λ and $\lambda+1$.

Lemma 2.3 is proved.

***Corollary* 2.1.** *Let M^n be a smooth closed manifold, $f\colon M^n \to [0,1]$ an S^1-Bott function, and $c_1, \ldots, c_k$ its critical values. Suppose $\varepsilon_i > 0$, $1 \le i \le k$, such that the interval $[c_i - \varepsilon_i, c_i + \varepsilon_i]$ has no other critical values. Then on a level surface $f^{-1}(c_i)$ there are only critical circles with trivial coordinate systems if and only if the nonzero homology groups $H_*\big(f^{-1}[c_i-\varepsilon_i, c_i+\varepsilon_i], f^{-1}(c_i-\varepsilon_i), \mathbf{Z}\big)$ are free Abelian groups.*

Thus we have a homological criterion when S^1-Bott functions do not have critical circle with twisted coordinate systems.

In the next section, we give another class of S^1-Bott function which do not possess the critical circle with twisted coordinate systems.

3. Diagrams of S^1-Bott functions and their applications. In this section we explore S^1-Bott functions. We recall the definition of **partitions of diagrams** [4]. Partition diagrams represent the construction of S^1-Bott functions, especially for simply connected manifolds.

Consider the decomposition of a closed smooth manifold M^n by handles $M_0^n \subset M_1^n \subset \ldots \ldots \subset M_n^n = M^n$, where the manifold M_i^n is obtained from the manifold M_{i-1}^n by attaching handles of index i. Assume that $C_i = H_i(M_i^n, M_{i-1}^n, \mathbf{Z}) \approx \underbrace{\mathbf{Z} \oplus \ldots \oplus \mathbf{Z}}_{k_i}$, where k_i is the number of handles of index i. Mean discs handles of index i form a basis for the homology groups $H_i(M_i^n, M_{i-1}^n, \mathbf{Z})$. Using the exact homology sequence for the triple $M_{i-1}^n \subset M_i^n \subset M_{i+1}^n$ we can construct a chain complex of free Abelian groups: $(C, \partial)\colon C_0 \leftarrow \ldots \leftarrow C_{i-1} \xleftarrow{\partial_i} C_i \xleftarrow{\partial_{i+1}} C_{i+1} \leftarrow \ldots \leftarrow C_n$, whose homology coincides with the homology of the manifold M^n. Suppose that manifold M^n is oriented. The choice of orientation allows to orient medium and comedium sphere of the handle which allows to determine the homology indices λ and $\lambda+1$ in manifolds ∂M^n_λ. Thus the homomorphism ∂_λ given by the matrix of indices of homologous intersections of right-hand and left-hand spheres of handles in the submanifold ∂M^n_λ.

If each handle determines a vertex, and bridge the edges of those vertices for which the corresponding handles have a non-zero intersection, then we obtain a graph. Note that the structure of this graph can be complicated. However, it can be simplified.

It is known [4] that with the addition of handles all the matrices of homomorphisms ∂_i, $0 \le i \le n$, can be made diagonal.

Suppose that M^n is a simply connected manifold, $n > 5$ and there are no handles of the indices 1 and $n-1$, then certainly the homology of the intersection indices of right-hand and left-hand spheres coincide with their geometric intersection indices.

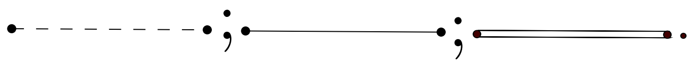
Thus a pair of adjacent handles with indices λ and $\lambda+1$ may either not intersect or have intersection ± 1, ± 2 or $\pm m$, where $|m| > 2$. Since the Euler characteristic of a closed smooth manifold M^n which admits a round handles decomposition is zero, it follows that the decomposition M^n by handles, we can introduce the following object, **a diagram**. A diagram is a disconnected graph whose vertices correspond to handles and whose edges connect vertices if and only if the intersection of the handle is nonzero. More precisely:

Definition 3.1. *Ω_n is called a diagram of length n, if the plane is given by $(n+1)$ set of points $\left(a^1_0, \ldots, a^1_{k_0}; a^1_1, \ldots, a^1_{k_1}; \ldots; a^n_1, \ldots, a^n_{k_n}\right)$, which satisfy the following conditions*:

1) *for some i the set $(a^i_1, \ldots, a^i_{k_i})$ may be empty,*

2) $k_0 - k_1 + k_2 - \ldots + (-1)^n k_n = 0$,

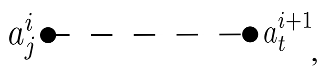
3) *point of the set $(a^i_1, \ldots, a^i_{k_i})$, $1 < i < n-1$, can be connected either with only one point from the set $\left(a^{i-1}_1, \ldots, a^{i-1}_{k_{i-1}}\right)$ or with only one point from $(a^{i+1}_1, \ldots, a^{i+1}_{k_{i+1}})$ in one of three ways.*



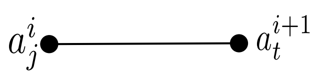
A set of points $a^0_1, \ldots, a^1_{k_0}; \ldots; a^i_1, \ldots, a^i_{k_i}$ is called an i-**skeleton diagram** Ω_n.

