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RINGS WITH FINITE DECOMPOSITION OF IDENTITY
КІЛЬЦЯ ІЗ СКІНЧЕННИМ РОЗКЛАДОМ ОДИНИЦІ

A criterion for semiprime rings with finite decomposition of identity to be prime is given. We also give a short survey on some finiteness conditions related to the decomposition of identity. We consider the notion of a net of a ring and show that the lattice of all two-sided ideals of a right semidistributive semiperfect ring is distributive. An application of decompositions of identity to groups of units is given.

Наведено критерій первинності напівпервинних кілець із скінченним розкладом одиниці, а також короткий огляд деяких умов скінченності відносно розкладу одиниці. Розглянуто поняття сітки кільця і показано, що решітка всіх двобічних ідеалів правого напівдистрибутивного напівдосконалого кільця є дистрибутивною. Наведено застосування розкладу одиниці до груп одиниць.

1. Introduction. Recall that a ring is called **semiprime** if it does not contain nilpotent nonzero ideals. A ring A is **prime** if a product of any two non-zero two-sided ideals of A is not equal to zero.

A ring is said to be FDI-ring if it has a decomposition of identity into a finite sum of pairwise orthogonal primitive idempotents. This class of rings includes right Artinian, right Noetherian, semiperfect, and Goldie rings.

The article is organized as follows. In Section 2 we give a survey of some results and examples on various finiteness conditions related to FDI-rings. In Section 3 a criterion for semiprime FDI-finite rings to decompose into a finite direct product of prime rings is given. From this main criterion we obtain corollaries which show that some classes of semiprime rings decompose into a finite direct products of prime rings. Examples of such classes of rings are right Bézout rings, right semidistributive rings, serial rings, primely triangular rings, piecewise domains, right hereditary (semihereditary) FDI-rings and right Noetherian right hereditary (semihereditary) rings. Section 4 is concerned with generalized matrix rings. First the formula for the Jacobson radical for such rings is given. Next, using the criterion for a decomposition of a semiprime ring obtained in the previous section, we give some corollaries for a decomposition of some classes of primely triangular matrix rings into a finite product of prime rings.

In Section 5 an application of decompositions of identity to Sylow subgroups in the unit group is given.

In Section 6 we consider the notion of a net of a ring and use it to observe that the lattice of all two-sided ideals of a semiperfect right semidistributive ring is distributive.

All rings in this paper are assumed to be associative (but not necessary commutative) with $1 \neq 0$ and all modules are assumed to be unitary. The Jacobson radical of A will be denoted by $\operatorname{rad} A$.

2. FDI-rings and finiteness conditions. In this section we give a survey of diverse finiteness conditions related to that of FDI mentioned in Introduction.

An important role in the theory of rings and modules is played by various finiteness conditions. In particular, minimal and maximal conditions, which can be formulated as chain conditions on submodules and one-sided ideals. The minimal condition on right (resp. left) ideals defines right Artinian (resp. left Artinian) rings. This condition is often written as d.c.c. (descending chain condition) on right ideals. Analogously, right (resp. left) Noetherian rings are defined as rings which satisfy the maximal condition, or the a.c.c. (ascending chain condition), on right (resp. left) ideals.

In the formulations of the finiteness conditions in ring theory some special classes of ideals are often chosen. One of the important classes of ideals is formed by the principal right (left) ideals. The famous Theorem P by H. Bass [3] (see also [24], Theorem 10.5.5) states that the d.c.c. on principal left ideals defines the right perfect rings.

Another important special class of ideals is formed by right (left) annihilators. One can show that the set of all right (resp. left) annihilators of a ring A forms a lattice. There is a lattice anti-isomorphism between the lattice of right annihilators of A and that of left annihilators of A. This easily implies the next fact.

Proposition 2.1. *The following conditions are equivalent for a ring* A:

(1) *A satisfies d.c.c. on right* (*resp. left*) *annihilators.*

(2) *A satisfies a.c.c. on left* (*resp. right*) *annihilators.*

A. W. Goldie considered another finiteness condition. A module M is said to satisfy the maximal condition for direct sums if M does not contain an infinite direct sum of submodules [21]. A. W. Goldie proved that a module M with maximum condition for direct sums has a finite uniform dimension u.dim$M < \infty$ (also called finite Goldie dimension, or, simply, finite dimension)[1]. He also proved in [21] that a module M is finite-dimensional if and only if M satisfies a.c.c. or d.c.c. on complements in M (recall that a submodule C is called a **complement** in M if there exists a submodule $S \subseteq M$ such that C is maximal with respect to the property that $C \cap S = 0$).

Summarizing the results obtained by Goldie in [21] we can write the following theorem which shows the equivalence of different kind of finiteness conditions for a module M.

Theorem 2.1. *The following conditions are equivalent for a module* M:

1. *M is finite-dimensional.*
2. u.dim $M < \infty$.
3. *M satisfies a.c.c. on complements.*
4. *M satisfies d.c.c. on complements.*
5. *M contains no infinite direct sum of submodules.*

One of the most frequently used finiteness condition for a ring A is connected with the cardinal number of a set of pairwise orthogonal nonzero idempotents. Following to A. A. Tuganbaev [42] we give the following definition.

Definition 2.1. *A ring A is called* **orthogonally finite** *if it contains no infinite set of pairwise orthogonal nonzero idempotents.*

Using this definition we can write the following theorem:

Theorem 2.2 ([30], Proposition 6.59)**.** *For any ring A the following statements are equivalent*:

1. *A is orthogonally finite.*
2. *A satisfies the ascending chain condition on right direct summands.*
3. *A satisfies the descending chain condition on left direct summands.*

From the above theorem and Definition 2.1 we obtain the following examples of orthogonally finite rings:

1. Semisimple rings.

[1] The term “finite-dimensional” for such M was used originally by A. W. Goldie.

