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ON THE BERNSTEIN – WALSH-TYPE LEMMAS IN REGIONS OF THE COMPLEX PLANE

ПРО ЛЕМИ ТИПУ БЕРНШТЕЙНА – УОЛША В ОБЛАСТЯХ КОМПЛЕКСНОЇ ПЛОЩИНИ

Let $G \subset \mathbb{C}$ be a finite region bounded by a Jordan curve $L := \partial G$, $\Omega := \operatorname{ext} \overline{G}$ (respect to $\overline{\mathbb{C}}$), $\Delta := \{z \colon |z| > 1\}$; $w = \Phi(z)$ be the univalent conformal mapping of Ω onto Δ normalized by $\Phi(\infty) = \infty$, $\Phi^{'}(\infty) > 0$.

Let $A_p(G)$, $p > 0$, denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|^p_{A_p(G)} := \iint\limits_G |f(z)|^p d\sigma_z < \infty, \qquad (*)$$

where σ is a two-dimensional Lebesque measure.

Let $P_n(z)$ be arbitrary algebraic polynomial of degree at most n. The well-known Bernstein – Walsh lemma says that

$$\|P_n(z)\| \leq |\Phi(z)|^{n+1} \|P_n\|_{C(\overline{G})}, \quad z \in \Omega. \qquad (**)$$

Firstly, we study the estimation problem $(**)$ for the norm $(*)$. Secondly, we continue studying the estimation $(**)$ when we replace the norm $\|P_n\|_{C(\overline{G})}$ by $\|P_n\|_{A_2(G)}$ for some regions of complex plane.

Припустимо, що $G \subset \mathbb{C}$ — скінченна область, що обмежена кривою Жордана $L := \partial G$, $\Omega := \operatorname{ext} \overline{G}$ (відносно $\overline{\mathbb{C}}$), $\Delta := \{z \colon |z| > 1\}$; $w = \Phi(z)$ — унівалентне конформне відображення Ω на Δ, нормоване з використанням $\Phi(\infty) = \infty$, $\Phi^{'}(\infty) > 0$.

Нехай $A_p(G)$, $p > 0$, позначає клас функцій f, які є аналітичними в G і задовольняють умову

$$\|f\|^p_{A_p(G)} := \iint\limits_G |f(z)|^p d\sigma_z < \infty, \qquad (*)$$

де σ — двовимірна міра Лебега.

Припустимо, що $P_n(z)$ — довільний алгебраїчний поліном степеня не більше n. У відомій лемі Бернштейна – Уолша стверджується, що

$$\|P_n(z)\| \leq |\Phi(z)|^{n+1} \|P_n\|_{C(\overline{G})}, \quad z \in \Omega. \qquad (**)$$

По-перше, розглянуто задачу оцінювання $(**)$ для норми $(*)$. По-друге, продовжено дослідження оцінювання $(**)$ у випадку, коли норма $\|P_n\|_{C(\overline{G})}$ замінюється нормою $\|P_n\|_{A_2(G)}$ для деяких областей комплексної площини.

1. Introduction and main results. Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $B := B(0,1) := \{z \colon |z| < 1\}$, $\Delta := \Delta(0,1) := \{w \colon |w| > 1\}$, $\Omega := \operatorname{ext} \overline{G}$ (respect to $\overline{\mathbb{C}}$); $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\Phi^{'}(\infty) > 0$, and $\Psi := \Phi^{-1}$. Let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most n.

Let σ be the two-dimensional Lebesque measure and let $h(z)$ be a weight function defined in G.

Let $A_p(h, G)$, $p > 0$ denote the class of functions f which are analytic in G and satisfy the condition

$$\|f\|_{A_p(h,G)} := \left(\iint\limits_G h(z)|f(z)|^p d\sigma_z \right)^{1/p} < \infty$$

and $A_p(1,G) \equiv A_p(G)$.

In case of when L is rectifiable, let $\mathcal{L}_p(L)$, $p > 0$, we denote the class of functions f which are integrable on L and satisfy the condition

$$\|f\|_{\mathcal{L}_p(L)} := \left(\int\limits_L |f(z)|^p |dz| \right)^{1/p} < \infty.$$

Well known Bernstein – Walsh lemma [1] says that

$$|P_n(z)| \leq |\Phi(z)|^{n+1} \|P_n\|_{C(\overline{G})}, \quad z \in \Omega. \tag{1.1}$$

For $R > 1$, let us set $L_R := \{z \colon |\Phi(z)| = R\}$, $G_R := \operatorname{int} L_R$, $\Omega_R := \operatorname{ext} L_R$. Then, (1.1) can be written as following:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^{n+1} \|P_n\|_{C(\overline{G})}. \tag{1.2}$$

For $R = 1 + \dfrac{c_1}{n}$, according to (1.2), we see that the C-norm of polynomials $P_n(z)$ in $\overline{G}_R$ and $\overline{G}$ is identical, i.e., the norm $\|P_n\|_{C(\overline{G})}$ increases with at most a constant.

