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SOME PROBLEMS OF THE LINEAR THEORY OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

ДЕЯКІ ПРОБЛЕМИ ЛІНІЙНОЇ ТЕОРІЇ СИСТЕМ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We consider problems of the linear theory of systems of ordinary differential equations related to the investigation of invariant hyperplanes of these systems, the notion of equivalence for these systems, and the Floquet – Lyapunov theory for periodic systems of linear equations. In particular, we introduce the notion of equivalence of systems of linear differential equations of different orders, propose a new formula of the Floquet form for periodic systems, and present the application of this formula to the introduction of amplitude-phase coordinates in a neighborhood of a periodic trajectory of a dynamical system.

Розглянуто проблеми лінійної теорії систем звичайних диференціальних рівнянь, пов'язані з дослідженням інваріантних гіперплощин таких систем, поняттям еквівалентності для вказаних систем та теорією Флоке – Ляпунова для періодичних систем лінійних рівнянь. Зокрема, введено поняття еквівалентності систем лінійних диференціальних рівнянь різних порядків, запропоновано нову формулу вигляду Флоке для періодичних систем, наведено застосування цієї формули для введення амплітудно-фазових координат в околі періодичної траєкторії динамічної системи.

Introduction. In this work, we consider problems of the linear theory of systems of ordinary differential equations related to the investigation of invariant hyperplanes of these systems, the notion of equivalence for these systems, and the Floquet – Lyapunov theory for periodic systems of linear equations.

The work is based on the preprints [1] and [2] and consists of three parts.

In the first part, we clarify conditions of invariance in the sense of Bogolyubov for two orthogonal hyperplanes with respect to a linear system of differential equations with variable coefficients. It is proved that the invariance of these hyperplanes is equivalent to the separation of the system of differential equations into two independent subsystems whose orders correspond to the dimensions of the hyperplanes.

On the basis of these investigations, we introduce the notion of equivalence of systems of differential equations of different orders.

In the second part, we consider a real T-periodic system of linear differential equations. We study the case, inadequately described by the Floquet – Lyapunov theory, of necessary period doubling under the reduction of this system to a system with constant coefficients by a real periodic matrix. We prove the real T-periodic equivalence of this system and a higher-order system of differential equations with constant coefficients.

In the third part, the results of the second part are used, first, to the reduction of a nonlinear system of differential equations with separated periodic linear part to a system of equations with constant coefficients with separated linear part of higher order and, second, for the introduction of local coordinates in the neighborhood of a periodic trajectory of an autonomous system of nonlinear differential equations.

1. Two lemmas. Suppose that matrices $\Phi_1(t)$ and $\Phi_2(t)$ are continuously differentiable for all $t \in \mathbb{R}$, $\Phi_1(t) \in \mathbf{M}_{nm}(\mathbb{R})$, $\Phi_2(t) \in \mathbf{M}_{pm}(\mathbb{R})$, where $m > n > p = m - n$, and the following condition is satisfied:

$$\det \begin{pmatrix} \Phi_1(t) \\ \Phi_2(t) \end{pmatrix} \neq 0.$$

Let $\Phi_1^+(t)$ and $\Phi_2^+(t)$ denote matrices pseudoinverse to $\Phi_1(t)$ and $\Phi_2(t)$ and defined by the condition

$$\begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}\left(\Phi_1^+(t), \Phi_2^+(t)\right) = E, \tag{1.1}$$

$$\Phi_1(t) \in \mathbf{M}_{n\,m}(\mathbb{R}), \quad \Phi_2(t) \in \mathbf{M}_{m-n\,m}(\mathbb{R}), \quad E \in \mathbf{M}_m(\mathbb{R}).$$

Let

$$M_1(t) = \Phi_1^+(t)\Phi_1(t),$$

$$M_2(t) = \Phi_2^+(t)\Phi_2(t).$$

If follows from (1.1) and the definitions of the matrices $M_1(t)$ and $M_2(t)$ that these matrices satisfy the conditions

$$M_\nu^2(t) = M_1(t), \quad \nu = 1, 2, \qquad \operatorname{rank} M_1(t) = n, \qquad \operatorname{rank} M_2(t) = p,$$

$$M_1(t)M_2(t) = M_2(t)M_1(t) = 0, \qquad M_1(t) + M_2(t) = E. \tag{1.2}$$

Let us prove equality (1.2). Indeed, since the matrix $(\Phi_1^+(t), \Phi_2^+(t))$ is inverse to the matrix $\begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}$, multiplying the latter from the left by the matrix $\left(\Phi_1^+(t), \Phi_2^+(t)\right)$ we obtain the equality

$$\Phi_1^+(t)\Phi_1(t) + \Phi_2^+(t)\Phi_2(t) = E,$$

which is equivalent to equality (1.2).

In the space $\mathbb{R}^m$, we define two subspaces by using the matrix $M_1(t)$, namely,

$$\begin{aligned} M^n(t) &= \{y \in \mathbb{R}^m : y = M_1(t)y\}, \\ M^{m-n}(t) &= \{y \in \mathbb{R}^m : M_1(t)y = 0\}, \end{aligned} \tag{1.3}$$

and two subspaces by using the matrix $M_2(t)$:

$$\begin{aligned} M_1^p(t) &= \{y \in \mathbb{R}^m : y = M_2(t)y\}, \\ M_1^{m-p}(t) &= \{y \in \mathbb{R}^m : M_2(t)y = 0\}. \end{aligned} \tag{1.4}$$

Lemma 1. *The subspaces $M^k(t)$ and $M_1^k(t)$, $k \in \{n, p\}$, satisfy the conditions*

$$M^n(t) = M_1^{m-p}(t) = \ker \Phi_2(t), \qquad M^{m-n}(t) = M_1^p(t) = \ker \Phi_1(t).$$

Indeed, according to properties of the matrix $M_1(t)$, the general solution of the equation defined by subspace (1.3) is the function

$$y = M_1(t)c(t), \tag{1.5}$$

where $c(t)$ is an arbitrary function with values in $\mathbb{R}^m$. According to properties of the matrix $M_2(t)$, the general solution of the equation defined by subspace (1.4) is the function

$$y = (E - M_2(t))c_1(t) = M_1(t)c_1(t).$$

Thus, the general solutions of the considered equations coincide for $c(t) = c_1(t)$, which proves the equality $M^n(t) = M_1^{m-p}(t)$.

It follows from the definition of M_1^{m-p} that

$$M_1^{m-p}(t) = \ker M_2(t) = \ker(\Phi_2^+(t)\Phi_2(t)).$$

Taking into account that the equality $\Phi_2(t)x = 0$ implies that

$$\Phi_2(t)\Phi_2^+(t)x = x = 0,$$

we conclude that $\ker \Phi_2^+(t) = 0$ and, hence, $\ker(\Phi_2^+(t)\Phi_2(t)) = \ker \Phi_2(t)$. This proves the first equality of Lemma 1. The second equality is proved by analogy.

Lemma 2. *The mapping $\Phi_1^+(t)\colon y = \Phi_1^+(t)x$ is a diffeomorphism of $\mathbb{R}^n$ into $M^n(t)$, and the mapping $\Phi_2^+(t)\colon y = \Phi_2^+(t)x$ is a diffeomorphism of $\mathbb{R}^n$ into $M^{m-n}(t)$.*

We prove only the first assertion of Lemma 2 because the second assertion is proved by analogy.

The matrix $\Phi_1^+(t)$, as a block of the matrix inverse to $\begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}$, is continuously differentiable and has a continuously differentiable pseudoinverse matrix, namely $\Phi_1(t)$. Therefore, to prove the first assertion of Lemma 2, it remains to prove that the mapping $\Phi_1^+(t)\colon \mathbb{R}^n \to M^n(t)$ is a homeomorphism.

Since $\ker \Phi_1^+(t) = 0$, we conclude that $\Phi_1^+(t)$ is a homeomorphism of $\mathbb{R}^n$ into the image of $\Phi_1^+(t;\mathbb{R}^n)$ under the mapping $\Phi_1^+(t)\colon \mathbb{R}^n \to \mathbb{R}^m$. It remains to prove that $\Phi_1^+(t;\mathbb{R}^n) = M^n(t)$.

Assume that this is not true. Then either there exists a point $y \in \Phi_1^+(t;\mathbb{R}^n)$ such that $y \notin M^n(t)$ or there exists a point $y \in \mathbb{R}^m$ such that $y \notin \Phi_1^+(t;\mathbb{R}^n)$.

In the first case, y is the image of a certain point from $\mathbb{R}^n$, namely the point $x = \Phi_1(t)y$, according to the equation $y = \Phi_1^+(t)x$. In this case, we have $y = \Phi_1^+(t)\Phi_1(t)y$, $y = M_1(t)y$, $y \in M^n(t)$, which contradicts the assumption.

In the second case, we have $y = M_1(t)y$ and, according to (1.5), $y = M_1(t)c(t)$ for a certain $c(t) \in \mathbb{R}^m$. Thus, $y = M_1(t)c(t) = \Phi_1^+(t)\Phi_1(t)c(t) = \Phi_1^+(t)x$, where $x = \Phi_1(t)c(t)$, i.e., y is the image of the point x under the mapping $\Phi_1^+(t)\colon \mathbb{R}^n \to \mathbb{R}^m$, whence $y \in \Phi_1^+(t;\mathbb{R}^n)$. This contradicts the assumption.

2. Main theorem. According to the results presented above, every continuously differentiable nonsingular matrix $\begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}$ defines, in the (t,y)-space $\mathbb{R}\times\mathbb{R}^m$, $m = \dim \begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}$, the two subspaces

$$\ker \Phi_1(t) = M_1^n(t), \quad n = \dim \Phi_1(t),$$

$$\ker \Phi_2(t) = M_1^{m-n}(t), \quad p = \dim \Phi_2(t) = m - n,$$

and two diffeomorphisms

$$\Phi_1^+(t)\colon \mathbb{R}^n \to M_1^n(t),$$

$$\Phi_2^+(t)\colon \mathbb{R}^p \to M_1^{m-n}(t).$$

Using a matrix $Q(t) \in \mathbf{M}_m(\mathbb{R})$ continuous for all $t \in \mathbb{R}$, we introduce, in the (t, y)-space $\mathbb{R} \times \mathbb{R}^m$, a linear vector field $(t, y' = Q(t)y)$ the integral curves of which are defined by the solutions $y = y(t)$ of the differential equation

$$\frac{dy}{dt} = Q(t)y. \tag{2.1}$$

If the union of the subspaces $M_1^n(t)$ and $M_1^{m-n}(t)$ (or one of these subspaces) is the union of integral curves of the vector field (t, y'), then these subspaces are called invariant manifolds of the differential equation (2.1) or the vector field (t, y'). If the subspace $M_1^k(t)$, $k \in \{n, p\}$, is an invariant manifold of Eq. (2.1), then "the motion of its points y in the (t, y)-space is independent of the motion of the points y outside the subspace $M^n(t)$ for both $t > 0$ and $t < 0$".

We pose the problem as follows: Find conditions under which the subspace $M_1^k(t)$ is an invariant manifold of Eq. (2.1). An equivalent statement of this problem is the following: Find conditions under which the solutions $y = y(t)$ of Eq. (2.1) satisfy one of the additional conditions

$$y = M_1(t)y$$

and

$$M_1(t)y = 0$$

for any $t \in \mathbb{R}$.

Finally, according to the terminology of Krylov – Bogolyubov nonlinear mechanics [3, 4], the invariant manifold $M^k(t)$ of Eq. (2.1) is an integral manifold of Eq. (2.1) if, for any solution $y = y(t)$ of Eq. (2.1), the fact that the inclusion

$$y(t) \in M^k(t),$$

holds for a certain $t = t_0$ implies that this inclusion is true for any $t \in \mathbb{R}$.