A point at which the chart is not linked to some other point is called **free**. If the chart has a fragment

then a^i_i is called a **semi-free** point (intersection of the handle is ± 2). If there is a fragment

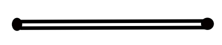


then a^i_i is called a **dependent** point (intersection of the handle is $\pm m$). Fragment

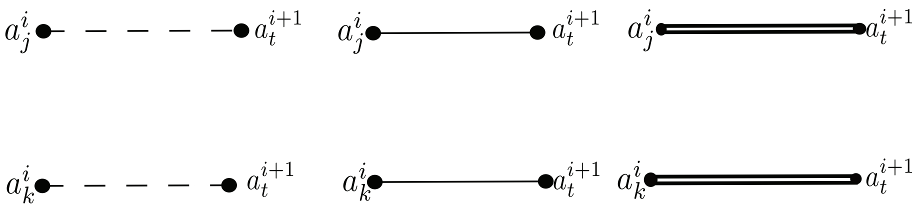


is called **inserted in dimension** i (the index of intersection of corresponding handles is equal to ± 1).

Definition 3.2. *A pair of points of dimension i and $i+1$ are independent in the dimension i, if there is no connection between them or if they form a fragment.*



In what follows we divide a chart into disjoint pairs of independent points. Let us make a restriction for the fragments of the diagram form



We do not allow breaking any of the fragments into a pair of the form (a_j^i, a_t^{i+1}), (a_k^i, a_l^{i+1}).

Definition 3.3. *If a chart Ω_n can be represented as the disjoint union of independent pairs of points, then it admits a partition. A pair of points (a_j^i, a_k^{i+1}) of this partition is called the* ***vertices*** *of the partition in dimension i.*

Let us fix a partition diagram Ω_n denoted by $\Omega_n(\sigma)$. It is possible that the diagram $\Omega_n(\sigma)$ does not admit a partition since in some dimensions it may not have enough points for the formation of independent pairs.

Definition 3.4. *The* ***base*** *of the diagram Ω_n is the diagram $\overline{\Omega}_n$ obtained from Ω_n by eliminating all inserts.*

Definition 3.5. *A* ***stabilization*** *of the diagram Ω_n in dimension i is a diagram of the form $\Omega_n^{S(i)} = \Omega_n \underset{i}{\cup} A_i$, where A_i is a new insert on dimension i.*

Lemma 3.1. *For each chart Ω_n there exists its stabilization in dimensions $i_1, \ldots, i_s$, denoted by $\Omega_n^{S(i_1,\ldots,i_s)}$, such that the diagram $\Omega_n^{S(i_1,\ldots,i_s)}$ admits a partition.*

Definition 3.6. *The number $\chi_i(\Omega_n) = k_i - k_{i-1} + \ldots + (-1)^{i+1}k_0$ is called i-th Euler characteristic of the diagram Ω_n.*

Obviously, the insertion of dimension i increases the i-th Euler characteristic of $\Omega_n^{S(i)}$ the unit and it does not change the values of the remaining j-Euler characteristics $\chi_j\left(\Omega_n^{S(i)}\right) = \chi_j(\Omega_n)$ for $j \neq i$.

Lemma 3.2. *If the diagram Ω_n admits a partition, then the number of vertices of the partition Ω_n in each dimension is the same for all of its possible partitions.*

Suppose that the diagram Ω_n admits a partition. Denote by $m_i(\Omega_n)$ the number of vertices in dimension i of a partition Ω_n and by $M(\Omega_n)$ the number $M(\Omega_n) = \sum_{j=0}^{i} m_j(\Omega_n)$.

In light of the lemma this numbers does not depend on the choice of a particular partition of the diagram Ω_n.

Definition 3.7. *Dimension λ of a chart Ω_n is called singular if $\chi_{\lambda-1}(\Omega_n) = \chi_{\lambda+1}(\Omega_n) = 0$, $\chi_\lambda(\Omega_n) = k > 0$ and chart Ω_n in dimensions λ does not consist of semi-free fragments.*

In the process of decomposition of the diagram Ω_n into a pair of independent points is necessary in this situation it to make one box of dimension $\lambda-1$ or in dimension $\lambda+1$ which leads to ambiguity. The result will be different number of pairs in dimension $\lambda+1$ or in dimension $\lambda+1$, depending on whether in any dimension we have made insertions.