Similar estimation to (1.2) in space $\mathcal{L}_p(L)$ was investigated in [2] and obtained as following:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq c_1 R^{n+1/p} \|P_n\|_{\mathcal{L}_p(L)}, \quad p > 0. \tag{1.3}$$

Here and throughout this paper, $c, c_0, c_1, c_2, \ldots$ are positive constants (in general, different in different relations), which are depended on G in general.

Definition 1.1 [3, p. 97; 4]. *The Jordan arc* (*or curve*) *L is called K-quasiconformal* ($K \geq 1$), *if there is a K-quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment* (*or circle*).

$F(L)$ denotes the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (*or circle*) *and defines*

$$K_L := \inf \{K(f) \colon f \in F(L)\},$$

where $K(f)$ is the maximal dilatation of a such mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a K-quasiconformal curve, if $K_L \leq K$.

We well know that there exists quasiconformal curve which is not rectifiable [3, p. 104].

Let L be a K-quasiconformal and $y(\cdot)$ be a regular quasiconformal reflection across L (for detail see Section 2). For $R > 1$, let $L^* := y(L_R)$, $G^* := \operatorname{int} L^*$, $\Omega^* := \operatorname{ext} L^*$; $w = \Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty) = \infty$, $\Phi'_R(\infty) > 0$, and $\Psi_R := \Phi_R^{-1}$; $d(\Gamma, L) := \inf \{|\zeta - z| \colon z \in \Gamma, \zeta \in L\}$.

The Bernstein – Walsh-type estimation in the space $A_p(h,G)$, $p > 0$ is contained in [5]. In particular,

$$\|P_n\|_{A_p(G_R)} \leq c_2 \, {R^*}^{n+1/p} \|P_n\|_{A_p(G)}, \quad p > 0, \tag{1.4}$$

where $R^* := 1 + c_3(R - 1)$. Therefore, if we choose $R = 1 + \dfrac{c_1}{n}$, then (1.4) can be shown that A_p-norm of polynomials $P_n(z)$ in G_R and G is identical.

N. Stylianopoulos in [9] obtained the following result by changing the norm $\|P_n\|_{C(\overline{G})}$ in (1.1) with the norm $\|P_n\|_{A_2(G)}$.

Lemma 1.1 [9]**.** *Assume that L is quasiconformal and rectifiable. Then, for any $P_n \in \wp_n$*

$$|P_n(z)| \leq \frac{c(L)}{d(z, L)} \sqrt{n} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega. \tag{1.5}$$

In this work, we study the similar problems to (1.5) for domains with K-quasiconformal (non-rectifiable!) boundary. Now, we give the main results.

Theorem 1.1. *Assume that L is K-quasiconformal. Then, for any $P_n \in \wp_n$ and $R > 1$ we have*

$$|P_n(z)| \leq \frac{c_4}{d(L, L_R)} \|P_n\|_{A_2(G_R)} |\Phi(z)|^{n+1}, \quad z \in \Omega. \tag{1.6}$$

Theorem 1.2. *Assume that L is K-quasiconformal. Then, for any $P_n \in \wp_n$ we have*

$$|P_n(z)| \leq \frac{c_5}{d(L, L_{1+\frac{c}{n}})} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \Omega. \tag{1.7}$$

***Remark* 1.1.** If $z \in \overline{G}_{1+c/n} \cap \Omega$ for some $c > 1$, then (1.7) is better than (1.5).

Theorem 1.3. *Assume that L is K-quasiconformal. Then, for any $P_n \in \wp_n$ we have*

$$|P_n(z)| \leq \frac{c_6}{d(z, L)} n^{\mu - \mu^{-1}} \|P_n\|_{A_2(G)} |\Phi_{1+1/n}(z)|^{n+1}, \quad z \in \Omega, \tag{1.8}$$

where $\mu := \min\left\{2, K^4\right\}$.

***Remark* 1.2.** For $K \leq \sqrt[4]{\dfrac{1+\sqrt{17}}{4}}$ and for $z \in \Omega$ such that far away from L, (1.8) is better than (1.5).

Theorem 1.4. *Assume that L is K-quasiconformal. Then, for any $P_n \in \wp_n$ we have*

$$|P_n(z)| \leq c_7 n^{\mu} \|P_n\|_{A_2(G)} |\Phi_{1+1/n}(z)|^{n+1}, \quad z \in \Omega, \tag{1.9}$$

where $\mu := \min\left\{2, K^4\right\}$.

Theorem 1.5. *Assume that L is K-quasiconformal. Then, for any $P_n \in \wp_n$ we have*

$$|P_n(z)| \leq \frac{c_8}{d(z, L)} n^{\nu - \nu^{-1}} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \overline{\Omega}_{1+1/n}, \tag{1.10}$$

where $\nu := \min\left\{2, K^2\right\}$.