Therefore, the posed problem is equivalent to the problem of finding conditions under which the subspace $M^k(t)$ is an integral manifold of Eq. (2.1).

Theorem 1. *Suppose that* $Q(t) \in \mathbf{M}_m(\mathbb{R})$, $\Phi(t) \in \mathbf{M}_{mn}(\mathbb{R})$, $m > n$, $\Phi^+(t) \in \mathbf{M}_{mn}(\mathbb{R})$, *and* $\operatorname{rank}\Phi(t) = n$. *Let* $Q(t)$ *be a continuous function and let* $\Phi(t)$ *and* $\Phi^+(t)$ *be continuously differentiable functions for all* $t \in \mathbb{R}$. *Also assume that* $\Phi^+(t)$ *is a matrix pseudoinverse to the matrix* $\Phi(t)$ and

$$M^n(t) = \{y \in \mathbb{R}^m \colon\ y = M(t)y\},$$

$$M^{m-n}(t) = \{y \in \mathbb{R}^m \colon\ M(t)y = 0\},$$

$$M(t) = \Phi^+(t)\Phi(t),$$

$$L(M, Q) = \frac{dM}{dt} + MQ - QM.$$

Then the following assertions are true:

1. *The subspaces* $M^n(t)$ *and* $M^{m-n}(t)$, *taken together, are invariant manifolds of the differential equation*

$$\frac{dy}{dt} = Q(t)y \tag{I}$$

if and only if

$$L(M(t), Q(t)) = 0.$$

2. *The subspace* $M^n(t)$ *is an invariant manifold of the differential equation* (I) *if and only if*

$$L(M(t), Q(t))M(t) = 0$$

for any $t \in \mathbb{R}$. *Moreover, if* $M^n(t)$ *is an invariant manifold of Eq.* (I), *then, on* $M^n(t)$ *defined by the diffeomorphism* $\Phi^+(t)$,

$$y = \Phi^+(t)x, \quad x \in \mathbb{R}^n,$$

Eq. (I) *is equivalent to the equation*

$$\frac{dx}{dt} = P(t)x \tag{II}$$

with the coefficient matrix

$$P(t) = \left(\frac{d\Phi(t)}{dt} + \Phi(t)Q(t)\right)\Phi^+(t),$$

i.e., the fundamental matrices of solutions of Eqs. (I) *and* (II) $Y(t)$ *and* $X(t)$, $Y(0) = E$, $X(0) = E \in \mathbf{M}_n(\mathbb{R})$, *satisfy the relations*

$$Y(t)\Phi^+(0) = \Phi^+(t)X(t),$$

$$X(t) = \Phi(t)Y(t)\Phi^+(0)$$

for any $t \in \mathbb{R}$.

3. *If* $M^{m-n}(t)$ *is an invariant manifold of Eq.* (I), *then*

$$\ker L(M(t), Q(t)) \supset M^{m-n}(t)$$

for any $t \in \mathbb{R}$.

We now pass to the proof of the theorem. Let

$$L(M(t), Q(t)) = 0 \quad \forall t \in \mathbb{R}.$$

Consider the function

$$r = (E - M(t))y(t), \tag{2.2}$$

where $y = y(t)$ is a solution of Eq. (2.1) corresponding to the initial conditions

$$y(t_0) = M(t_0)c \tag{2.3}$$

and c is an arbitrary point of the space $\mathbb{R}^m$. According to definition (2.2), the function r is equal to zero for $t = t_0$:

$$r = r^0 = (E - M(t_0))y(t_0) = (E - M(t_0))M(t_0)c = 0. \tag{2.4}$$

Differentiating function (2.2), we obtain

$$\frac{dr}{dt} = -\frac{dM(t)}{dt} + (E - M(t))Q(t)y(t) =$$

$$= Q(t)y(t) - \left(\frac{dM(t)}{dt} + M(t)Q(t) - Q(t)M(t)\right) y(t) - Q(t)M(t)y(t) =$$

$$= -L(M(t), Q(t)y(t) + Q(t)(y(t) - M(t)y(t)) = Q(t)r. \tag{2.5}$$

According to (2.4), it follows from (2.5) that $r(t) = 0$. Therefore,

$$y(t) = M(t)y(t) \quad \forall t \in \mathbb{R}. \tag{2.6}$$

On the one hand, we have

$$\operatorname{rank} M(t) = \operatorname{rank}(\Phi^+(t)\Phi(t)) \leq \min(\operatorname{rank}\Phi^+(t), \qquad \operatorname{rank}\Phi(t)) = n,$$

while, on the other hand,

$$n = \operatorname{rank}(\Phi(t)\Phi^+(t)\Phi(t)\Phi^+(t)) = \operatorname{rank}(\Phi(t)M(t)\Phi^+(t)) \leq \operatorname{rank} M(t).$$

Therefore, $\operatorname{rank} M(t) = n$ for any $t \in \mathbb{R}$. Then the subspace of $\mathbb{R}^m$ defined by points (2.2) is n-dimensional. Since points (2.3) belong to the subspace $M^n(t_0)$, the subspace $M^n(t_0)$ coincides with the subspace defined by Eq. (2.3). In this case, equality (2.6) means that

$$y(t) \in M^n(t) \quad \forall t \in \mathbb{R}$$

for any solution of Eq. (2.1) with initial value $y(t_0) = M(t_0)c$ for an arbitrary $c \in \mathbb{R}^m$. Thus, the subspace $M^n(t)$ is an invariant manifold of Eq. (2.1).

We now find the solution $y = y(t)$ of Eq. (2.1) with the initial conditions

$$y(t_0) = (E - M(t_0))c, \tag{2.7}$$

where c is an arbitrary point of $\mathbb{R}^m$, and consider the function

$$r_1 = M(t)y(t).$$

Differentiating this function, we get

$$\frac{dr_1}{dt} = \frac{dM(t)}{dt}y(t) + M(t)Q(t)y(t) =$$

$$= \left(\frac{dM(t)}{dt} + M(t)Q(t) - Q(t)M(t)\right) y(t) + Q(t)M(t)y(t) =$$

$$= L(M(t), Q(t))y(t) + Q(t)r = Q(t)r. \tag{2.8}$$

By definition, the function r_1 is equal to zero at the point $t = t_0$:

$$r_1 = r_1^0 = M(t_0)y(t_0) = M(t_0)(E - M(t_0))c = 0. \tag{2.9}$$

Therefore, it follows from (2.8) and (2.9) that $r_1(t) = 0$ for any $t \in \mathbb{R}$. Thus,

$$M(t)y(t) = 0 \quad \forall t \in \mathbb{R},$$

which completes the proof of the inclusion

$$y(t) \in M^{m-n}(t) \tag{2.10}$$

for any $t \in \mathbb{R}$.

Consider $\operatorname{rank}(E - M(t_0)) = m - \operatorname{rank} M(t_0) = m - n$. Thus, the subspace formed by points (2.7) is $(m-n)$-dimensional and coincides with the subspace $M^{m-n}(t_0)$. In this case, inclusion (2.10) means that the subspace $M^{m-n}(t)$ is an invariant manifold of Eq. (2.1).

We have proved that the condition $L(M(t), Q(t)) = 0$ is sufficient for the subspaces $M^n(t)$ and $M^{m-n}(t)$, taken together, to be invariant manifolds of Eq. (2.1).

Let the subspace $M^n(t)$ be an invariant manifold of Eq. (2.1). Consider the solutions of Eq. (2.1)

$$y = Y(t)\Phi^+(0)c,$$

where c is an arbitrary point of $\mathbb{R}^m$. The relation

$$y(0) = Y(0)\Phi^+(0)c = \Phi^+(0)c \in M^n(0)$$

yields the inclusion

$$Y(t)\Phi^+(0)c \in M^n \quad \forall t \in \mathbb{R}.$$

This means that

$$Y(t)\Phi^+(0)c = M(t)Y(t)\Phi^+(0)c \tag{2.11}$$

for any $c \in \mathbb{R}^m$ and, hence, for unit vectors of the space $\mathbb{R}^m$. It follows from (2.11) that

$$Y(t)\Phi^+(0) = M(t)Y(t)\Phi^+(0) \tag{2.12}$$

for any $t \in \mathbb{R}$.

Let X_t denote the matrix

$$X_t = \Phi(t)Y(t)\Phi^+(0). \tag{2.13}$$

We rewrite (2.12) in the form of the relation

$$Y(t)\Phi^+(0) = \Phi^+(t)X_t, \tag{2.14}$$

which is true for any $t \in \mathbb{R}$. Differentiating (2.14) with regard for (2.12) and (2.13), we obtain

$$Q(t)Y(t)\Phi^+(0) = \frac{d\Phi^+(t)}{dt}X_t + \Phi^+(t)\frac{dX_t}{dt}, \tag{2.15}$$

$$Q(t)\Phi^+(t)X_t = \frac{d\Phi^+(t)}{dt}X_t + \Phi^+(t)\left(\frac{d\Phi(t)}{dt}X_t\Phi^+(0) + \Phi(t)Q(t)Y(t)\Phi^+(0)\right) =$$

$$= \frac{d\Phi^+(t)}{dt}X_t + \Phi^+(t)\left(\frac{d\Phi(t)}{dt} + \Phi(t)Q(t)\right)\Phi^+(t)X_t. \tag{2.16}$$

Subtracting (2.15) from (2.16), we get

$$\Phi^+(t)\frac{dX_t}{dt} = \Phi^+(t)\left(\frac{d\Phi(t)}{dt} + \Phi(t)Q(t)\right)\Phi^+(t)X_t.$$

This proves that

$$\Phi^+(t)\left[\frac{dX_t}{dt} - \left(\frac{d\Phi(t)}{dt} + \Phi(t)Q(t)\right)\Phi^+(t)X_t\right] = 0 \tag{2.17}$$

for any $t \in \mathbb{R}$. Since $\ker \Phi^+(t) = 0$, equality (2.17) is possible only if

$$\frac{dX_t}{dt} = P(t)X_t, \tag{2.18}$$

where

$$P(t) = \left(\frac{d\Phi(t)}{dt} + \Phi(t)Q(t)\right)\Phi^+(t). \tag{2.19}$$

By definition, we have

$$X_0 = \Phi(0)Y(0)\Phi^+(0) = E, \quad E \in \mathbf{M}_n.$$

Therefore,

$$X_t = X(t) \quad \forall t \in \mathbb{R}.$$

Thus, if $M^n(t)$ is an invariant manifold of Eq. (2.1), then (2.14) takes the form

$$Y(t)\Phi^+(0) = \Phi^+(t)X(t), \tag{2.20}$$

where $X(t)$, $X(0) = E$, is the fundamental matrix of solutions of Eq. (2.18).

Multiplying (2.20) from the left by $\Phi(t)$, we get

$$X(t) = \Phi(t)Y(t)\Phi^+(0).$$

Differentiating (2.12), we obtain

$$QY(t)\Phi^+(0) = \left(\frac{dM(t)}{dt} + M(t)Q(t) - Q(t)M(t)\right)Y(t)\Phi^+(0) + QM(t)Y(t)\Phi^+(0). \tag{2.21}$$

In view of (2.12), relation (2.21) yields

$$L(M(t), Q(t))Y(t)\Phi^+(0) = 0. \tag{2.22}$$

Using (2.22) and (2.14), we get

$$L(M(t), Q(t))\Phi^+(t)X(t) = 0.$$

Multiplying this equality from the right by $X^{-1}(t)$, we obtain the final result

$$L(M(t), Q(t))\Phi^+(t) = 0$$

for any $t \in \mathbb{R}$.