Lemma 3.3. *The diagram $\Omega_n = \left(a_0^1, \ldots, a_{k_0}^1; a_1^1, \ldots, a_{k_1}^1; \ldots; a_1^n, \ldots, a_{k_n}^n\right)$ admit a partition if and only if it has no negative i-th Euler characteristics and singular dimensions. If the diagram Ω_n admit partition then the number of vertices in dimension i of a partition is equal $m_i(\Omega_n) = \chi_i(\Omega_n)$.*

If the diagram Ω_n not admit partition then there exists its stabilization Ω_n^S, such that the diagram Ω_n^S admits a partition. Raises the question of the minimum possible number of the vertices in dimension i among stabilized diagram $\Omega_n^{S_j}$ have a partition.

For diagram Ω_n denote by $m_i^s(\Omega_n)$ the **minimum** possible number of the vertices in dimension i among stabilized diagram $\Omega_n^{S_j}$ have a partition.

Let **N** be the set of integers, put $\rho(n) = \dfrac{1}{2}\bigl(n + |n|\bigr)$, where $n \in \mathbf{N}$.

Theorem 3.1. *Let Ω_n be an arbitrary diagram, then $m_i^s(\Omega_n)$ of the diagram Ω_n is equal $m_i^s(\Omega_n) = \rho\bigl(\chi_i(\Omega_n)\bigr)$. If Ω_n^S is a stabilization of diagram Ω_n, then $m_i^s(\Omega_n^S) \geq m_i^s(\Omega)$.*

Definition 3.8. *For the diagram Ω_n its i-th Morse number $M_i(\Omega_n)$ is the the number $m_i^s(\overline{\Omega}_n)$, where $\overline{\Omega}_n$ is the base of the diagram Ω_n.*

Definition 3.9. *A diagram Ω_n is called exact if there exists a stabilization $\Omega_n^{S_*}$ of Ω_n, such that $\Omega_n^{S_*}$ admit partition with the number of vertices in dimension i is equal $m_i(\Omega_n^{S_*}) = M_i(\Omega_n)$ simultaneously for all i .*

Theorem 3.2. *The diagram Ω_n is exact if and only if it does not have singular dimensions.*

A stabilization of diagram Ω_n is called economical if

1) when $\chi_i(\Omega_n) = k < 0$, perform k insert in the dimension i,

2) when i is singular dimension, then perform on insert in the dimension $i-1$ or in dimension $i+1$.

We now describe how we can construct a diagram Ω_n on a decomposition of a smooth closed manifold M^n on round handles.

Let $M_0^n(R) \subset M_1^n(R) \subset \ldots \subset M_{n-1}^n(R) = M^n$, be a round handle decomposition M^n. By Lemma 2.2, we replace each hand index λ on two ordinary handles of indices λ and $\lambda+1$. As a result, we obtain an expansion of the manifold M^n by handles: $M_0^n \subset M_1^n \subset \ldots \subset M_n^n = M^n$. Using this handle decomposition of M^n we can construct a chain complex of free abelian groups: $(C, \partial)\colon C_0 \leftarrow \ldots \leftarrow C_{i-1} \xleftarrow{\partial_i} C_i \xleftarrow{\partial_{i+1}} C_{i+1} \leftarrow \ldots \leftarrow C_n$. Citing the matrix of differentials to the diagonal form, we construct a diagram Ω_n. The following fact holds:

Proposition 3.1. *Let $M_0^n(R) \subset M_1^n(R) \subset \ldots \subset M_{n-1}^n(R) = M^n$ be a round handle decomposition of manifold M^n and Ω_n be a diagram associated to this decomposition. Assume that the diagram Ω_n has no semi-free vertices. If the diagram Ω_n is the economical stabilization of its base $\overline{\Omega}_n$, then the original decomposition on the round handles had missing twisted round handles.*

Proof. Indeed, in this case, the diagram Ω_n does not allow the insertion of the round twisted handle. All the points of insertion involved for the formation of vertices with other points of the diagram. And by condition of the proposition semi-free vertices are not present.

Remark **3.1.** It is easy to construct a decomposition of a manifold M^n by the round handles, among which are twisted round handles, but at the same time have associated with this decomposition diagram no semi-free vertices.