***Remark* 1.3.** For any $K \leq \dfrac{1}{2}\sqrt{1+\sqrt{17}}$ and the points $z \in \overline{\Omega}_{1+1/n}$, (1.10) is better than (1.5).

Theorem 1.6. *Assume that L is K-quasiconformal. Then, for any $P_n \in \wp_n$ we have*

$$|P_n(z)| \leq c_9 n^{\nu} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \quad z \in \overline{\Omega}_{1+1/n}, \tag{1.11}$$

where $\nu := \min\left\{2, K^2\right\}$.

2. Some auxiliary results. Let $G \subset \mathbb{C}$ be a finite region bounded by Jordan curve L and let $w = \varphi(z)$ be the univalent conformal mapping of G onto the B normalized by $\varphi(0) = 0$, $\varphi'(0) > 0$ and $\psi := \varphi^{-1}$.

The level curve (interior or exterior) can be defined for $t > 0$ as

$$L_t := \{z \colon |\varphi(z)| = t, \text{ if } t < 1, \ |\Phi(z)| = t, \text{ if } t > 1\}, \quad L_1 \equiv L,$$

and let $G_t := \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$.

We note that, the region D in Definition 1.1 may be taken as $D \subset \mathbb{C}$ or $D \equiv \mathbb{C}$. Case $D \equiv \mathbb{C}$ gives the global definition of a K-quasiconformal arc or curve consequently. At the same time, we can consider the domain $D \supset L$ as the neighborhood of the curve L. In this case, Definition 1.1 will be called local definition. This local definition has an advantage in determining the coefficients of quasiconformality for some simple arcs or curves.

Let us denote natural representation of L by $z = z(s)$, $s \in [0, \operatorname{mes} L]$.

Definition 2.1 [12]. *We say that $G \in C_\theta$ if $L := \partial G$ has a continuous tangent $\theta(z) := \theta(z(s))$ for every points $z(s)$.*

According to [4], we have the following facts:

***Corollary* 2.1.** *If $G \in C_\theta$, then $K = 1 + \varepsilon$, for all $\varepsilon > 0$.*

***Corollary* 2.2.** *If L is an analytic curve or arc, then $K = 1$.*

For $a > 0$ and $b > 0$, we shall use the notations "$a \prec b$" (order inequality), if $a \le cb$ and "$a \asymp b$" are equivalent to $c_1 a \le b \le c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively. Throughout this paper $\varepsilon, \varepsilon_1, \varepsilon_2, \ldots$ are sufficiently small positive constants (in general, different in different relations), which depend on G in general.

Let L be a K-quasiconformal curve and $D = \mathbb{C}$. Then [8] there exists a quasiconformal reflection $y(\cdot)$ across L such that $y(G) = \Omega$, $y(\Omega) = G$ and $y(\cdot)$ fixes the points of L. The quasiconformal reflection $y(\cdot)$ is such that it satisfied the following condition [8, 7; p. 26]:

$$|y(\zeta) - z| \asymp |\zeta - z|, \quad z \in L, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon},$$

$$|y_{\overline{\zeta}}| \asymp |y_\zeta| \asymp 1, \quad \varepsilon < |\zeta| < \frac{1}{\varepsilon}, \tag{2.1}$$

$$|y_{\overline{\zeta}}| \asymp |y(\zeta)|^2, \quad |\zeta| < \varepsilon, \quad |y_{\overline{\zeta}}| \asymp |\zeta|^{-2}, \quad |\zeta| > \frac{1}{\varepsilon},$$

and for the Jacobian $J_y = |y_z|^2 - |y_{\overline{z}}|^2$ of $y(\cdot)$ the relation $J_y \asymp 1$ is hold.

On the other hand, let L be a K-quasiconformal curve and $D \subset \mathbb{C}$. Then the region D in the Definition 1.1 can be chosen to be the region $D := G_{R_0} \backslash G_{r_0}$, for a certain number $1 < R_0 \le 2$, depending on φ, Φ, f, and $r_0 = R_0^{-1}$. In this case, it is known that the function $\alpha(\cdot) = f^{-1}\left\{\left[\overline{f(\cdot)}\right]^{-1}\right\}$ is a K^2-quasiconformal reflection across L as shown in [11, p. 28] by analogously in [8, p. 75], that is, $\alpha(\cdot)$ is a K^2-quasiconformal mapping leaving the points on L fixed and satisfying the conditions $\alpha(G_{\widetilde{R}} \backslash \overline{G}) \subset G \backslash \overline{G}_{r_0}$, $\alpha(G \backslash \overline{G}_{\widetilde{r}}) \subset G_{R_0} \backslash \overline{G}$ for some $1 < \widetilde{R} < R_0$, $r_0 < \widetilde{r} < 1$. Therefore, by means of the extension theorem of a quasiconformal mapping [3, p. 98], without loss of generality, we may assume that