Thus, the fact that the subspace $M^n(t)$ is integral implies that all conditions of the theorem related to this case are satisfied; to this end, it suffices to rewrite equality (2.20) as an equation of the subspace $M^n(t)$ in the parametric form:

$$y = \Phi^+(t)x, \qquad \in \mathbb{R}^n, \quad t \in \mathbb{R}.$$

Assume that the condition

$$L\big(M(t), Q(t)\big)\Phi^+(t) = 0 \quad \forall t \in \mathbb{R}, \tag{2.23}$$

is satisfied. Multiplying (2.23) from the right by the matrix $\Phi(t)X(t)$, where $X(t)$ is the fundamental matrix of solutions of Eq. (2.18) with coefficient matrix (2.19), $X(0) = E$, we obtain

$$L\big(M(t), Q(t)\big)M(t)X(t) = 0 \quad \forall t \in \mathbb{R}.$$

Now consider the function

$$r = y(t) - \Phi^+(t)X(t)c, \tag{2.24}$$

where $y = y(t)$ is the solution of Eq. (2.23) such that

$$y(t_0) = \Phi^+(t_0)X(t_0)c \tag{2.25}$$

and $X(t)$, $X(0) = E$, is the fundamental matrix of solutions of the equation

$$\frac{dx}{dt} = P(t)x$$

with coefficient matrix (2.19). Differentiating function (2.24), we obtain

$$\frac{dr}{dt} = Q(t)y(t) - \left(\frac{d\Phi^+(t)}{dt}X(t)c + \Phi^+(t)P(t)X(t)c\right) =$$

$$= Q(t)\big[(y(t) - \Phi^+(t)X(t)c)\big] +$$

$$+ Q(t)\Phi^+(t)X(t)c - \left(\frac{d\Phi^+(t)}{dt}X(t)c + \Phi^+(t)P(t)X(t)c\right) =$$

$$= Q(t)r - \left(\frac{d\Phi^+(t)}{dt} + \Phi^+(t)P(t) - Q(t)\Phi^+(t)\right) X(t)c\ . \tag{2.26}$$

Let us prove that the second term in (2.26) is $0 \in \mathbf{M}_{mn}$. Indeed, taking (2.19) into account, we get

$$\frac{d\Phi^+(t)}{dt} + \Phi^+(t)\left(\frac{d\Phi(t)}{dt} + \Phi(t)Q(t)\right)\Phi^+(t) - Q(t)\Phi^+(t) =$$

$$= \frac{d\Phi^+(t)}{dt} + \Phi^+(t)\frac{d\Phi(t)}{dt}\Phi^+(t) + M(t)Q(t)\Phi^+(t) - Q(t)\Phi^+(t) =$$

$$= \left(\frac{d\Phi^+(t)}{dt}\Phi(t) + \Phi^+(t)\frac{d\Phi(t)}{dt}\right)\Phi^+(t), \tag{2.27}$$

$$M(t)Q(t)\Phi^+(t) - Q(t)M(t)\Phi^+(t) + Q(t)M(t)\Phi^+(t) - Q(t)\Phi^+(t) =$$

$$= \left(\frac{dM(t)}{dt} + M(t)Q(t) - Q(t)M(t)\right)\Phi^+(t) +$$

$$+ Q(t)\Phi^+(t)\Phi(t)\Phi^+(t) - Q(t)\Phi^+(t) = 0.$$

With regard for (2.27), equality (2.26) takes the form

$$\frac{dr}{dt} = Q(t)r. \tag{2.28}$$

For $t = t_0$, according to (2.25), function (2.24) is equal to zero:

$$r(t_0) = y(t_0) - \Phi^+(t_0)X(t_0)c = 0.$$

Therefore, it follows from (2.28) that

$$r(t) = 0 \quad \forall t \in \mathbb{R},$$

and, hence,

$$y(t) = \Phi^+(t)X(t)c \quad \forall t \in \mathbb{R}, \tag{2.29}$$

where c is an arbitrary point of the space $\mathbb{R}^n$.

Since (2.29) is the parametric representation of the equation of the subspace $M^n(t)$, it follows from (2.29) that the condition that

$$y(t) \in M^n(t) \tag{2.30}$$

for $t = t_0$ implies that inclusion (2.30) holds for any $t \in \mathbb{R}$. This proves that condition (2.23) is not only necessary but also sufficient for the subspace $M^n(t)$ to be an invariant manifold of Eq. (2.1).

This completes the proof of the second assertion of Theorem 1.

Let the subspace $M^{m-n}(t)$ be an invariant manifold of Eq. (2.1). Consider the solutions $y = y(t)$ of Eq. (2.1) defined by the relation

$$y(t) = Y(t)(E - M(0))c,$$

where c is an arbitrary point of $\mathbb{R}^m$.

It follows from the relation

$$(0)y(0) = M(0)Y(0)(E - M(0))c = M(0)(E - M(0))c = 0$$

that $y(0) \in M^{m-n}(0)$ and, hence, $y(t) \in M^{m-n}(t) \quad \forall t \in \mathbb{R}$. This proves that

$$M(t)Y(t)(E - M(0))c = 0 \quad \forall t \in \mathbb{R}.$$

Differentiating this equality, we get

$$\left(\frac{dM(t)}{dt} + M(t)Q(t) - Q(t)M(t)\right)Y(t)(E - M(0))c +$$

$$+ Q(t)M(t)Y(t)(E - M(0))c = L(M(t), Q(t))Y(t)(E - M(0))c = 0. \tag{2.31}$$

The points $y = (E - M(0))c$ define the subspace $M^{m-n}(0)$. Therefore, the equation

$$y = Y(t)(E - M(0))c, \qquad c \in \mathbb{R}^m, \quad t \in \mathbb{R},$$

defines the subspace $M^{m-n}(t)$ in the parametric form. Therefore, equality (2.31) means that

$$\ker L(M(t), Q(t)) \supset M^{m-n}(t) \quad \forall t \in \mathbb{R}. \tag{2.32}$$

Thus, inclusion (2.32) is a necessary condition for the subspace $M^{m-n}(t)$ to be an invariant manifold of Eq. (2.1).

Assume that the subspaces $M^n(t)$ and $M^{m-n}(t)$, taken together, are invariant manifolds of Eq. (2.1). Then equality (2.22) yields

$$\ker\ L(M(t), Q(t)) \supset Y(t)M(0)c \quad \forall t \in \mathbb{R}, \tag{2.33}$$

where c is an arbitrary point of $\mathbb{R}^m$. Moreover, since the equation

$$y = Y(t)M(0)c, \qquad c \in \mathbb{R}^m, \quad t \in \mathbb{R},$$

defines the subspace $M^n(t)$ in the parametric form, it follows from (2.33) that

$$\ker L(M(t), Q(t)) \supset M^n(t) \quad \forall t \in \mathbb{R}.$$

Thus, if the subspaces $M^n(t)$ and $M^{m-n}(t)$, taken together, are invariant manifolds of Eq. (2.1), then

$$\ker L\big(M(t), Q(t)\big) \supset (M^n(t) \cup M^{m-n}(t)). \tag{2.34}$$

Since $\operatorname{rank} M^n(t) = n$, $\operatorname{rank} M^{m-n}(t) = m - n$, and $M^n(t) \cap M^{m-n}(t) = \{0\}$, we conclude that the union on the right-hand side of expression (2.34) contains a basis of the space $\mathbb{R}^m$. Therefore, relation (2.34) yields

$$\ker L(M(t), Q(t)) \supset \mathbb{R}^m \quad \forall t \in \mathbb{R}. \tag{2.35}$$

Since $\ker L(M(t), Q(t)) \in M_m(\mathbb{R})$ $\forall t \in \mathbb{R}$, inclusion (2.35) is possible only if

$$\ker L(M(t), Q(t)) = 0. \tag{2.36}$$

Thus, condition (2.36) is not only sufficient but also necessary for the subspaces $M^n(t)$ and $M^{m-n}(t)$, taken together, to be invariant manifolds of Eq. (2.1).

3. Equivalence of linear differential equations of different orders. We prove the following theorem.

Theorem 2. *If the subspaces $M^n(t)$ and $M^{m-n}(t)$, taken together, are invariant manifolds of the differential equation*

$$\frac{dy}{dt} = Q(t)y, \tag{I}$$

then the change of variables

$$y = \Phi_1^+(t)x + \Phi_2^+(t)z \tag{II}$$

reduces this equation to the system of differential equations

$$\frac{dx}{dt} = P(t)x, \qquad \frac{dz}{dt} = G(t)z \tag{III}$$

with coefficient matrices

$$P(t) = \left(\frac{d\Phi_1(t)}{dt} + \Phi_1(t)Q(t)\right)\Phi_1^+(t), \tag{IV}$$

$$G(t) = \left(\frac{d\Phi_2(t)}{dt} + \Phi_2(t)Q(t) \right) \Phi_2^+(t), \tag{V}$$

and vice versa, if the differential equation (I) *can be reduced by the change of variables* (II) *to the system of differential equations* (III), *then the subspaces* $M^n(t)$ *and* $M^{m-n}(t)$, *taken together, are invariant manifolds of Eq.* (I), *and the coefficient matrices of system* (III) *are defined by relations* (IV) *and* (V).

Indeed, let the subspaces $M^n(t)$ and $M^{m-n}(t)$, taken together, be invariant manifolds of Eq. (I). Then, according to assertion 2 of Theorem 1, Eq. (I) is equivalent on $M^n(t)$ and $M^{m-n}(t)$ to the corresponding first and second equations of system (III) with the coefficient matrices defined by relations (IV) and (V), respectively. Let $Y(t)$, $X(t)$, and $Z(t)$, where $Y(0) = E$, $X(0) = E$, and $Z(0) = E$, be the fundamental matrices of solutions of Eqs. (I) and (III) and let E be the identity matrices of the corresponding orders. According to assertion 2 of Theorem 1, we have

$$Y(t)\Phi_1^+(0) = \Phi_1^+(t)X(t), \qquad Y(t)\Phi_2^+(0) = \Phi_2^+(t)Z(t) \tag{3.1}$$

for all $t \in \mathbb{R}$. Thus, according to (3.1),

$$Y(t)(\Phi_1^+(0), \Phi_2^+(0)) = (\Phi_1^+(t), \Phi_2^+(t)) \begin{pmatrix} X(t) & 0 \\ 0 & Z(t) \end{pmatrix} \tag{3.2}$$

for all $t \in \mathbb{R}$. The equality

$$Y(t) = \Phi_1^+(t)X(t)\Phi_1(0) + \Phi_2^+(t)Z(t)\Phi_2(0)$$

for all $t \in \mathbb{R}$ follows from (3.2). Thus, for an arbitrary $y_0 \in \mathbb{R}^m$, we have

$$Y(t)y_0 = \Phi_1^+(t)X(t)x_0 + \Phi_2^+(t)Z(t)z_0 \tag{3.3}$$

for all $t \in \mathbb{R}$ and x_0 and z_0 chosen according to the condition

$$x_0 = \Phi_1(0)y_0, \qquad z_0 = \Phi_2(0)y_0.$$

Equality (3.3) means that the change of variables (II) reduces the differential equation (I) to the system of differential equations (III).

Now assume that the differential equation (I) can be reduced to the system of differential equations (III) by the change of variables (II). Taking into account that the subspaces $z = 0$ and $x = 0$ are invariant manifolds of system (III) and using (II), we obtain relations (3.1), which yield

$$X(t) = \Phi_1(t)Y(t)\Phi_1^+(0), \qquad Z(t) = \Phi_2(t)Y(t)\Phi_2^+(0) \tag{3.4}$$

for all $t \in \mathbb{R}$.