Definition 3.10. *Let M^n be a smooth closed manifold. The number $\chi_i(M^n) = \mu\big(H_i(M^n, \mathbf{Z})\big) - \mu\big(H_{i-1}(M^n, \mathbf{Z})\big) + \ldots + (-1)^{i+1}\mu\big(H_0(M^n, \mathbf{Z})\big)$ is called the i-th Euler characteristic of M^n, where $\mu(H)$ is a minimal number of generators H.*

Definition 3.11. *A dimension λ of closed manifold M^n is called singular if $H_\lambda(M^n, \mathbf{Z})$ is a nonzero finite group distinct from $\mathbf{Z}_2 \oplus \ldots \oplus \mathbf{Z}_2$ and $\chi_{\lambda-1}(M^n) = \chi_{\lambda+1}(M^n) = 0$.*

Definition 3.12. *Let M^n be a smooth closed manifold. A round handle decomposition is called quasiminimal, if one of the following holds*:

1) *the number of round handles of index i equals to $\rho(\chi_i\big(M^n)\big) + \varepsilon_i$, where $\varepsilon_i = 0$, if dimension $i+1$ is nonsingular and $\varepsilon_i = 1$, if dimension $i+1$ is singular*;

2) *the number of round handles of index i equals to $\rho\big(\chi_i(M^n)\big)$, if dimension $i+1$ is singular, then there is only one handle of index $i+2$.*

In both cases, the number of round handles of index $i+1$ equals to $\rho\big(\chi_{i+1}(M^n)\big)$. A round handle decomposition is called minimal, if number of round handles of index i equals to $\rho\big(\chi_i(M^n)\big)$ for all i.

Using the decomposition of manifold on handles and the diagram technique, we can easily prove the following fact [4].

Proposition 3.2. *Let M^n be a smooth closed simply-connected manifold $(n > 5)$. Then M^n admits a* ***quasiminimal*** *decomposition into round handles. If manifold M^n have not singular dimensions, then M^n admits a* ***minimal*** *decomposition into round handles.*

Definition 3.13. *Let the manifold M^n admits S^1-Bott function, then S^1**-Morse number** $M_i^{S^1}(M^n)$ **of index** i is the minimum number of singular circles of index i taken over all S^1-Bott functions on M^n.*

Lemma 3.4. *Let on a closed manifold M^n exist a smoth function $f\colon M^n \to \mathbb{R}$ such that each connected component of the singular set Σ_f of f is either a nondegenerate critical point p_i, $i = 1, \ldots, k$, or a nondegenerate critical circle S_j^1, $j = 1, \ldots, l$. Then the Euler characteristic of the manifold M^n is equal to $\chi(M^n) = \sum_{i=1}^{k} (-1)^{\text{index}(p_i)}$.*

Proof. It is known that for any Morse function on the manifold M^n $g\colon M^n \to \mathbb{R}$ with critical points p_i, $i = 1, \ldots, q$, there is the formula $\chi(M^n) = \sum_{i=1}^{q} (-1)^{\text{index}(p_i)}$. By small perturbation of the function f any nondegenerate critical circle S_j^1 of index λ can be replaced by nondegenerate critical points of idexes λ and $\lambda+1$ [1]. Therefore the contribution in the formula for Euler characteristic of these critical points will be zero and we obtain the desired formula.

4. Manifolds with free S^1-action. Let on smooth manifold M^n there is smooth free circle action. Then of course the set M^n/S^1 is a manifold and natural projection $p\colon M^n \to M^n/S^1$ is fibre bundle. Any smooth S^1-invariant function $f : M^n \to \mathbb{R}$ on a manifold M^n is called an S^1**-invariant Bott function** if each connected component of the singular set Σ_f is nondegenerate critical circle.

It is clear that if f be a S^1-invariant Bott function on the manifold M^n then it projection $\pi_*(f)\colon M^n/S^1 \to \mathbb{R}$, is a Morse function. And conversly, if $g\colon M^n/S^1 \to \mathbb{R}$ be a Morse function on the manifold M^n/S^1 then $\pi_*^{-1}(g) = g\circ\pi\colon M^n \to \mathbb{R}$ is S^1-invariant Bott function on the manifold

M^n. The critical point of the index λ of the function g correspond to critical circle of the index λ of the function $\pi_*^{-1}(g)$.

In this situation for the manifold M^n S^1**-equivariant Morse number of index** i, $M_i^{eqS^1}(M^n)$ is the minimum number of singular circles of index i taken over all S^1-invariant Bott functions on M^n.

For the manifold M^n/S^1 **Morse number of index** i, $M_i(M^n/S^1)$ is the minimum number of critical points of index i taken over all Morse functions on M^n/S^1.

Therefore for a calculation of the S^1-equivariant Morse number of the index i there is possible to use a Morse functions on the manifold M^n/S^1. The following fact is clear:

***Corollary* 4.1.** *Let on smooth manifold M^n there is smooth free circle action. Then for the manifold M^n S^1-equivariant Morse number of index i is equal Morse number of index i for the manifold M^n/S^1.*

Good example in this direction is the fibre bundle $p\colon S^{2n+1} \to \mathbb{CP}^n$. For this S^1-action a S^1-equivariant Morse number of even indexes equal 1 and for odd indexes equal 0.