$$y(z) = \alpha(z), \quad z \in D.$$

Then, taking $y(z)$ that satisfies (2.1) and denoting the restriction of $y(z)$ in D as $\alpha^*(\cdot)$, we see that the following conditions are also satisfied for $\alpha^*(\cdot)$:

$$|z_1 - \alpha^*(z)| \asymp |z_1 - z|, \quad z_1 \in L, \quad \varepsilon < |z| < \frac{1}{\varepsilon},$$

$$|\alpha^*_{\bar z}| \asymp |\alpha^*_z| \asymp 1, \quad \varepsilon < |z| < \frac{1}{\varepsilon}, \tag{2.2}$$

$$|\alpha^*_{\bar z}| \asymp |\alpha^*(z)|^2, \quad |z| < \varepsilon, \ |\alpha^*_{\bar z}| \asymp |z|^{-2}, \quad |z| > \frac{1}{\varepsilon},$$

and for the Jacobian $J_{\alpha^*} = |\alpha^*_z|^2 - |\alpha^*_{\bar z}|^2$ of $\alpha^*(\cdot)$ the relation $J_{\alpha^*} \asymp 1$ is hold.

For simplicity of notation, we denote the $\alpha^*(\cdot)$ also as $\alpha(\cdot)$. Throughout this paper we assume that $D \subset \mathbb{C}$.

For $R > 1$, we denote $L^* := y(L_R)$, $G^* := \operatorname{int} L^*$, $\Omega^* := \operatorname{ext} L^*$; $w = \Phi_R(z)$ be the conformal mapping of Ω^* onto the Δ normalized by $\Phi_R(\infty) = \infty$, $\Phi'_R(\infty) > 0$; $\Psi_R := \Phi_R^{-1}$. For $t > 1$, let $L^*_t := \{z\colon |\Phi_R(z)| = t\}$, $G^*_t := \operatorname{int} L^*_t$, $\Omega^*_t := \operatorname{ext} L^*_t$.

According to [10], for all $z \in L^*$ and $t \in L$ such that $|z - t| = d(z, L)$ we have

$$d(z, L) \asymp d(t, L_R) \asymp d(z, L^*_R).$$
$$|\Phi_R(z)| \le |\Phi_R(t)| \le 1 + c(R-1). \tag{2.3}$$

Lemma 2.1 [11]. *Let L be a K-quasiconformal curve, $z_1 \in L$, z_2, $z_3 \in \Omega \cap \cap \{z\colon |z - z_1| \prec d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*:

a) *the statements $|z_1 - z_2| \prec |z_1 - z_3|$ and $|w_1 - w_2| \prec |w_1 - w_3|$ are equivalent; so are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$;*

b) *if $|z_1 - z_2| \prec |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2},$$

where $0 < r_0 < 1$ a constant, depending on G.

In particular, for arbitrary $z_1 \in L$, $1 < R < R_0$ and fixed $z_3 \in L_{R_0}$ we have

$$(R-1)^{K^2} \prec d(z_1, L_R) \prec (R-1)^{1/K^2}. \tag{2.4}$$

***Remark* 2.1.** The left part of (2.4) for arbitrary continuum can be replaced by (see, for instance, [7])

$$(R-1)^2 \prec d(z_1, L_R). \tag{2.5}$$

Let $\{z_j\}_{j=1}^m$ be a fixed system of the points on L and the weight function $h(z)$ is defined as the following:

$$h(z) = h_0(z) \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \tag{2.6}$$

where $\gamma_j > -2$ for $j = \overline{1, m}$ and $h_0(z)$ is uniformly separated from zero in G:

$$h_0(z) \ge c_0 > 0 \quad \forall z \in G.$$

Lemma 2.2 [6]. *Let L be a K-quasiconformal curve; $h(z)$ is defined in* (2.6). *Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n = 1, 2, \ldots$, we have*

$$\|P_n\|_{A_p(h,G_{1+c(R-1)})} \leq c_1 R^{n+1/p} \|P_n\|_{A_p(h,G)}, \quad p > 0, \tag{2.7}$$

where c, c_1 are independent of n and R.

3. Proof of theorems. ***3.1. Proof of Theorem*** **1.1.** First of all, we note the estimation

$$|P_n(z)| \prec \frac{1}{d(z, L_R)} \|P_n\|_{A_2(G_R)}, \quad z \in \overline{G}. \tag{3.1}$$

Since L is a K-quasiconformal, we conclude that any L_R, $R > 1$, is also quasiconformal. Therefore, we can construct a $c\,(K)$-quasiconformal reflection $y_R\,(z)\,, y_R\,(0) = \infty$, across L_R such that $y_R\,(G_R) = \Omega_R$, $y_R\,(\Omega_R) = G_R$ and $y_R(\cdot)$ fixes the points of L_R that satisfies conditions (2.1) described for $y_R\,(z)$. By using this constructed $y_R\,(z)$, we can write the following integral representations for $P_n(z)$ [7, p. 105]:

$$P_n\,(z) = -\frac{1}{\pi} \iint\limits_{G_R} \frac{P_n\,(\zeta)\, y_{R,\bar{\zeta}}}{(y_R\,(\zeta) - z)^2} d\sigma_\zeta\,, \quad z \in G_R. \tag{3.2}$$