Substituting (3.4) into relations (3.1), we obtain

$$Y(t)\Phi_1^+(0) = M_1(t)Y(t)\Phi_1^+(0), \qquad Y(t)\Phi_2^+(0) = M_2(t)Y(t)\Phi_2^+(0) \tag{3.5}$$

for all $t \in \mathbb{R}$.

It follows from the first relation in (3.5) that

$$y(t) = M_1(t)y(t) \tag{3.6}$$

for any solution $y(t)$ of Eq. (I) that satisfies the condition

$$y(0) = \Phi_1^+(0)c, \tag{3.7}$$

where c is an arbitrary constant from $\mathbb{R}^n$. Since points (3.7) fill the subspace $M^n(0)$, we conclude that, according to (3.6), the integral curves $(t, y(t))$ of Eq. (I) that pass through points of the subspace $M^n(0)$ for $t = 0$ belong to the subspace $M^n(t)$ for any $t \in \mathbb{R}$. This is sufficient for the subspace $M^n(t)$ to be an invariant manifold of Eq. (I).

It follows from the second relation in (3.5) that

$$y(t) = M_2(t)y(t)$$

for any solution $y(t)$ of Eq. (I) that satisfies the condition

$$y(0) = \Phi_2^+(0)c,$$

where c is an arbitrary constant from $\mathbb{R}^{m-n}$.

By analogy, we prove that the subspace

$$M_2^{m-n}(t) = \{y \in \mathbb{R}^{m-n} \colon y = M_2(t)y\}$$

is an invariant manifold of Eq. (I).

According to Lemma 1, the equality

$$M_2^{m-n}(t) = M^{m-n}(t)$$

holds for any $t \in \mathbb{R}$. This proves that the subspace $M^{m-n}(t)$ is an invariant manifold of the differential equation (I). Thus, the subspaces $M^n(t)$ and $M^{m-n}(t)$, taken together, are invariant manifolds of Eq. (I). According to assertion 2 of Theorem 1, this is sufficient for relations (IV) and (V) to be true.

Let $F(t) \in \mathbf{M}_{pn}(\mathbb{R})$, $n > p$, $F^+(t) \in \mathbf{M}_{pn}(\mathbb{R})$, $\operatorname{rank} F(t) = p$, and let $F(t)$ and $F^+(t)$ be continuously differentiable functions for all $t \in \mathbb{R}$. Also assume that $F^+(t)$ is a matrix pseudoinverse to the matrix $F(t)$ and $K(t) = F^+(t)F(t)$. Finally, let the subspace

$$K^p(t) = \{x \in \mathbb{R}^n \colon x = K(t)x\}$$

be an invariant manifold of the differential equation

$$\frac{dx}{dt} = P(t)x, \tag{VI}$$

which is equivalent on $K^p(t)$ to the differential equation

$$\frac{dz}{dt} = R(t)z. \tag{VII}$$

The system of differential equations (III) is called a *decomposition* of the differential equation (I) if the change of variables (II) reduces this equation to the system of differential equations (III).

The differential equation (VII) is called a *restriction* of the differential equation (VI) to the subspace $K^p(t)$ if the subspace $K^p(t)$ is an invariant manifold of Eq. (VI), and this equation is equivalent to Eq. (VII) on $K^p(t)$.

We say that the differential equations (I) and (VI) are equivalent if Eq. (VI), together with its restriction to $K^{m-n}(t)$ (VII), is a decomposition of Eq. (I).

By definition, the fundamental matrices of solutions of equivalent differential equations are expressed in terms of one another via the matrices that define the invariant subspaces of these differential equations. Indeed, using the definitions presented above and taking into account that

$$G(t) = R(t)$$

for all $t \in \mathbb{R}$, we conclude that relation (3.1) and the relation

$$X(t)F^+(0) = F^+(t)Z(t) \tag{3.8}$$

for the fundamental matrices of the solutions $Y(t)$, $X(t)$, and $Z(t)$ of the differential equations (I), (VI), and (VII) are true.

It follows from (3.1) and (3.8) that

$$Y(t) = (\Phi_1^+(t)X(t) + \Phi_2^+(t)F(t)X(t)F^+(0))\begin{pmatrix}\Phi_1(0)\\ \Phi_2(0)\end{pmatrix} =$$

$$= \Phi_1^+(t)X(t)\Phi_1(0) + \Phi_2^+(t)F(t)X(t)F^+(0)\Phi_2(0), \tag{3.9}$$

$$X(t) = \Phi_1(t)Y(t)\Phi_1^+(0) \tag{3.10}$$

for all $t \in \mathbb{R}$. Relations (3.9) and (3.10) describe the relationship between the fundamental matrices of solutions of the equivalent differential equations (I) and (VI).

The notion of equivalence of differential equations of orders m and n defined above for

$$m > n > m - n$$

can easily be generalized to the case

$$m = 2n. \tag{3.11}$$

Indeed, since the space $\mathbb{R}^n$ is an invariant manifold of the differential equation (VI), and Eq. (VI) is equivalent on it to the differential equation (VII) with the same coefficient matrix, we conclude that, in case (3.11), the equivalence of the differential equations (I) and (VI) is determined by the decomposition of Eq. (I) into the system of equations

$$\frac{dx}{dt} = P(t)x, \qquad \frac{dz}{dt} = P(t)z.$$

The results presented above yield the following statement:

Corollary. *The differential equations* (I) *and* (VI) *are equivalent if and only if*

$$L(M(t), Q(t)) = 0, \qquad L(K(t), P(t))K(t) = 0, \tag{VIII}$$

$$P(t) = \left(\frac{d\Phi_1(t)}{dt} + \Phi_1(t)Q(t)\right)\Phi_1^+(t), \tag{IX}$$

$$\left(\frac{d\Phi_2(t)}{dt} + \Phi_2(t)Q(t)\right)\Phi_2^+(t) = \left(\frac{dF(t)}{dt} + F(t)P(t)\right)F^+(t) \qquad \text{(X)}$$

for all $t \in \mathbb{R}$.

Indeed, assume that the differential equations (I) and (VI) are equivalent. Then we have the decomposition of Eq. (I) into the system of equations (III) the second equation of which is the restriction of the differential equation (VI) to $K^{m-n}(t)$. It follows from the definition of decomposition and Theorem 2 that the subspaces $M^n(t)$ and $M^{m-n}(t)$ are invariant manifolds of the differential equation (I). It follows from the definition of the restriction of the differential equation (VI) to the subspace $K^{m-n}(t)$ that $K^{m-n}(t)$ is an invariant manifold of this equation. According to assertions 1 and 2 of Theorem 1, this is sufficient for relations (VIII) and (IX) to be true. Moreover, this is sufficient for the coefficient matrices of the differential equations (I), (III), and (VII) to satisfy the relations

$$G(t) = \left(\frac{d\Phi_2(t)}{dt} + \Phi_2(t)Q(t)\right)\Phi_2^+(t), \qquad (3.12)$$

$$R(t) = \left(\frac{dF(t)}{dt} + F(t)P(t)\right)F^+(t), \qquad (3.13)$$

and

$$G(t) = R(t) \qquad (3.14)$$

for all $t \in \mathbb{R}$.

The last relation proves equality (X).

Let relations (VIII) – (X) be true. Then, according to assertions 1 and 2 of Theorem 1, the subspaces $M^n(t)$ and $M^{m-n}(t)$ are invariant manifolds of the differential equation (I), and the subspace $K^{m-n}(t)$ is an invariant manifold of the differential equation (VI); furthermore, the coefficient matrices of the corresponding differential equations $G(t)$ and $R(t)$ are defined by relations (3.12) and (3.13), and, hence, according to condition (X), they satisfy equality (3.14). According to Theorem 2, this implies that the system of differential equations (III) the second equation of which is the restriction of the differential equation (VI) to the subspace $K^{m-n}(t)$ is a decomposition of the differential equation (I). This proves that relations (VIII) – (X) yield the equivalence of the differential equations (I) and (VI).

Note that, for $m = 2n$, conditions (VIII) – (X) are simplified because, in this case, $F(t)$ and $K(t)$ are the identity matrices. In this case, these conditions take the form

$$L(M(t), Q(t)) = 0,$$

$$P(t) = \left(\frac{d\Phi_1(t)}{dt} + \Phi_1(t)Q(t)\right)\Phi_1^+(t) = \left(\frac{d\Phi_2(t)}{dt} + \Phi_2(t)Q(t)\right)\Phi_2^+(t)$$

for any $t \in \mathbb{R}$.

Also note that the equivalence of the differential equations (I) and (VI) means that the relations

$$Y(t)\Phi_1^+(0) = \Phi_1^+(t)X(t), \qquad Y(t)\Phi_2^+(0) = \Phi_2^+(t)F(t)X(t)F^+(0) \qquad (3.15)$$

for the fundamental matrices of solutions of Eqs. (I) and (VI) $Y(t)$ and $X(t)$, as well as the other relations that can be obtained from (3.15) by the corresponding transformations, are true.

4. Addition to the Floquet – Lyapunov theory. Consider the linear differential equation

$$\frac{dx}{dt} = P(t)x, \tag{I}$$

where $x \in \mathbb{R}^n$, $P(t) \in \mathbf{M}_n(\mathbb{R})$, and $P(t)$ is a continuous periodic matrix with period T.

According to the well-known Floquet theorem [5], the fundamental matrix of solutions of Eq. (I) $X(t)$, $X(0) = E$, can be represented in the form

$$X(t) = \Phi(t)e^{Ht}, \tag{II}$$

where $\Phi(t)$ is a matrix periodic in t with period T, and H is the constant matrix defined by the monodromy matrix $X(T)$ of Eq. (I) according to the formula

$$H = \frac{1}{T} \ln X(T). \tag{III}$$

The logarithm is a multivalued function whose real value does not always exist. Thus, relation (I) with matrix (III) such that

$$H \in \mathbf{M}_n(\mathbb{R}) \tag{IV}$$

is not always true. According to the theory of matrices [6], condition (IV) is satisfied if and only if every elementary divisor corresponding to the negative eigenvalues of the matrix $X(T)$ is repeated an even number of times. Thus, only in this case does equality (II) hold with matrices $\Phi(t)$ and H from the space of real matrices $\mathbf{M}_n(\mathbb{R})$.

In the case where condition (IV) cannot be satisfied, the Floquet representation (II) exists only with matrices $\Phi(t)$ and H from the space $\mathbf{M}_n(\mathbb{C})$, where $\mathbb{C}$ is the plane of complex numbers, or this representation transforms into equality (II) with real matrices $\Phi(t)$ and H, the first of which is periodic with period $2T$ and the second is defined by the relation

$$H = \frac{1}{2T} \ln X(2T). \tag{V}$$

The Floquet representation (II) with matrix (V) is a consequence of the presence of negative eigenvalues of the monodromy matrix of Eq. (I).

We consider in detail the differential equation (I) whose monodromy matrix possesses this property and prove several previously unknown statements for this equation.

Theorem 3. *Suppose that the coefficient matrix of the differential equation* (I) $P(t)$ *belongs to* $\mathbf{M}_n(\mathbb{R})$ *for any* $t \in \mathbb{R}$ *and is continuous on* $\mathbb{R}$ *and periodic in* t *with period* T.