The next example to show that S^1-equivariant Morse number of the manifold M^n depends on circle action.

Let $p\colon S^3 \to S^2$ be the Hopf fibre bundle. Suppose that circle trivial act on S^1. Using Hopf fibre bundle and trivial circle action on S^1 we shall construct new fibre bundle $p \times \mathrm{id}\colon S^3 \times S^1 \to S^2 \times S^1$. It is clear, that on manifold $S^2 \times S^1$ there is a Morse function with one critical point of the indexes 0, 1, 2, 3. Therefore for such circle action on manifold $S^2 \times S^1$ the S^1-equivariant Morse number for any index equal 1.

On the other side, let circle act trivially on S^3 and q is rotation on S^1. Consider fibre bundle $\mathrm{id} \times q\colon S^3 \times S^1 \to S^3$. On S^3 there is a Morse function with one critical point of the indexes 0 and 3. Therefore in this situation for the manifold $S^3 \times S^1$ the S^1-equivariant Morse number for indexes 0 and 3 equal 1 and 0 for other indexes.

***Remark* 4.1.** This example shows that for manifold S^1-equivariant Morse number and S^1-Morse number of some index may be different.

Definition 4.14. *Let on smooth manifold M^n there be a smooth free circle action. Then this free circle action is* ***minimal*** *if for all indexes S^1-equivariant Morse number is equal S^1-Morse number for the manifold M^n.*

***Corollary* 4.2.** *Let on smooth simply-connected manifold M^n there be a smooth minimal free circle action. Then manifold M^n have not singular dimensions.*

Proof. Obviously, for dimension three corolary is valid.

A manifold, which allows free circle action has Euler characteristic zero. If $n = 4$, then free action on simply-connected manifolds M^4 non exist, since the Euler characteristic of a simply connected four-dimensional manifold is always positive.

From the structure of the homology groups follows that simply-connected manifold M^n, $8 \geq \geq n \geq 5$, have not singular dimensions.

Let $n \geq 9$. Suppose that on M^n there be a minimal smooth free circle action. Obviosly, that there be a equality $M_i(M^n/S^1) = M_i^{eqS^1}(M^n)$. By Smale theorem [10] on manifold $M_i(M^n/S^1)$ there be a Morse function with the number of critical points of index i is equal $M_i(M^n/S^1)$ for all i simultaneously. Therefore on the manifold M^n there exist S^1-invariant Bott function f with the number of critical circles of index i is equal $M_i(M^n/S^1) = M_i^{eqS^1}(M^n)$ for all i simultaneously. Since free circle action is minimal, therefore $M_i^{S^1}(M^n) = M_i^{eqS^1}(M^n)$. If simply-connected manifold M^n have singular dimension then S^1-Bott function on M^n can not have the number of critical circles of index i equal to i-th S^1-Morse number $M_i^{S^1}(M^n)$ for all i simultaneously. Consequently, the manifold M^n has no singular dimensions.

Corollary 4.2 is proved.

Theorem 4.1. *Let on smooth simply-connected manifold M^n there be a smooth free circle action. Then this circle action is minimal if and only if* $\mu(H_i(M^n/S^1, Z) + \mu(\text{Tors } H_{i-1}(M^n/S^1, Z) = \rho\big(\chi_i(M^n)\big)$ *for all i.*

Proof. From exact homotopy sequence of fibration follow that manifold M^n/S^1 is simply-connected. Let $n = 3$, using results about three dimension Poincare conjecture [13] can obtained that $M^3 = S^3$, $M^3/S^1 = S^2$ and we have Hopf fibre bundle $p\colon S^3 \to S^2$. Therefore, Theorem 4.1 is proved.

If $n = 4$, then free action on simply-connected manifolds M^4 non exist.

Let $n \geq 5$. *Necessary.* Suppose that on M^n there be a minimal smooth free circle action. If $n \geq 5$ from results of Smale and Barden [3, 10] it follows that Morse number in dimension i of the manifold M^n/S^1 is equal $M_i(M^n/S^1) = \mu(H_i\big(M^n/S^1, Z)\big) + \mu\big(\text{Tors } H_{i-1}(M^n/S^1, Z)\big)$. There is equality $M_i(M^n/S^1) = M_i^{eqS^1}(M^n)$. Because of the condition of minimal free circle action there is equality $M_i(M^n/S^1) = M_i^{eqS^1}(M^n) = M_i^{S^1}(M^n) = \rho\big(\chi_i(M^n)\big)$.

Sufficiently. Consider on manifold M^n/S^1 Morse function with the number of critical points of index i equal $M_i(M^n/S^1) = \mu(H_i(M^n/S^1, Z)) + \mu(\text{Tors } H_{i-1}(M^n/S^1, Z))$. By the construction and condition of the theorem we have the equalities $M_i(M^n/S^1) = M_i^{eqS^1}(M^n) = \rho(\chi_i(M^n))$. But $M_i^{S^1}(M^n) = \rho\big(\chi_i(M^n)\big)$ and therefore free action of S^1 is minimal.