For $\varepsilon > 0$, let us set $U_\varepsilon(z) := \{\zeta \colon |\zeta - z| < \varepsilon\}$ and without loss of generality we may take $U_\varepsilon := U_\varepsilon(0) \subset G^*$. For arbitrary fixed point $z \in L$ we have

$$|P_n\,(z)\,| \leq \frac{1}{\pi} \iint\limits_{U_\varepsilon} \frac{|P_n\,(\zeta)\,||y_{R,\bar{\zeta}}|}{|y_R\,(\zeta) - z|^2} d\sigma_\zeta + \frac{1}{\pi} \iint\limits_{G_R \setminus U_\varepsilon} \frac{|P_n\,(\zeta)\,||y_{R,\bar{\zeta}}|}{|y_R\,(\zeta) - z|^2} d\sigma_\zeta =: J_1 + J_2. \tag{3.3}$$

To estimate the integral J_1, applying the Hölder inequality we get

$$J_1^2 \leq \iint\limits_{U_\varepsilon} |P_n\,(\zeta)\,|^2 d\sigma_\zeta \iint\limits_{U_\varepsilon} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R\,(\zeta) - z|^4} d\sigma_\zeta \prec \|P_n\|^2_{A_2(G)} \iint\limits_{U_\varepsilon} \frac{|y_{R,\bar{\zeta}}|^2}{|y_R\,(\zeta) - z|^4} d\sigma_\zeta.$$

According to (2.1), $|y_{R,\overline{\zeta}}| \asymp |y_R(\zeta)|^2$, for all $\zeta \in U_\varepsilon$, because of $|\zeta - z| \geq \varepsilon$, $|y_R(\zeta) - z| \asymp |y_R(\zeta)|$ for $z \in L$ and $\zeta \in U_\varepsilon$. On the other hand, if $J_{y,R} := |y_{R,\zeta}|^2 - |y_{R,\overline{\zeta}}|^2$ is Jacobian of the reflection $y_R(\zeta)$, we can obtain

$$|J_{y,R}| \succ |y_{R,\overline{\zeta}}|^2$$

as in [12]. Then, we can find

$$J_1^2 \prec \|P_n\|^2_{A_2(G)} \iint\limits_{y_R(U_\varepsilon)} \frac{|y_{R,\bar{\zeta}}|^2}{|J_{y,R}||\zeta - z|^4} d\sigma_\zeta \prec$$

$$\prec \|P_n\|^2_{A_2(G)} \iint\limits_{|\zeta - z| \geq c_1} \frac{d\sigma_\zeta}{|\zeta - z|^4} \prec \|P_n\|^2_{A_2(G)}. \tag{3.4}$$

For the J_2, we get

$$J_2^2 = \iint\limits_{G_R \setminus U_\varepsilon} \frac{|y_{R,\bar{\zeta}}|^2 d\sigma_\zeta}{|y_R\,(\zeta) - z|^4} \iint\limits_{G_R \setminus U_\varepsilon} |P_n\,(\zeta)\,|^2 d\sigma_\zeta =: J_{21} J_{22}. \tag{3.5}$$

For the integral J_{21}, we get

$$J_{21} \prec \iint\limits_{y(G_R \setminus U_\varepsilon)} \frac{d\sigma_\zeta}{|\zeta - z|^4} \leq \iint\limits_{|\zeta - z| \geq d(z, L_R)} \frac{d\sigma_\zeta}{|\zeta - z|^4} \prec d^{-2}(z, L_R) \tag{3.6}$$

and for J_{22} :

$$J_{22} = \iint\limits_{G_R \setminus U_\varepsilon} |P_n(\zeta)|^2 d\sigma_\zeta \leq \iint\limits_{G_R} |P_n(\zeta)|^2 d\sigma_\zeta = \|P_n\|^2_{A_2(G_R)}.$$

Then,

$$J_2^2 = J_{21} J_{22} \prec d^{-2}(z, L_R) \|P_n\|^2_{A_2(G_R)}. \tag{3.7}$$

Combining (3.3), (3.4), (3.5) and (3.7), we prove the estimation (3.1). To complete the proof of Theorem 1.1, according the maximum modulus principle, for any $z \in \Omega$ we have

$$\left| \frac{P_n(z)}{\Phi^{n+1}(z)} \right| \leq \max_{z \in \overline{\Omega}} \left| \frac{P_n(z)}{\Phi^{n+1}(z)} \right| = \max_{z \in L} |P_n(z)| \prec \frac{1}{d(z, L_R)} \|P_n\|_{A_2(G_R)},$$

or

$$|P_n(z)| \prec \frac{1}{d(L, L_R)} \|P_n\|_{A_2(G_R)} |\Phi(z)|^{n+1}.$$

Taking $R = 1 + 1/n$, according to Lemma 2.2, we obtain the proof of Theorem 1.2.