Then the following assertions are true:

1. *The algebraic number* p *of negative eigenvalues of the monodromy matrix* $X(T)$ *of Eq.* (I) *is even.*

2. *Equality* (II) *holds for the matrix*

$$H = \frac{1}{T} \ln(X(T)I), \tag{VI}$$

where I *is the real matrix defined by the conditions*

$$I^2 = E, \qquad \ln(X(T)I) \in \mathbf{M}_n(\mathbb{R}),$$

and for the periodic matrix $\Phi(t)$ *such that*

$$\Phi(t+T)I_1 = \Phi(t)I_1, \qquad \Phi(t+T)I_2 = -\Phi(t)I_2 \tag{VII}$$

for all $t \in \mathbb{R}$, *where*

$$I_1 = \frac{E+I}{2}, \qquad I_2 = \frac{E-I}{2}.$$

3. *There exists a nonsingular matrix* $(U(t), V(t))$ *continuously differentiable and real for all* $t \in \mathbb{R}$, *periodic with period* T, *and such that the change of variables*

$$x = U(t)z_1 + V(t)z_2$$

reduces the differential equation (I) *to the system of differential equations*

$$\frac{dz_1}{dt} = H_1 z_1, \qquad \frac{dz_2}{dt} = G(t)z_2, \tag{VIII}$$

where H_1 *is a constant matrix,* $G(t)$ *is a periodic matrix with period* T, *and the set of eigenvalues of the monodromy matrix* $Z_2(T)$ *of the second equation of the system is either the set of all negative eigenvalues of the matrix* $X(T)$ *or its subset.*

To prove the theorem, we use the representation of the matrix $X(T)$ in terms of its Jordan form $J(\lambda)$, namely

$$X(T) = SJ(\lambda)S^{-1},$$

and obtain the equality

$$\det X(T) = \prod_{\nu=1}^{n} \lambda_\nu, \tag{4.1}$$

which associates the determinant of the matrix $X(T)$ with its eigenvalues λ_ν, $\nu = \overline{1, n}$.

We now use the Liouville – Ostrogradskii – Jacobi formula and represent the determinant of the matrix $X(T)$ in terms of the trace of the coefficient matrix of Eq. (I):

$$\det X(T) = \exp\left\{\operatorname{tr} P(t)dt\right\}. \tag{4.2}$$

Equating the right-hand sides of relations (4.1) and (4.2), we obtain an equality that proves that

$$\prod_{\nu=1}^{n} \lambda_\nu > 0. \tag{4.3}$$

Since each pair of complex conjugate eigenvalues of the matrix $X(T)$ in the product of all its eigenvalues gives a positive number, it follows from relation (4.3) that the product of all negative eigenvalues of the matrix $X(T)$ also gives a positive number. Thus, the algebraic number of negative eigenvalues of the matrix $X(T)$, i.e., the sum of multiplicities of the roots of characteristic equations for all different negative eigenvalues of the matrix $X(T)$, is an even number.

Prior to the proof of assertion 2 of Theorem 3, note that, in the case where the logarithm of the matrix $X(T)$ is real, by setting $I = E$ one can reduce equalities (II) and (VI) to the Floquet relations (II) and (III) with a matrix $\Phi(T)$ that possesses properties that follow from these relations and are indicated in assertion 2 of Theorem 3.

It remains to consider the case where the matrix $X(T)$ has negative eigenvalues and does not have a real logarithm. In this case, the real canonical form of the matrix $X(T)$ can be represented in the form of decomposition into two blocks A and B, where A either is empty or has a real logarithm, and B has only negative eigenvalues and does not have a real logarithm.

Let $B \in \mathbf{M}_d(\mathbb{R})$, where

$$n > d.$$

Then the following equality is true:

$$X(T) = S \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} S^{-1}, \tag{4.4}$$

where S, A, and B are real matrices with properties indicated above for A and B.

We set

$$Y(t) = S^{-1} X(t) S, \quad B_1 = -B. \tag{4.5}$$

According to properties of the fundamental matrix of solutions of Eq. (I), we have

$$X(t+T) = X(t) X(T). \tag{4.6}$$

Therefore, it follows from (4.4), (4.5), and (4.6) that

$$Y(t+kT) = S^{-1} X(t) X^k(T) S = S^{-1} X(t) S S^{-1} X^k(T) S =$$

$$= Y(t) \begin{pmatrix} A^k & 0 \\ 0 & B^k \end{pmatrix} = Y(t) \begin{pmatrix} A^k & 0 \\ 0 & (-1)^k B_1^k \end{pmatrix} \tag{4.7}$$

for any integer k.

We represent $Y(t)$ in the block form

$$Y(t) = (Y_1(t), Y_2(t)) \tag{4.8}$$

consistent with decomposition (4.4) of the matrix $X(T)$ into the blocks A and B. Using relations (4.7), we get

$$Y_1(t+kT) = Y_1(t) A^k, \qquad Y_2(t+kT) = (-1)^k Y_2(t) B_1^k \tag{4.9}$$

for any integer k.

Since the eigenvalues of the matrix B_1 are positive by virtue of the definition (4.5) of this matrix, both matrices A and B_1 have real logarithms $\ln A$ and $\ln B_1$.

In view of the arguments presented above, relation (4.9) yields

$$Y_1(t) = Y_1 \left(t - \left[\frac{t}{T} \right] T + \left[\frac{t}{T} \right] T \right) = Y_1 \left(t - \left[\frac{t}{T} \right] T \right) A^{[t/T]} =$$

$$= Y_1 \left(t - \left[\frac{t}{T} \right] T \right) \exp \left\{ \left(\left[\frac{t}{T} \right] T - t \right) \frac{\ln A}{T} \right\} \exp \left\{ \frac{t}{T} \ln A \right\}, \tag{4.10}$$

$$Y_2(t) = Y_2 \left(t - \left[\frac{t}{T} \right] T + \left[\frac{t}{T} \right] T \right) = Y_2 \left(t - [t/T]\, T \right) (-1)^{[t/T]} B_1^{[t/T]} =$$

$$= (-1)^{[t/T]} Y_2 \left(t - \left[\frac{t}{T} \right] T \right) \exp \left\{ \left(\left[\frac{t}{T} \right] T - t \right) \frac{\ln B_1}{T} \right\} \exp \left\{ \frac{t}{T} \ln B_1 \right\} \tag{4.11}$$

for all $t \in \mathbb{R}$; here, $[\,t\,]$ denotes the integer part of the number t.

Let $\Phi_1(t)$ and $\Phi_2(t)$ denote the coefficients of $\exp\left\{\frac{t}{T}\ln A\right\}$ and $\exp\left\{\frac{t}{T}\ln B_1\right\}$ in relations (4.10) and (4.11), respectively. Then, using (4.8), (4.10), and (4.11), we obtain

$$Y(t) = \big(\Phi_1(t), \Phi_2(t)\big) \begin{pmatrix} \exp\left\{\frac{t}{T}\ln A\right\} & 0 \\ 0 & \exp\left\{\frac{t}{T}\ln B_1\right\} \end{pmatrix} \tag{4.12}$$

for all $t \in \mathbb{R}$. This equality implies that the matrices $\Phi_1(t)$ and $\Phi_2(t)$ are continuously differentiable on $\mathbb{R}$. Furthermore, it follows from the introduced notation that the matrix $\Phi_1(t)$ is periodic with period T, and the matrix $\Phi_2(t)$, which is the product of the function $(-1)^{[t/T]}$ and a periodic matrix with period T, satisfies the condition

$$\Phi_2(t+T) = -\Phi_2(t)$$

for all $t \in \mathbb{R}$.

Let I_0 denote the matrix

$$\begin{pmatrix} E_1 & 0 \\ 0 & -E_2 \end{pmatrix},$$

where E_1 and E_2 are the identity matrices from $\mathbf{M}_{n-d}(\mathbb{R})$ and $\mathbf{M}_d(\mathbb{R})$, respectively. Then

$$Y(T)I_0 = \begin{pmatrix} A & 0 \\ 0 & B_1 \end{pmatrix}$$

and relation (4.12) takes the form

$$Y(t) = \big(\Phi_1(t), \Phi_2(t)\big) \exp\left\{\frac{t}{T}\ln(Y(T)I_0)\right\}. \tag{4.13}$$

Using (4.13) and the first equality in (4.5), we obtain

$$X(t) = S\big(\Phi_1(t), \Phi_2(t)\big)S^{-1} \exp\left\{\frac{t}{T} S(\ln(Y(T)I_0))S^{-1}\right\}. \tag{4.14}$$

Since

$$S\big(\ln(Y(T)I_0)\big)S^{-1} = \ln(SY(T)S^{-1}SI_0S^{-1}) = \ln(X(T)I),$$

where

$$I = SI_0S^{-1}, \tag{4.15}$$

relation (4.14) takes the form of the required representation (II) under the condition that

$$H = \frac{1}{T}\ln(X(T)I),$$

$$\Phi(t) = S(\Phi_1(t), \Phi_2(t))S^{-1}.$$

Taking (4.15) into account, we get

$$I^2 = E, \qquad I_1 = S\begin{pmatrix} E_1 & 0 \\ 0 & 0 \end{pmatrix}S^{-1}, \qquad I_2 = S\begin{pmatrix} 0 & 0 \\ 0 & E_2 \end{pmatrix}S^{-1},$$

where E_1 and E_2 are the identity matrices of the corresponding orders.

Using the expressions for I_1 and I_2, we obtain

$$\Phi(t)I_1 = S(\Phi_1(t), 0)S^{-1}, \qquad \Phi(t)I_2 = S(0, \Phi_2(t))S^{-1} \tag{4.16}$$

for all $t \in \mathbb{R}$. In view of properties of the matrices $\Phi_1(t)$ and $\Phi_2(t)$, relation (4.16) yields

$$\Phi(t+T)I_1 = \Phi(t)I_1, \qquad \Phi(t+T)I_2 = -\Phi(t)I_2$$

for all $t \in \mathbb{R}$, which completes the proof of assertion 2 of Theorem 3 in the case considered.

Let $d = n$. In this case, we obtain the equality

$$X(T) = SBS^{-1}$$

instead of (4.4), the equality

$$Y(t+kT) = (-1)^k Y(t)B_1^k$$

instead of (4.7), and the equality

$$Y(t) = \Phi_2(t)\exp\left\{\frac{t}{T}\ln B_1\right\}$$

and condition

$$\Phi_2(t+T) = -\Phi_2(t)$$

for all $t \in \mathbb{R}$ instead of (4.12).

We set

$$I_0 = -E.$$

Using the last two formulas, we obtain equality (II) of the form

$$Y(t) = \Phi_2(t)\exp\left\{\frac{t}{T}\ln(-Y(t))\right\},$$

where

$$H = \frac{1}{T}\ln(-X(T)), \qquad H \in \mathbf{M}_n\mathbb{R},$$

$$\Phi(t) = S\Phi_2(t)S^{-1}, \qquad \Phi(t+T) = -\Phi(t), \qquad \Phi(t) \in \mathbf{M}_n\mathbb{R},$$

for all $t \in \mathbb{R}$, which completes the proof of assertion 2 of Theorem 3.

We now pass to the proof of assertion 3 of Theorem 3. In this assertion, we separate two limiting cases, namely, the case where the matrix $X(t)$ has a real logarithm and the second case where all eigenvalues of the matrix $X(t)$ are negative and their elementary divisors are different.

In the first case, assertion 3 of Theorem 3 follows from the Floquet relations (II) and (III), according to which the change of variables

$$x = \Phi(t)z$$

reduces the differential equation (I) to the differential equation

$$\frac{dz}{dt} = Hz$$

and guarantees the properties of the matrices H and $\Phi(t)$ indicated in Theorem 3.