Theorem 4.1 is proved.

***Corollary* 4.3.** *Let on smooth manifold M^n there be a smooth free circle action. Suppose that manifold M^n/S^1 is*

a) $\pi_1(M^n/S^1) \approx \mathbb{Z}$, *or* $\pi_1(M^n/S^1) \approx \mathbb{Z} \oplus \mathbb{Z}$, $n > 6$;

b) $\pi_1(M^n/S^1)$ *is infinite*, $n > 8$.

Then S^1-equivariant Morse number of index i for manifold M^n equal

a) $\widehat{S}^i_{(2)}(M^n/S^1) + \widehat{S}^{i+1}_{(2)}(M^n/S^1) + \dim_{N(Z[\pi])}\big(H^i_{(2)}(M^n/S^1, \mathbb{Z})\big)$;

b) $\mathbb{D}^i(M^n/S^1) + \widehat{S}^i_{(2)}(M^n/S^1) + \widehat{S}^{i+1}_{(2)}(M^n/S^1) + \dim_{N(Z[\pi])}\big(H^i_{(2)}(M^n/S^1, \mathbb{Z})\big)$, *for* $3 < i < n-3$.

Proof. It follows from results of [4, 14] that on M^n there is Morse functions with the number of critical points of index i equal Morse number of the manifold M^n/S^1.

5. Manifolds with semi-free S^1-action. Let M^{2n} be a closed smooth manifold with semi-free S^1-action which has only isolated fixed points. It is known that every isolated fixed point p of a semi-free S^1-action has the following important property: near such a point the action is equivalent to a certain linear $S^1 = SO(2)$-action on $\mathbb{R}^{2n}$. More precisely, for every isolated fixed point p there exist an open invariant neighborhood U of p and a diffeomorphism h from U to an open unit disk D in $\mathbb{C}^n$ centered at origin such that h is conjugate to the given S^1-action on U to the S^1-action on $\mathbb{C}^n$ with weight $(1, \ldots, 1)$. We will use both complex, $(z_1, \ldots, z_n)$, and real coordinates $(x_1, y_1, \ldots, x_n, y_n)$

on $\mathbb{C}^n = \mathbb{R}^{2n}$ with $z_j = x_j + \sqrt{-1}y_j$. The pair (U, h) will be called a **standard chart** at the point p. Let $f\colon M^{2n} \to \mathbb{R}$ be a smooth S^1-invariant function on the manifold M^{2n}. Denote by Σ_f the set of singular points of the function f. It is clear that the set of isolated singular points $\Sigma_f(p_j) \subset \Sigma_f$ of f coincides with the set of fixed points M^{S^1}.

For a nondegenerate critical point p_j there exist a standard chart (U_j, h_j) such that on U_j the function f is given by the following formula:

$$f = f(p) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2.$$

Notice that the index of nondegenerate critical point p_j is always even.

Denote by $\Sigma_f(S^1)$ the set singular points of the function f that are disconnected union of circles. These circles will be called singular.

A circle $s \in \Sigma f_(S^1)$ is called nondegenerate if there is an S^1-invariant neighborhood U of s on which S^1 acts freely and such that the point $\pi(s)$ is nondegenerate for the function $\pi_*(f)\colon U/S^1 \to$ $\to \mathbb{R}$, induced on U/S^1 by the natural map $\pi\colon U \to U/S^1$. An invariant version of Morse lemma says that there exist an S^1-invariant neighborhood U of the circle s and coordinates $(x_1, \ldots, x_{2n-1})$ on U/S^1 such that the function $\pi_*(f)$ has the following presentation:

$$\pi_*(f) = \pi_*\big(f(\pi(s))\big) - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_{2n-1}^2.$$

By definition λ is the **index** of singular circle s.

Definition 5.1. *A smooth S^1-invariant function $f\colon M^{2n} \to \mathbb{R}$ on a manifold M^{2n} with a semi-free circle action which has isolated fixed points is called*: *S^1_*-Bott function if each connected component of the singular set Σ_f is either a nondegenerate fixed point or a nondegenerate critical circle.*

Theorem 5.1. *Assume that M^{2n} is the closed manifold with a smooth semi-free circle action which has isolated fixed points $p_1, \ldots, p_k$. Let for any fixed point p_j consider standard chart (U_j, h_j) and function*

$$f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_{\lambda_j}|^2 + |z_{\lambda_j+1}|^2 + \ldots + |z_n|^2$$

*on U_j, where λ_j is an **arbitrary integer** from $0, 1, \ldots, n$.*

Then there exist an S^1-invariant S^1_*-Bott function f on M^{2n} such that $f = f_j$ on U_j.