***3.2. Proof of Theorem* 1.3.** For the arbitrary fixed $R > 1$, let us set $L^* := y(L_R)$. According to (2.3), the number ε_1 (consuquently $\rho_1 := 1 + \varepsilon_1(R - 1)$) can be chosen such that $\overline{G}^*_{\rho_1} \subseteq G$. Let $R_1 := 1 + \dfrac{\rho_1 - 1}{2}$.

For $z \in \Omega$ and $w = \Phi_R(z)$ let us get

$$h_R(w) := \frac{P_n(\Psi_R(w))}{w^{n+1}}.$$

Cauchy integral representation for unbounded region gives

$$h_R(w) = -\frac{1}{2\pi i} \int\limits_{|t| = R_1} h_R(t) \frac{dt}{t - w}.$$

For all $|t| = R_1 > 1$, $|t|^{n+1} = R_1^{n+1} > 1$, then

$$A_n := |P_n(\Psi_R(w))| \leq |w|^{n+1} \frac{1}{2\pi} \int\limits_{|t| = R_1} |P_n(\Psi_R(t))| \frac{|dt|}{|t - w|}. \tag{3.8}$$

Applying the Hölder inequality, we get

$$A_n \prec |w|^{n+1} \left(\int\limits_{|t| = R_1} |P_n(\Psi_R(t)) \Psi'_R(t)|^2 |dt| \right)^{1/2} \times$$

$$\times \left(\int\limits_{|t| = R_1} \frac{1}{|\Psi'_R(t)|^2 |t - w|^2} |dt| \right)^{1/2} =: |w|^{n+1} (A_n^1 B_n^1)^{1/2}. \tag{3.9}$$

Let us set

$$f_n(t) := P_n(\Psi_R(t))\,\Psi_R'(t).$$

Now, we separate the circle $|t| = R_1$ to n equal partδ_n with $\operatorname{mes}\delta_n = \dfrac{2\pi R_1}{n}$ and by applying the mean value theorem to the integral A_n^1 we get

$$A_n^1 = \sum_{k=1}^{n} \int_{\delta_k} |f_n(t)|^2 |dt| = \sum_{k=1}^{n} \left|f_n(t_k')\right|^2 \operatorname{mes}\delta_k, \quad t_k' \in \delta_k.$$

On the other hand, by applying mean value estimation,

$$\left|f_n\left(t_k'\right)\right|^2 \le \frac{1}{\pi\left(|t_k'|-1\right)^2} \iint\limits_{|\xi - t_k'|<|t_k'|-1} |f_n(\xi)|^2 d\sigma_\xi,$$

we obtain

$$A_n^1 \prec \sum_{k=1}^{n} \frac{\operatorname{mes}\delta_k}{\pi\left(|t_k'|-1\right)^2} \iint\limits_{|\xi - t_k'|<|t_k'|-1} |f_n(\xi)|^2 d\sigma_\xi, \qquad t_k' \in \delta_k.$$

Taking into account that discs with origin at the points t_k' at most two may be crossing, we have

$$A_n^1 \prec \frac{\operatorname{mes}\delta_1}{\left(|t_1'|-1\right)^2} \iint\limits_{1<|\xi|<\rho_1} |f_n(\xi)|^2 d\sigma_\xi \prec n \iint\limits_{1<|\xi|<\rho_1} |f_n(\xi)|^2 d\sigma_\xi.$$

According to (2.2), for A_n^1, we get

$$A_n^1 \prec n \iint\limits_{G_{\rho_1}^* \setminus G^*} |P_n(z)|^2 d\sigma_z \prec n\|P_n\|_{A_2(G)}^2. \tag{3.10}$$

To estimate the integral B_n^1, taking into account the estimation for the Ψ_R' (see, for instance, [7], Theorem 2.8) and Lemma 2.1 written for $\Phi_R(z)$, from (2.4) and (2.5) we get

$$B_n^1 \prec \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{d^2(\Psi_R(t), L^*)} \frac{|dt|}{|t-w|^2} =$$

$$= \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{d^2(\Psi_R(t), L^*)} \frac{|dt|}{|t-w|^{2-\frac{2}{\mu}} |t-w|^{\frac{2}{\mu}}} \prec$$

$$\prec \frac{1}{d^2(z, L_{R_1}^*)} \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{(|t|-1)^{2\mu}} \frac{|dt|}{|t-w|^{2-\frac{2}{\mu}}} =$$

$$= \frac{1}{d^2(z, L_{R_1}^*)} \int\limits_{|t|=R_1} \frac{1}{(|t|-1)^{2(\mu-1)}} \frac{|dt|}{|t-w|^{2(1-\frac{1}{\mu})}} \prec$$

$$\prec \frac{1}{d^2(z, L^*_{R_1})} n^{2(\mu-\mu^{-1})-1} \prec \frac{1}{d^2(z, L)} n^{2(\mu-\mu^{-1})-1}, \tag{3.11}$$

where $\mu := \min\left\{2, K^4\right\}$. Relations (3.8), (3.9), (3.10), and (3.11) yield

$$|P_n(z)| \prec |w|^{n+1}\sqrt{n}\|P_n\|_{A_2(G)} \frac{1}{d(z, L)} n^{(\mu-\mu^{-1})-1/2} =$$

$$= \frac{n^{(\mu-\mu^{-1})}}{d(z, L)} |\Phi_R(z)|^{n+1} \|P_n\|_{A_2(G)}, \quad z \in \Omega.$$

Theorem 1.3 is proved.