In the second case, assertion 3 of Theorem 3 is trivial: the change of variables

$$x = z$$

reduces the differential equation (I) to a differential equation with the same coefficient matrix:

$$\frac{dz}{dt} = P(t)z.$$

Associating these limiting cases with the representation of the matrix $X(T)$ via its real canonical form (4.4), we establish that the first case corresponds to

$$X(T) = SAS^{-1}$$

and the second case corresponds to

$$X(T) = SBS^{-1}.$$

Thus, the only nonlimiting case in assertion 3 of Theorem 3 is the case where

$$A \in \mathbf{M}_{n-d}(\mathbb{R}), \qquad B \in \mathbf{M}_d(\mathbb{R}), \quad n > d > 1.$$

Assume that these conditions are satisfied. Then it follows from the proof of assertion 2 of Theorem 3 that the matrix $Y(t)$ associated with the matrix $X(t)$ by relation (4.5) has the form (4.12). Denoting

$$U(t) = \Phi_1(t), \qquad V(t) = \Phi_2(t), \qquad H_1 = \frac{\ln A}{T}, \qquad H_2 = \frac{\ln B_1}{T},$$

we represent (4.12) in the form

$$Y(t) = (U(t), V(t)) \begin{pmatrix} e^{H_1 t} & 0 \\ 0 & e^{H_2 t} \end{pmatrix}. \tag{4.17}$$

It follows from (4.17) that

$$Y(t) \begin{pmatrix} E_1 \\ 0 \end{pmatrix} = (U(t), V(t)) \begin{pmatrix} e^{H_1 t} \\ 0 \end{pmatrix} = U(t)e^{H_1 t}, \tag{4.18}$$

where E_1 is the identity matrix of order $n - d$.

Differentiating equality (4.18) with regard for the first relation in (4.5), we get

$$S^{-1}P(t)SY(t) \begin{pmatrix} E_1 \\ 0 \end{pmatrix} = S^{-1}P(t)SU(t)e^{H_1 t} = \frac{dU(t)}{dt} e^{H_1 t} + U(t)H_1 e^{H_1 t}.$$

Thus,

$$\frac{dU(t)}{dt} + U(t)H_1 = S^{-1}P(t)SU(t) \tag{4.19}$$

for all $t \in \mathbb{R}$.

The matrix $Y(t)$ is the fundamental matrix of solutions of the differential equation

$$\frac{dy}{dt} = S^{-1}P(t)Sy. \tag{4.20}$$

Let $W(t) \in \mathbf{M}_{nd}(\mathbb{R})$ for all $t \in \mathbb{R}$ and let this matrix be continuously differentiable on $\mathbb{R}$, periodic with period T, and such that

$$\det\big(U(t), W(t)\big) \neq 0$$

for all $t \in \mathbb{R}$.

The existence of this matrix follows from the theorem on a quasiperiodic basis in $\mathbb{R}^n$ presented in [7].

In the differential equation (4.20), we perform the change of variables according to the formula

$$y = U(t)y_1 + W(t)y_2\,. \tag{4.21}$$

Using equality (4.19), we obtain the differential equation

$$U(t)\left(\frac{dy_1}{dt} - H_1y_1\right) + W(t)\frac{dy_2}{dt} = \left(S^{-1}P(t)SW(t) - \frac{dW(t)}{dt}\right)y_2\,.$$

Solving this equation with the use of the matrix

$$\begin{pmatrix} L_1(t) \\ L_2(t) \end{pmatrix} \tag{4.22}$$

that is inverse to the matrix $(U(t), W(t))$, we obtain the following system of differential equations for $\dfrac{dy_1}{dt}$ and $\dfrac{dy_2}{dt}$:

$$\frac{dy_1}{dt} = H_1y_1 + L_1(t)\left(S^{-1}P(t)SW(t) - \frac{dW(t)}{dt}\right)y_2\,, \tag{4.23}$$

$$\frac{dy_2}{dt} = L_2(t)\left(S^{-1}P(t)SW(t) - \frac{dW(t)}{dt}\right)y_2. \tag{4.24}$$

Since the coefficient matrix of system (4.23), (4.24) has a block-triangular form, the fundamental matrix of solutions of this system is the matrix

$$\begin{pmatrix} e^{H_1t} & Y_1(t) \\ 0 & Y_2(t) \end{pmatrix} \tag{4.25}$$

the second column of which is formed by solutions of the system of differential equations (4.23), (4.24) with given initial values $y_1 = Y_1(0)$ and $y_2 = Y_2(0)$ such that

$$\det Y_2(0) \neq 0.$$

In view of (4.21), the matrix

$$(U(t), W(t)) \begin{pmatrix} e^{H_1 t} & Y_1(t) \\ 0 & Y_2(t) \end{pmatrix}$$

is a fundamental matrix of solutions of Eq. (4.20). Moreover, relation (4.17) also determines a fundamental matrix of solutions of Eq. (4.20). According to the theory of linear differential equations, there exists the following relation between these two fundamental matrices of solutions:

$$\big(U(t), V(t)\big) \begin{pmatrix} e^{H_1 t} & 0 \\ 0 & e^{H_2 t} \end{pmatrix} C = \big(U(t), W(t)\big) \begin{pmatrix} e^{H_1 t} & Y_1(t) \\ 0 & Y_2(t) \end{pmatrix} \tag{4.26}$$

for all $t \in \mathbb{R}$, where C is a nonsingular constant matrix. Substituting $t = 0$ into (4.26), we obtain the following algebraic equation for the determination of the matrix C:

$$(U(0), V(0))C = (U(0), W(0)) \begin{pmatrix} E_1 & Y_1(0) \\ 0 & Y_2(0) \end{pmatrix}. \tag{4.27}$$

Multiplying (4.27) by the matrix $\begin{pmatrix} L_1(0) \\ L_2(0) \end{pmatrix}$, we obtain

$$\begin{pmatrix} E_1 & L_1(0)V(0) \\ 0 & L_2(0)V(0) \end{pmatrix} C = \begin{pmatrix} E_1 & Y_1(0) \\ 0 & Y_2(0) \end{pmatrix}.$$

This equality implies that, first,

$$\det \big(L_2(0), V(0)\big) \neq 0$$

and, second, for

$$Y_1(0) = L_1(0)V(0), \qquad Y_2(0) = L_2(0)V(0), \tag{4.28}$$

we have

$$C = E. \tag{4.29}$$

Thus, determining the solutions $\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix}$ of the system of differential equations (4.23), (4.24) with initial values (4.28), we obtain the following equality from (4.26) and (4.29):

$$\big(U(t), V(t)\big) \begin{pmatrix} e^{H_1 t} & 0 \\ 0 & e^{H_2 t} \end{pmatrix} = \big(U(t), W(t)\big) \begin{pmatrix} e^{H_1 t} & Y_1(t) \\ 0 & Y_2(t) \end{pmatrix} \tag{4.30}$$

for all $t \in \mathbb{R}$.

Multiplying (4.30) by matrix (4.22), we get

$$\begin{pmatrix} E_1 & L_1(t)V(t) \\ 0 & L_2(t)V(t) \end{pmatrix} \begin{pmatrix} e^{H_1 t} & 0 \\ 0 & e^{H_2 t} \end{pmatrix} = \begin{pmatrix} e^{H_1 t} & Y_1(t) \\ 0 & Y_2(t) \end{pmatrix}.$$

Thus,

$$Y_1(t) = L_1(t)V(t)e^{H_2 t}, \tag{4.31}$$

$$Y_2(t) = L_2(t)V(t)e^{H_2 t} \tag{4.32}$$

for all $t \in \mathbb{R}$. Since the matrix $Y_2(t)$ is nonsingular, we can determine the value of $e^{H_2 t}$ from (4.32). Substituting this value into (4.31), we establish that

$$Y_1(t) = L_1(t)V(t)(L_2(t)V(t))^{-1}Y_2(t) \tag{4.33}$$

for all $t \in \mathbb{R}$.

We rewrite the system of differential equations (4.23), (4.24) in the form of the system

$$\frac{dy_1}{dt} = H_1 y_1 + R_1(t)y_2, \tag{4.34}$$

$$\frac{dy_2}{dt} = G(t)y_2, \tag{4.35}$$

where

$$R_1(t) = L_1(t)\left(S^{-1}P(t)SW(t) - \frac{dW(t)}{dt}\right),$$

$$G(t) = L_2(t)\left(S^{-1}P(t)SW(t) - \frac{dW(t)}{dt}\right).$$

Using the matrix

$$F(t) = L_1(t)V(t)(L_2(t)V(t))^{-1}, \tag{4.36}$$

we rewrite equality (4.33) in the form

$$Y_1(t) = F(t)Y_2(t). \tag{4.37}$$

Differentiating equality (4.37) and taking into account that the matrix $\begin{pmatrix} Y_1(t) \\ Y_2(t) \end{pmatrix}$ is a block of the fundamental matrix (4.25) of solutions of the system of differential equations (4.23), (4.24) [and, hence, of system (4.34), (4.35)], we get

$$\frac{dF(t)}{dt} + F(t)G(t) = H_1 F(t) + R_1(t) \tag{4.38}$$

for all $t \in \mathbb{R}$. Finally, performing the change of variables

$$y_1 = z_1 + F(t)z_2, \qquad y_2 = z_2,$$

we obtain the system

$$\frac{dz_1(t)}{dt} + \frac{dF(t)}{dt}z_2 + F(t)G(t)z_2 = H_1 z_1 + H_1 F(t)z_2 + R_1(t)z_2,$$

$$\frac{dz_2(t)}{dt} = G(t)z_2$$

instead of the system differential equations (4.34), (4.35). By virtue of (4.38), this system takes the form

$$\frac{dz_1(t)}{dt} = H_1 z_1, \qquad \frac{dz_2(t)}{dt} = G(t)z_2. \tag{4.39}$$

Since the second equation of system (4.39) coincides (to within notation) with Eq. (4.35), the matrix $Y_2(t)$ is a fundamental matrix of solutions of the second equation of system (4.39). Then, according to relation (4.32), the matrix

$$L_2(t)V(t)e^{H_2t}(L_2(0)V(0))^{-1}$$

is a fundamental matrix of solutions of the second equation of system (4.39) and is equal to the identity matrix for $t = 0$. Thus, the matrix

$$(L_2(T)V(T))e^{H_2T}(L_2(0)V(0))^{-1} \tag{4.40}$$

is the monodromy matrix of the second equation of system (4.39).

By definition, the matrix $L_2(t)$ is periodic with period T, the matrix $V(t)$ satisfies the condition

$$V(t+T) = -V(t), \tag{4.41}$$

and the matrix H_2 has the form

$$H_2 = \frac{1}{T}\ln(-B).$$

Taking into account the properties of the matrices $L_2(t)$, $V(t)$, and H_2 presented above, we conclude that matrix (4.40) has the form

$$(-L_2(0)V(0))(-B)(L_2(0)V(0))^{-1} = L_2(0)V(0)B(L_2(0)V(0))^{-1}.$$

Thus, it follows from the results presented above and the definition of the matrix B that the set of eigenvalues of matrix (4.40) is either the set of all negative eigenvalues of the matrix $X(T)$ or its subset.

Consider the matrix $F(t)$. The definition of this matrix [see (4.36)] and the fact that the matrices $L_1(t)$ and $L_2(t)$ are periodic with period T and the matrix $V(t)$ satisfies condition (4.41) imply that

$$F(t+T) = (-L_1(t)V(t))(-L_2(t)V(t))^{-1} = F(t)$$

for all $t \in \mathbb{R}$.

Thus, the matrix $F(t)$ is periodic with period T.

To complete the proof of assertion 3 of Theorem 3, it remains to take into account that the change of variables

$$x = Sy \tag{4.42}$$

transforms the differential equation (I) into the differential equation (4.20). Therefore, the superposition of changes (4.42), (4.21), and (4.20) transforms the differential equation (I) into the system of differential equations (4.39), and both the change of variables and the differential equations of system (4.39) themselves possess the properties indicated in Theorem 3.

We now make several remarks on assertions 2 and 3 of Theorem 3.