Proof. Consider on U_j the function f_j. Let $\pi_*(f_j)\colon U_j/S^1 \to \mathbb{R}$, continuos function induced on U_j/S^1 by the natural map $\pi\colon U_j \to U_j/S^1$. It is clear that function $\pi_*(f_j)$ is smooth on manifold $(U_j \setminus p_j)/S^1$. Denote by g smooth extension functions $\pi_*(f_j)$ on M^{2n}/S^1. By small deformation of the function g, that is fixed on U_j/S^1, we shall find function g_1 on M^{2n}/S^1 such that g_1 equal $\pi_*(f_j)$ on U_j/S^1 and g_1 have only nondegenerate critical points on $M^{2n} \setminus \bigcup(U_j/S^1)$. Then the function $f = g_1 \circ p$ satisfics conditions of the theorem.

Theorem 5.2. *The number of fixed points of any smooth semi-free circle action on M^{2n} with isolated fixed points is always even and equal to the Euler characteristic of the manifold M^{2n}.*

Proof. In first we consider following functions:

$$f_1 = f_1(p_1) + |z_1|^2 + \ldots + |z_n|^2 \qquad \text{on} \quad U_1 \qquad \text{and} \quad f_j = f_j(p_i) - |z_1|^2 - \ldots - |z_n|^2$$

on U_j, $2 \leq j \leq l$, and extend such functions to S^1-invariant Bott function f on manifold $M^{2n} \setminus U_1 \bigcup U_2 \bigcup \ldots \bigcup U_l$. We suppose that U_j is diffeomorfic to open disk D^{2n} for any j. Consider manifold $V^{2n} = W^{2n} \setminus \bigcup U_j$. The boudary of manifod V^{2n} is disconnected union of spheres S^{2n-1}. By construction of manifold V^{2n} there is free cirle action. The boundary of the manifold V^{2n}/S^1 is disconnected union of complex projective spaces $\mathbb{CP}^{n-1}$. If the number of the boundary components of the manifold V^{2n}/S^1 is odd then we glue pairwise boundary components and obtain compact smoth manifold with with boundary $\mathbb{CP}^{n-1}$. From the well known fact that the manifold $\mathbb{CP}^{n-1}$ is non-cobordant to zero it follows that the number of fixed points of any smooth semi-free circle action on M^{2n} with isolated fixed points is even. The value of the Euler characteristic $\chi(M^{2n}) = 2k$ is follow from Lemma 3.4.

Definition 5.2. *Let f be an S^1-invariant S^1_*-Bott function for smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$ on a closed manifold M^{2n}. Denote by λ_j the index of a critical point p_j of the function f. The* ***state*** *of the function f is the collection of numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, which we will be denoted by $St_f(\Lambda)$. It is clear that all numbers λ_j are even and $(0 \leq \lambda_j \leq 2n)$.*

***Remark* 5.1.** It follows from Theorem 5.1 that for every smooth semi-free circle action on a closed manifold M^{2n} with isolated fixed points $p_1, \ldots, p_{2k}$ and any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$ there exists an S^1-invariant S^1_*-Bott functions f on M^{2n} with state $St_f(\Lambda)$.

Definition 5.3. *Let M^{2n} be a closed smooth manifold with smooth semi-free circle action which has finitely many fixed points $p_1, \ldots, p_{2k}$. Fix any collection even numbers $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2k})$, such that $0 \leq \lambda_j \leq 2n$.*

The S^1-Morse number $\mathcal{M}_i^{S^1}\big(M^{2n}, St(\Lambda)\big)$ of index i is the minimum numbers of singular circles of index i taken over all S^1-invariant S^1_*-Bott functions f on M^{2n} with state $St_f(\Lambda)$.

The following is an unsolved problem: for a manifold M^{2n} with a semi-free circle action which has finitely many fixed points **find exact values** of numbers $\mathcal{M}_i^{S^1}\big(M^{2n}, St(\Lambda)\big)$.

6. About S^1-equivariant Morse numbers $\mathcal{M}_i^{S^1}\big(M^{2n}, St(\Lambda)\big)$. Let M^{2n} be a compact closed manifold of dimension with semi-free circle action which has finite many fixed points $p_1, , \ldots, p_{2k}$. Denote by $\pi\colon M^{2n} \to M^{2n}/S^1$ the canonical map. The set M^{2n}/S^1 is manifold with singular points $\pi(p_1), \ldots, \pi(p_{2k})$. It is clear that neighborhood of any singular point is a cone over $\mathbb{CP}^{n-1}$. If $f\colon M^{2n} \to \mathbb{R}$ is a smooth S^1-invariant S^1_*-Bott function on the manifold M^{2n}, then $\pi_*(f)\colon M^{2n}/S^1 \to \mathbb{R}$ is a continuos function such that on smooth non-compact manifold $N^{2n-1} = M^{2n}/S^1 \setminus \bigcup_{j=1}^{2k} \pi(p_j)$ it is Morse function.