***3.3. Proof of Theorem* 1.4.** Proof of the Theorem 1.4 will be similar to proof of Theorem 1.3. The term in (3.11) will be treated as the following:

$$B^1_n \prec \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{d^2(\Psi_R(t), L^*)} \frac{|dt|}{|t-w|^2} \prec$$

$$\prec \int\limits_{|t|=R_1} \frac{1}{(|t|-1)^{2\mu-2}} \frac{|dt|}{|t-w|^2} \prec \int\limits_{|t|=R_1} \frac{|dt|}{(|t|-1)^{2\mu}} \prec n^{2\mu-1}. \tag{3.12}$$

And, consequently,

$$|P_n(z)| \prec |w|^{n+1}\sqrt{n}\|P_n\|_{A_2(G)} n^{\mu-1/2} = n^{\mu}\|P_n\|_{A_2(G)} |\Phi_R(z)|^{n+1}, \quad z \in \Omega.$$

***3.4. Proof of Theorem* 1.5.** Let $R > 1$ be arbitrary fixed and let $R_1 := 1 + \dfrac{R-1}{2}$. For $z \in \overline{\Omega}_R$ and $w = \Phi(z)$ let us get

$$h(w) := \frac{P_n(\Psi(w))}{w^{n+1}}.$$

Cauchy integral representation for unbounded region gives

$$h(w) = -\frac{1}{2\pi i} \int\limits_{|t|=R_1} h(t) \frac{dt}{t-w}.$$

Following the method used in proof of Theorem 1.3, similar terms are treated as below:

$$\widetilde{A_n} := |P_n(\Psi(w))| \le |w|^{n+1} \frac{1}{2\pi} \int\limits_{|t|=R_1} |P_n(\Psi(t))| \frac{|dt|}{|t-w|} \prec$$

$$\prec |w|^{n+1} \left(\int\limits_{|t|=R_1} |P_n(\Psi(t))\Psi'(t)|^2 |dt| \right)^{1/2} \times$$

$$\times \left(\int\limits_{|t|=R_1} \frac{1}{|\Psi'(t)|^2 |t-w|^2} |dt| \right)^{1/2} =: |w|^{n+1} (\widetilde{A}^1_n \widetilde{B}^1_n)^{1/2}. \tag{3.13}$$

Let us set

$$\widetilde{f_n}(t) := P_n(\Psi(t))\,\Psi'(t).$$

Now, we separate the circle $|t| = R_1$ to n equal part δ_n with $\operatorname{mes}\delta_n = \dfrac{2\pi R_1}{n}$ and by applying the mean value theorem to the integral $\widetilde{A}_n^1$ we get

$$\widetilde{A}_n^1 = \sum_{k=1}^{n} \int\limits_{\delta_k} |\widetilde{f_n}(t)|^2 |dt| = \sum_{k=1}^{n} |\widetilde{f_n}\left(t_k'\right)|^2 \operatorname{mes}\delta_k, \quad t_k' \in \delta_k.$$

On the other hand, applying mean value estimation, we obtain

$$\widetilde{A}_n^1 \prec \sum_{k=1}^{n} \frac{\operatorname{mes}\delta_k}{\pi\left(|t_k'| - 1\right)^2} \iint\limits_{|\xi - t_k'| < |t_k'| - 1} |\widetilde{f_n}(\xi)|^2 d\sigma_\xi, \quad t_k' \in \delta_k,$$

$$\widetilde{A}_n^1 \prec \frac{\operatorname{mes}\delta_1}{\left(|t_1'| - 1\right)^2} \iint\limits_{1<|\xi|<R} |\widetilde{f_n}(\xi)|^2 d\sigma_\xi \prec n \iint\limits_{1<|\xi|<R} |\widetilde{f_n}(\xi)|^2 d\sigma_\xi.$$

According to (2.2), for $\widetilde{A}_n^1$, we get

$$\widetilde{A}_n^1 \prec n \iint\limits_{G_R \setminus G} |P_n(z)|^2 d\sigma_z \prec n \|P_n\|_{A_2(G_R)}^2. \tag{3.14}$$