The first remark deals with relation (VI), which defines the matrix H. It follows from the proof of Theorem 3 that H is not always uniquely defined. This nonuniqueness is caused by the condition of decomposition of the canonical form of the matrix $X(T)$ into blocks A and B according to which the matrix B can be either a block of the

Jordan form of the matrix $X(T)$ formed by all its Jordan cells corresponding to its negative eigenvalues or a block of this form obtained from the block indicated above by elimination of an arbitrary number of pairs of identical Jordan cells.

The second remark deals with the minimum possible value of the order of the second differential equation of system (VIII). It follows from the proof of Theorem 3 that this order is also related to the condition of decomposition of the real canonical form of the matrix $X(T)$ into blocks A and B and is equal to the minimum possible order of the matrix B of this decomposition. It follows from the first remark that the minimum possible value of the order of the second equation of system (VIII) is equal to the order of the matrix obtained from the Jordan form of the matrix $X(T)$ by elimination of all Jordan cells corresponding to negative eigenvalues of the matrix $X(T)$ and the maximum possible even number of identical Jordan cells of this matrix that correspond to its negative eigenvalues.

Also note that, according to the proof of Theorem 3, the matrix B is the Jordan form of the monodromy matrix $Z_2(T)$ of the second equation of system (VIII), and, hence, the fundamental matrix of solutions $Z_2(t)$ of this equation possesses all the corresponding properties.

Finally, note that, by virtue of Theorem 2 and assertion 3 of Theorem 3, the differential equation (I) has the invariant manifolds

$$K^{n-d}(t) = \{x \in \mathbb{R}^n\colon\ U(t)L_1(t)x = x\},$$

$$K^{d}(t) = \{x \in \mathbb{R}^n\colon\ V(t)L_2(t)x = x\}$$

periodic with period T,

$$K^{\nu}(t+T) = K^{\nu}(t), \quad \nu \in \{(n-d) \vee d\},$$

for all $t \in \mathbb{R}$. Moreover, Eq. (I) is equivalent on $K^{n-d}(t)$ to the first differential equation of system (VIII) and on $K^d(t)$ to the second differential equation of this system.

Corollary. *The fundamental matrix of solutions of the differential equation* (I) $X(t)$ *admits the representation*

$$X(t) = \Phi(t)e^{Ht}\Phi^{+}(0), \tag{IX}$$

where

$$H = \frac{1}{T}\ln\begin{pmatrix} X(T) & 0 \\ 0 & Z(T)\end{pmatrix}, \tag{X}$$

$H \in \mathbf{M}_m(\mathbb{R})$, $Z(T)$ *is the monodromy matrix of the restriction of* (I) *to its periodic invariant manifold* $K^d(t)$, $\Phi(t)$ *is a periodic matrix with period* T *that satisfies the equation*

$$\frac{d\Phi}{dt} + \Phi H = P(t)\Phi, \tag{XI}$$

$\Phi(t) \in \mathbf{M}_{nm}(\mathbb{R})$ *for all* $t \in \mathbb{R}$, $\Phi^{+}(0)$ *is a matrix pseudoinverse to the matrix* $\Phi(0)$, *and* $m = n + d$, $n \geq d \geq 0$.

Indeed, according to the last remark, the differential equation (I) has the periodic invariant manifold $K^d(t)$ on which Eq. (I) is equivalent to the second differential equation of system (VIII). Consider the system of differential equations

$$\frac{dx}{dt} = P(t)x, \qquad \frac{dz}{dt} = G(t)z, \tag{4.43}$$

which is formed of Eq. (I) and the second equation of system (VIII). According to the proof of assertion 3 of Theorem 3, the real canonical form of the monodromy matrix of this system

$$\begin{pmatrix} X(T) & 0 \\ 0 & Z(T) \end{pmatrix} \tag{4.44}$$

is the matrix

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}, \tag{4.45}$$

where A and B are the blocks of decomposition of the real canonical form of the matrix $X(T)$ such that the matrix A has a real logarithm. Since the matrix

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

is formed by pairwise identical Jordan cells, it has a real logarithm. Thus, the logarithm of matrix (4.45) can be chosen real. Therefore, we can choose the real logarithm of matrix (4.44) and define the matrix H according to relation (X) so that it satisfies the condition $H \in \mathbf{M}_{n+d}(\mathbb{R})$, where d is the order of the matrix B. Applying the Floquet formula (II) to the fundamental matrix of solutions of the system of differential equations (4.43)

$$\begin{pmatrix} X(t) & 0 \\ 0 & Z(t) \end{pmatrix},$$

we obtain

$$\begin{pmatrix} X(t) & 0 \\ 0 & Z(t) \end{pmatrix} = \begin{pmatrix} \Phi_1(t) \\ \Phi_2(t) \end{pmatrix} e^{Ht}, \tag{4.46}$$

where H is matrix (X) from the space $\mathbf{M}_{n+d}(\mathbb{R})$, $\Phi_1(t)$ and $\Phi_2(t)$ are periodic matrices with period T, and $\Phi_1(t) \in \mathbf{M}_{n\, n+d}(\mathbb{R})$ and $\Phi_2(t) \in \mathbf{M}_{d\, n+d}(\mathbb{R})$ for all $t \in \mathbb{R}$. Differentiating equality (4.46), we obtain the following matrix differential equation for the matrix $\Phi(t) = \begin{pmatrix} \Phi_1(t) \\ \Phi_2(t) \end{pmatrix}$:

$$\frac{d\Phi}{dt} + \Phi H = \begin{pmatrix} P(t) & 0 \\ 0 & G(t) \end{pmatrix} \Phi.$$

This equation implies that the matrix $\Phi_1(t)$ satisfies the differential equation (XI). Finally, multiplying equality (4.46) by the matrix $\Phi_1^+(0)$, which is pseudoinverse to the matrix $\Phi_1(0)$, we obtain the equality

$$X(t) = \Phi_1(t) e^{Ht} \Phi_1^+(0),$$

which coincides (to within notation) with (IX).

5. Two applications of obtained results. ***5.1.*** Let $x \in \mathbb{R}^n$, let $P(t)$ be a continuous matrix periodic with period T, let $P(t) \in \mathbf{M}_n(\mathbb{R})$ for all $t \in \mathbb{R}$, and let $X(t,x)$ be a function of variables $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$ that takes values in $\mathbb{R}^n$ and is continuous for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Consider the differential equation

$$\frac{dx}{dt} = P(t)x + X(t,x). \tag{5.1}$$

Let

$$X(t,x) \equiv 0.$$

Then the differential equation (5.1) has a fundamental matrix of solutions $X(t)$, which can be represented in the form

$$X(t) = \Phi(t)e^{Ht}\Phi^{+}(0), \tag{5.2}$$

and, moreover, the properties of the matrices $\Phi(t)$ and H are determined in the corollary in the last section.

To simplify the differential equation (5.1), we use relation (5.2). To this end, we change the variables in (5.1) by introducing a variable $y \in \mathbb{R}^m$ instead of $x \in \mathbb{R}^n$ according to the relation

$$x = \Phi(t)y. \tag{5.3}$$

Taking into account that the matrix $\Phi(t)$ is a solution of the differential equation (XI), we obtain the following equality from (5.1) and (5.3):

$$\Phi(t)\left(\frac{dy}{dt} - Hy\right) = X(t, \Phi(t)y).$$

We represent this equality in the form

$$\frac{dy}{dt} - Hy = \Phi^{+}(t)X(t, \Phi(t)y), \tag{5.4}$$

where $\Phi^{+}(t)$ is a matrix pseudoinverse to $\Phi(t)$ that has the same smoothness and period as $\Phi(t)$. In particular, as $\Phi^{+}(t)$, we can take the first block of the matrix $\big(\Phi_1^{+}(t), \Phi_2^{+}(t)\big)$, which is inverse to the matrix $\begin{pmatrix}\Phi_1(t)\\ \Phi_2(t)\end{pmatrix}$ defined by relation (4.46) of the previous section. Solving Eq. (5.4) with respect to $\dfrac{dy}{dt} - Hy$, we get

$$\frac{dy}{dt} = Hy + \Phi^{+}(t)X(t, \Phi(t)y). \tag{5.5}$$

The selected linear part of Eq. (5.5) has a constant coefficient matrix, and the general part preserves the properties of the corresponding part of the original equation (5.1).

5.2. Consider the differential equation

$$\frac{dx}{dt} = X(x) + X_1(t,x), \tag{5.6}$$

where $X(x)$ is a continuously differentiable function of x and $X_1(t,x)$ is a continuous function of t and x that takes values in $\mathbb{R}^n$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, $n \geq 2$.

Assume that, under the condition

$$X_1(t,x) \equiv 0, \tag{5.7}$$

Eq. (5.6) has a T-periodic solution

$$x(t) = \xi(\omega t), \tag{5.8}$$

where $\xi(\varphi)$ is a function periodic in φ with period 2π and $\omega = \dfrac{2\pi}{T}$ is the frequency of the periodic solution.

The variational equation corresponding to the periodic solution (5.8) of the differential equation (5.6) with condition (5.7) has the form

$$\frac{d\delta\xi}{dt} = \frac{\partial X(\xi(\omega t))}{\partial x}\,\delta\xi. \tag{5.9}$$

This equation has the solution

$$\delta\xi = \xi'(\omega t),$$

where " $'$ " stands for the derivative with respect to the variable φ.

Indeed, by definition, we have

$$\xi'(\varphi)\omega = X(\xi(\varphi)). \tag{5.10}$$

Thus,

$$\xi''(\varphi)\omega = \frac{\partial X(\xi(\varphi))}{\partial x}\,\xi'(\varphi) \tag{5.11}$$

for all $\varphi \in \mathbb{R}$. Substituting ωt for φ in (5.11), we obtain the identity

$$\frac{d}{dt}\,\xi'(\omega t) = \frac{\partial X(\xi(\omega t))}{\partial x}\,\xi'(\omega t),$$

which proves the required statement.

Let $B(\varphi)$ be a continuously differentiable periodic matrix with period 2π, let $B(\varphi) \in$ $\in \mathbf{M}_{n\,n-1}(\mathbb{R})$, and let

$$\det\big(\xi'(\varphi), B(\varphi)\big) \neq 0$$

for all $\varphi \in \mathbb{R}$.

Using the change of variables

$$\delta\xi = \xi'(\omega t)c + B(\omega t)g, \tag{5.12}$$

we reduce the variational equation (5.9) to the differential equation

$$\xi''(\omega t)\omega c + \xi'(\omega t)\frac{dc}{dt} + B'(\omega t)\omega g + B(\omega t)\frac{dg}{dt} = \frac{\partial X(\xi(\omega t))}{\partial x}(\xi'(\omega t)c + B(\omega t)g),$$

or, with regard for (5.11), to the equation

$$\xi'(\omega t)\frac{dc}{dt} + B(\omega t)\frac{dg}{dt} = \left(\frac{\partial X(\xi(\omega t))}{\partial x}B(\omega t) - B\,'(\omega t)\omega\right)g.$$

Solving this equation with respect to the derivatives $\dfrac{dc}{dt}$ and $\dfrac{dg}{dt}$ with the use of the matrix

$$\begin{pmatrix} (\xi'(\omega t))^+ \\ B^+(\omega t) \end{pmatrix}, \tag{5.13}$$

which is inverse to $(\xi'(\omega t), B(\omega t))$, we reduce (5.9) to the system of differential equations

$$\frac{dc}{dt} = (\xi'(\omega t))^+ \left(\frac{\partial X(\xi(\omega t))}{\partial x} B(\omega t) - B\,'(\omega t)\omega \right) g,$$

$$\tag{5.14}$$

$$\frac{dg}{dt} = B^+(\omega t) \left(\frac{\partial X(\xi(\omega t))}{\partial x} B(\omega t) - B\,'(\omega t)\omega \right) g.$$

According to the change of variables (5.12), the monodromy matrix of the system of differential equations (5.14) is similar to the monodromy matrix of the variational equation (5.9). Thus, the eigenvalues of both monodromy matrices coincide.