Choose an invariant neighborhood U_i of the point p_j diffeomorphic to the open unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_{j=1}^{2k} U_j$. Consider compact manifold $V^{2n-1} = (M^{2n} \setminus U)/S^1$, its boundary is a disconnected union of complex projective spaces $\partial V^{2n-1} = \mathbb{CP}_1^{n-1} \cup \ldots \cup \mathbb{CP}_{2k}^{n-1}$. It is clear that manifold $V^{2n-1} \setminus \partial V^{2n-1}$ and manifold N^{2n-1} are diffeomorphic. We use a manifold V^{2n-1} for the study of S^1-invariant S^1_*-Bott functions on the manifold M^{2n} with state $St(\Lambda) =$

$= (0, \ldots, 0, 2n, \ldots, 2n)$. Let $\partial_0 V^{2n-1}$ be a part of boundary of V^{2n-1} consist from r component $\mathbb{C}P^{2n-2}$, $2k-1 \geq r \geq 1$, and $\partial_1 V^{2n-1} = \partial V^{2n-1} \backslash \partial_0 V^{2n-1}$. On the manifold with boundary V^{2n-1} constructed Morse function $f\colon V \to [0,1]$, such that $f^{-1}(0) = \partial_0 V^{2n-1}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f we constructed on the manifold M^{2n} S^1-equivariant S^1_*-Bott function F with the state $St(0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction $\pi_*(F)$ on V coinside with f. Therefore Morse number of index i $M_i(V^{2n-1}, \partial_0 V^{2n-1})$ of manifold with boundary V^{2n-1} is equal $\mathcal{M}_i^{S^1}(M^{2n}, St(0, \ldots, 0, 2n, \ldots, 2n)$.

Theorem 6.1. *Let M^{2n} $(2n > 8)$ be a closed smooth manifold admits a smooth semi-free circle action with isolated fixed points $p_1, \ldots, p_{2k}$. Then for the manifold M^{2n} with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$*

$$\mathcal{M}_i^{S^1}(M^{2n}, St(\Lambda) = \mathbb{D}^i(V^{2n-1}, \partial_0 V^{2n-1}) + \widehat{S}^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) +$$

$$+ \widehat{S}^{i+1}_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) + \dim_{N(Z[\pi])} \left(H^i_{(2)}(V^{2n-1}, \partial_0 V^{2n-1}) \right)$$

for $3 \leq i \leq 2n-4$.

Proof. Choose an invariant neighborhood U_i of the point p_i diffeomorphic to the unit disc $D^{2n} \subset \mathbb{C}^n$ and set $U = \bigcup_i U_i$. Let f_i be a function on U_i equal

$$f_i = |z_1|^2 + \ldots + |z_n|^2 \qquad \text{and} \qquad f_j \quad \text{on} \quad U_j \quad \text{equal} \quad f_j = 1 - |z_1|^2 - \ldots - |z_n|^2,$$

for $i = 1, \ldots, r$, $j = r+1, \ldots, 2k-r$. Consider the manifold $V^{2n} = (M^{2n} \setminus U)/S^1$. It is clear that its boundary is a disconnected union of complex projective spaces $\partial V^{2n} = \mathbb{C}P_1^{2n-2} \cup \ldots \cup \mathbb{C}P_{2k}^{2n-2}$.

Let $\partial_0 V^{2n}$ be a part of boundary of V^{2n} consist from r component $\mathbb{C}P^{2n-2}$, that correspondent U_i and $\partial_1 V^{2n}$ be a part of boundary consist from component $\mathbb{C}P^{2n-2}$, that correspondent U_j. On manifold $V^{2n} = (M^{2n} \setminus U)/S^1$ constructed Morse function $f\colon V \to [0,1]$, such that $f^{-1}(0) = \partial_0 V^{2n}$ and $f^{-1}(1) = \partial_1 V^{2n}$. Using the function f we constructed on manifold M^{2n} S^1-equivariant S^1_*-Bott function F with the state $St(\Lambda) = (0, \ldots, 0, 2n, \ldots, 2n)$, such that restriction F on U_i coinside with f_i, restriction F on U_j coinside with f_j and restriction $\pi_*(F)$ on V coinside with f. Therefore Morse number of cobordism V equal $\mathcal{M}^{\lambda}_{S^1}\big(M^{2n}, St(\Lambda)\big)$. In the paper [14] there is value of Morse number of a cobordism.

Theorem 6.1 is proved.

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