To estimate the integral $\widetilde{B}_n^1$, taking into account that the estimation for the Ψ' (see, for instance, [7], Theorem 2.8) and Lemma 2.1, we get

$$\widetilde{B}_n^1 \prec \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{d^2(\Psi(t), L)} \frac{|dt|}{|t-w|^2} =$$

$$= \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{d^2(\Psi(t), L)} \frac{|dt|}{|t-w|^{2-2/\nu}|t-w|^{2/\nu}} \prec$$

$$\prec \frac{1}{d^2(z, L_{R_1})} \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{(|t|-1)^{2\nu}} \frac{|dt|}{|t-w|^{2-2/\nu}} =$$

$$= \frac{1}{d^2(z, L_{R_1})} \int\limits_{|t|=R_1} \frac{1}{(|t|-1)^{2(\nu-1)}} \frac{|dt|}{|t-w|^{2(1-1/\nu)}} \prec$$

$$\prec \frac{1}{d^2(z, L_{R_1})} n^{2(\nu-\nu^{-1})-1}, \tag{3.15}$$

where $\nu := \min\left\{2, K^2\right\}$.

Let us denote $\zeta = \Psi(\tau) \in L$, $\zeta_1 = \Psi(\tau_1) \in L_{R_1}$ such that $d(z, L) = |z - \zeta|$, $d(z, L_{R_1}) = |z - \zeta_1|$, and denote this image from $\tau = \Phi(\zeta)$, $\tau_1 = \Phi(\zeta_1)$. Also we denote points $|\tau^*| = 1$, $|w - \tau^*| = |w| - 1$, $|\tau_1^*| = R_1$, $|w - \tau_1^*| = |w| - R_1$. According to $R_1 := 1 + \dfrac{R-1}{2}$ we have $|w - \tau_1| \asymp |w - \tau_1^*| \succ |w - \tau^*| \asymp |w - \tau|$. Then, by Lemma 2.1 we get $d(z, L_{R_1}) \succ d(z, L)$. Therefore, we obtain

$$\widetilde{B}_n^1 \prec \frac{1}{d^2(z,L)} n^{2(\nu-\nu^{-1})-1}. \tag{3.16}$$

Relations (3.13), (3.14), (3.16) and Lemma 2.2 yield

$$|P_n(z)| \prec |w|^{n+1}\sqrt{n}\|P_n\|_{A_2(G_R)} \frac{1}{d(z,L)} n^{(\nu-\nu^{-1})-1/2} =$$

$$= \frac{n^{(\nu-\nu^{-1})}}{d(z,L)} |\Phi(z)|^{n+1}\|P_n\|_{A_2(G)}, \quad z \in \overline{\Omega}_{1+1/n}.$$

Theorem 1.5 is proved.

***3.5. Proof of Theorem* 1.6.** Analogous to proof of the Theorem 1.4, the proof of the Theorem 1.6 is identical to proof of the proof Theorem 1.5. In this case, the following the method used in the proof of Theorem 1.5 and (3.15) will be treated as the following:

$$B_n^1 \prec \int\limits_{|t|=R_1} \frac{(|t|-1)^2}{d^2(\Psi(t),L)} \frac{|dt|}{|t-w|^2} \prec$$

$$\prec \int\limits_{|t|=R_1} \frac{1}{(|t|-1)^{2\nu-2}} \frac{|dt|}{|t-w|^2} = \int\limits_{|t|=R_1} \frac{|dt|}{(|t|-1)^{2\nu}} \prec n^{2\nu-1}.$$

And, consequently,

$$|P_n(z)| \prec |w|^{n+1}\sqrt{n}\|P_n\|_{A_2(G_R)} n^{\nu-1/2} \prec n^{\nu}\|P_n\|_{A_2(G)}|\Phi(z)|^{n+1}, \quad z \in \overline{\Omega}_{1+1/n}.$$

Theorem 1.6 is proved.

We note that the Theorems 1.2 – 1.6 are sharp. This can be clearly seen by the example $P_n(z) = \sum_{j=0}^{n}(j+1)z^j$, $G = B$. In this case,

$$\|P_n\|_{C(\overline{G})} = \frac{(n+1)(n+2)}{2}, \qquad \|P_n\|_{A_2(G)} = \sqrt{\frac{\pi(n+1)(n+2)}{2}}.$$

Then, for all $z \in L_{1+1/n}$ such that $|P_n(z)| = \|P_n\|_{C(\overline{G_R})}$, we have

$$|P_n(z)| \geq \|P_n\|_{C(\overline{G})} \geq \frac{1}{\sqrt{2\pi}} n\|P_n\|_{A_2(G)} =$$

$$= \frac{1}{\sqrt{2\pi}} \frac{d(L, L_{1+1/n})}{d(L, L_{1+1/n})} n\|P_n\|_{A_2(G)} \frac{|\Phi(z)|^{n+1}}{|\Phi(z)|^{n+1}} \geq$$

$$\geq c_{10} \frac{1}{d(L, L_{1+1/n})} \|P_n\|_{A_2(G)}|\Phi(z)|^{n+1}.$$

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