It follows from system (5.14) that one of the eigenvalues of its monodromy matrix is equal to 1, whereas the other eigenvalues are eigenvalues of the monodromy matrix of the second differential equation of system (5.14). Thus, the same is true for the eigenvalues of the monodromy matrix of the variational equation (5.9).

We denote the coefficient matrix of the second differential equation of system (5.14) by $Q(\omega t)$, where $Q(\varphi)$ is a periodic matrix with period 2π, and consider the differential equation

$$\frac{dg}{dt} = Q(\omega t)g. \tag{5.15}$$

By virtue of the corollary in the previous section, the fundamental matrix of solutions of Eq. (5.15) $G(t)$ admits the representation

$$G(t) = \Phi(\omega t)e^{Ht}\Phi^+(0), \tag{5.16}$$

where

$$H = \frac{1}{T} \ln \begin{pmatrix} G(T) & 0 \\ 0 & Z(T) \end{pmatrix} \in \mathbf{M}_m(\mathbb{R}), \tag{5.17}$$

$2(n-1) \geq m \geq (n-1)$, $Z(t)$ is the fundamental matrix of the restriction of the differential equation (5.15) to its periodic invariant manifold $K^{m-(n-1)}(t)$, $\Phi(\varphi)$ is a periodic matrix with period 2π, $\Phi(\varphi) \in \mathbf{M}_{n-1\,m}(\mathbb{R})$ for all $\varphi \in \mathbb{R}$, $\Phi(\varphi)$ satisfies the differential equation

$$\frac{d\Phi}{d\varphi}\omega + \Phi H = Q(\varphi)\Phi, \tag{5.18}$$

and $\Phi^+(0)$ is a matrix pseudoinverse to $\Phi(0)$.

We now use the results obtained above for the introduction of amplitude – phase coordinates in the neighborhood of the closed curve

$$x = \xi(\varphi), \quad \varphi \in \mathbb{R}, \tag{5.19}$$

and for the reduction of the differential equation (5.6) in the neighborhood of this curve to a simpler amplitude – phase system of differential equations.

To this end, we change the variables in Eq. (5.6) according to the relation

$$x = \xi(\varphi) + B(\varphi)g, \tag{5.20}$$

where $B(\varphi)$ is the matrix defined above.

Using equality (5.10), we obtain the following differential equation instead of (5.6):

$$(\xi'(\varphi) + B'(\varphi)g)\left(\frac{d\varphi}{dt} - \omega\right) + B(\varphi)\frac{dg}{dt} =$$

$$= X(\xi(\varphi) + B(\varphi)g) - X(\xi(\varphi)) - B'(\varphi)\omega g + X_1(t, \xi(\varphi) + B(\varphi)g). \tag{5.21}$$

We solve Eq. (5.21) with respect to $\dfrac{d\varphi}{dt} - \omega$ and $\dfrac{dg}{dt}$ by using the matrix

$$\begin{pmatrix} L_1(\varphi, g) \\ L_2(\varphi, g) \end{pmatrix} \tag{5.22}$$

that is inverse to $(\xi'(\varphi) + B'(\varphi)g, B(\varphi))$.

Choosing a sufficiently small $\delta > 0$, one can easily construct matrix (5.22) for all

$$\varphi \in \mathbb{R}, \qquad \|g\| \le \delta,$$

on the basis of matrix (5.13) by setting

$$\begin{pmatrix} L_1(\varphi, 0) \\ L_2(\varphi, 0) \end{pmatrix} = \begin{pmatrix} (\xi'(\varphi))^+ \\ B^+(\varphi) \end{pmatrix}.$$

Using (5.21), we obtain the system of differential equations

$$\frac{d\varphi}{dt} = \omega + L_1(\varphi, g)\big[X(\xi(\varphi) + B(\varphi)g) - X(\xi(\varphi)) + X_1(t, \xi(\varphi) + B(\varphi)g)\big], \tag{5.23}$$

$$\frac{dg}{dt} = L_2(\varphi, g)\big[X(\xi(\varphi) + B(\varphi)g) - X(\xi(\varphi)) + X_1(t, \xi(\varphi) + B(\varphi)g)\big]. \tag{5.24}$$

We rewrite the differential equation (5.24) in the form

$$\frac{dg}{dt} = B^+(\varphi)\frac{\partial X(\xi(\varphi))}{\partial x}B(\varphi)g + G(\varphi, g) + L_2(\varphi, g)X_1(t, \xi(\varphi) + B(\varphi)g), \tag{5.25}$$

where $G(\varphi, g)$ denotes the function

$$L_2(\varphi, g)(X(\xi(\varphi) + B(\varphi)g) - X(\xi(\varphi)) - \frac{\partial X(\xi(\varphi))}{\partial x}B(\varphi)g) +$$

$$+ (L_2(\varphi, g) - L_2(\varphi, 0))\frac{\partial X(\xi(\varphi))}{\partial x}B(\varphi)g,$$

which satisfies the conditions

$$G(\varphi, 0) = 0, \qquad \frac{\partial G(\varphi, 0)}{\partial g} = 0.$$

According to the definition of the matrix $Q(\omega t)$, the coefficient matrix of the selected linear part of the differential equation (5.25) coincides with the matrix $Q(\varphi)$. Thus, Eq. (5.25) takes the form

$$\frac{dg}{dt} = Q(\varphi)g + G(\varphi, g) + L_2(\varphi, g)X_1(t, \xi(\varphi) + B(\varphi)g). \tag{5.26}$$

Let $\Phi(\varphi)$ and H be the matrices determined from representation (5.16) of a fundamental matrix of solutions of the differential equation (5.15). With the use of these matrices, we transform the system of differential equations (5.23), (5.24) by setting

$$g = \Phi(\varphi)h. \tag{5.27}$$

As a result, instead of (5.26), we obtain

$$\Phi'(\varphi)h + \Phi(\varphi)\left(\frac{dh}{dt} - Hh\right) + \Phi(\varphi)Hh = Q(\varphi)\Phi(\varphi)h + G(\varphi, \Phi(\varphi)h) +$$

$$+ L_2(\varphi, \Phi(\varphi)h)X_1(t, \xi(\varphi) + B(\varphi)\Phi(\varphi)h),$$

or, taking into account that $\Phi(\varphi)$ is a solution of the differential equation (5.18),

$$\Phi(\varphi)\left(\frac{dh}{dt} - Hh\right) = G(\varphi, \Phi(\varphi)h) + L_2(\varphi, \Phi(\varphi)h)X_1(t, \xi(\varphi) + B(\varphi)\Phi(\varphi)h). \tag{5.28}$$

Solving Eq. (5.28) with respect to $\dfrac{dh}{dt} - Hh$ with the use of the matrix $\Phi^+(\varphi)$ that is pseudoinverse to $\Phi(\varphi)$, we obtain

$$\frac{dh}{dt} = Hh + \Phi^+(\varphi)\big[G(\varphi, \Phi(\varphi)h) + L_2(\varphi, \Phi(\varphi)h)X_1(t, \xi(\varphi) + B(\varphi)\Phi(\varphi)h)\big].$$

By virtue of the results presented above, the change of variables (5.27) reduces the system of differential equations (5.23), (5.24) to the system

$$\frac{d\varphi}{dt} = \omega + f(t, \varphi, \Phi(\varphi)h), \tag{5.29}$$

$$\frac{dh}{dt} = Hh + \Phi^+(\varphi)F(t, \varphi, \Phi(\varphi)h), \tag{5.30}$$

where H is a matrix of the form (5.17),

$$f(t, \varphi, g) = L_1(\varphi, g)\big[X(\xi(\varphi) + B(\varphi)g) - X(\xi(\varphi)) + X_1(t, \xi(\varphi) + B(\varphi)g)\big],$$

and

$$F(t, \varphi, g) = L_2(\varphi, g)\left[X(\xi(\varphi) + B(\varphi)g) - X(\xi(\varphi)) - \frac{\partial X(\xi(\varphi))}{\partial x}B(\varphi)g\right] +$$

$$+ (L_2(\varphi, g) - L_2(\varphi, 0))\frac{\partial X(\xi(\varphi))}{\partial x}B(\varphi)g.$$

The system of differential equations (5.29), (5.30) is the required one.

Thus, by using the superposition of changes (5.20) and (5.27), and, hence, the change of variables

$$x = \xi(\varphi) + B(\varphi)\Phi(\varphi)h,$$

the original differential equation (5.6) can be reduced in the neighborhood of the closed curve (5.19) to the system of differential equations (5.29), (5.30), where the functions $f(t, \varphi, g)$ and $F(t, \varphi, g)$ are continuous in the variables t, φ, and g for $t \in \mathbb{R}$, $\varphi \in \mathbb{R}$, and $g \in \mathbb{R}^{n-1}$, $\|g\| \leq \delta$, take values in $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively, and are periodic in φ with period 2π, the matrix $\Phi(\varphi)$ belongs to $\mathbf{M}_{n-1\,m}(\mathbb{R})$ for all $\varphi \in \mathbb{R}$ and is periodic with period 2π, the matrix H belongs to $\mathbf{M}_m(\mathbb{R})$, its eigenvalues are the numbers

$$\frac{1}{T}\ln\lambda_j, \quad j = \overline{1,\, n-1},$$

and their p-fold repetitions, $1 \geq p_j \geq 0$, $\sum_{j=1}^{n-1} p_j = m - (n-1),\ 2(n-1) \geq m \geq$ $\geq (n-1)$, and 1 and $\lambda_1, \ldots, \lambda_{n-1}$ are the eigenvalues of the monodromy matrix of the variational equation (5.9).

The reduction of the differential equations considered above to equations with the constant matrix of coefficients in their separated linear part is essential for the subsequent investigation of these equations. A confirmation of this statement can be found, e.g., in [4, 8], where, however, the problem of this reduction was only partially solved.

1. *Samoilenko A. M.* On invariant manifolds of linear differential equations (in Ukrainian). – Kyiv, 2009. – 10 p. – (Preprint / Nat. Acad. Sci. Ukraine. Inst. Math.; № 2009.7).
2. *Samoilenko A. M.* On invariant manifolds of linear differential equations. II (in Ukrainian). – Kyiv, 2010. – 43 p. – (Preprint / Nat. Acad. Sci. Ukraine. Inst. Math.; № 2010.3).
3. *Bogolyubov N. N.* On some statistical methods in mathematical physics (in Russian). – Kiev: Acad. Sci. Ukr.SSR, 1945. – (Collection of Scientific Works (in Russian) / N. N. Bogolyubov. – Moscow: Nauka, 2006. – Vol. 4. – 432 p.).
4. *Bogolyubov N. N., Mitropol'skii Yu. A.* Asymptotic methods in the theory of nonlinear oscillations (in Russian). – Moscow: Fizmatgiz, 1963. – (Collection of Scientific Works (in Russian) / N. N. Bogolyubov. – Moscow: Nauka, 2005. – Vol. 3. – 605 p.).
5. *Floquet G.* Sur les 'equations diff'erentielles lin'eaires 'a coefficients p'eriodiques // Ann. 'Ecole Norm. Super. – 1883. – № 12. – P. 47 – 88.
6. *Gantmakher F. R.* Theory of matrices (in Russian). – Moscow: Nauka, 1988. – 552 p.
7. *Samoilenko A. M.* Elements of the mathematical theory of multifrequency oscillations. Invariant tori (in Russian). – Moscow: Nauka, 1987. – 304 p.
8. *Samoilenko A. M., Recke L.* Conditions for synchronization of one oscillating system // Ukr. Mat. Zh. – 2005. – **57**, № 7. – P. 922 – 945.

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