2. Noetherian rings.
3. Artinian rings.
4. Local rings (they have only one nonzero idempotent).
5. Rings with finite Goldie dimension.
6. Semiperfect rings. This follows by Theorem 10.3.7 from [24] which states that a semiperfect ring is decomposed into a direct sum of right ideals each of which has exactly one maximal submodule.
7. Perfect rings. This follows from Theorem P of H. Bass [3].

We proceed with the next:

Definition 2.2. *A ring A is called* **Dedekind-finite** (*or* **von Neumann-finite***, or* **directly finite**) *if $ba = 1$ whenever $ab = 1$ for $a, b \in A$. Otherwise A is said to be* **non-Dedekind-finite** (*or* **Dedekind-infinite**). *In this case there are elements a and b such that $ab = 1$, but $ba \neq 1$.*

The following easy fact gives simple examples of Dedekind-finite rings.

Proposition 2.2. *A ring A is Dedekind-finite if it satisfies any of the following conditions*:

1. *A is a unital subring*[2] *of a Dedekind-finite ring.*
2. *A is a direct product of Dedekind-finite rings.*
3. *A is an epimorphic image of a Dedekind-finite ring.*
4. *A has no right or left zero divisors.*

From this proposition and Definition 2.2 we immediately obtain the following examples of Dedekind-finite rings:

1. A commutative ring.
2. A domain.
3. A division ring.
4. A ring which is a finite dimensional vector space over a division ring.
5. The ring of endomorphisms of a finitely dimensional vector space over a division ring.
6. The ring of matrices $M_n(D)$ over a division ring D.
7. Any finite product of Dedekind-finite rings.
8. A semisimple ring.

Definition 2.3. *A ring A is called* **unit-regular** *if for any $a \in A$ there exists a unit $u \in \mathcal{U}(A)$ such that $a = aua$.*

We shall use the following easy fact:

Lemma 2.1. *Any unit-regular ring is Dedekind-finite.*

Proof. Suppose $ab = 1$. Since there exists $u \in \mathcal{U}(A)$ such that $a = aua$, we obtain $1 = ab = auab = au$ which means that $a = u^{-1} \in \mathcal{U}(A)$, and so $b = u \in \mathcal{U}(A)$. Therefore $ba = 1$, i.e., A is Dedekind-finite.

The following example shows the existence of non-Dedekind finite rings.

***Example* 2.1** (see [29, p. 4]). Let V be the k-vector space $ke_1 \oplus ke_2 \oplus \ldots$ with a countably infinite basis $\{e_i \colon i \geq 1\}$ over a field k, and let $A = \mathrm{End}_k(V)$ be the k-algebra of all vector space endomorphisms of V. Let $a, b \in A$ be defined by

$$b(e_i) = e_{i+1} \quad \text{for all} \quad i \geq 1$$

and

[2]Recall that a subring A_1 of A is said to be a unital subring if the identity of A is also the identity of A_1.

$$a(e_1) = 0, a(e_i) = e_{i-1} \quad \text{for all} \quad i \geq 2,$$

then $ab = 1 \neq ba$, so a is right-invertible without being left-invertible. Therefore A gives an example of a non-Dedekind-finite ring.

With respect to the above example notice the next fact due to N. Jacobson [25].

Proposition 2.3 (N. Jacobson [25])**.** *Any non-Dedekind-finite ring A contains a countable infinite set of pairwise orthogonal nonzero idempotents in A.*

The proof can be also found in [20] (Proposition 5.5), [29] (Proposition 21.26).

Proof. Let A be a ring which is non-Dedekind-finite, i.e., there exist elements $a, b \in A$ such that $ab = 1$ but $ba \neq 1$. Denote $e = ba$. Then $e^2 = b(ab)a = ba = e$, so e is a non-trivial idempotent. For $i, j \geq 0$ let

$$e_{ij} = b^i(1-e)a^j.$$

Then $\{e_{ij}\}$ is a set of matrix units in the sense that $e_{ij}e_{kl} = \delta_{jk}e_{il}$. To see this note that $a^ib^i = 1$ for all i, and $a(1-e) = 0 = (1-e)b$. If $j \neq k$, then a^jb^k is either $a^{|j-k|}$ or $b^{|j-k|}$, so

$$e_{ij}e_{kl} = b^i(1-e)a^jb^k(1-e)a^l = 0.$$

On the other hand, since $1-e$ is an idempotent,

$$e_{ij}e_{jl} = b^i(1-e)a^jb^j(1-e)a^l = b^i(1-e)a^l = e_{il}.$$

Note that each $e_{ij} \neq 0$. Indeed, if $b^i(1-e)a^j = 0$, then $0 = a^ib^i(1-e)a^jb^j = (1-e)$, a contradiction.

In particular, $\{e_{ii} \colon i \geq 1\}$ is a countable infinite set of pairwise orthogonal nonzero idempotents in A, and so A contains an infinite direct sum of nonzero right ideals $\bigoplus\limits_{i \geq 0} e_{ii}A$.

***Remark* 2.1.** From the proof of the above proposition it follows, in particular, that there is a countable infinite set of pairwise orthogonal nonzero idempotents $\{e_{ii} \colon i \geq 1\}$ in A such that A contains an infinite direct sum of nonzero right ideals $\bigoplus\limits_{i \geq 0} e_{ii}A$.

***Corollary* 2.1.** *Let A be a ring with a.c.c. or d.c.c. on principal left* (*right*) *ideals generated by idempotents. Then A is Dedekind-finite.*

***Corollary* 2.2** (T. Y. Lam [29], Corollary 21.27)**.** *Let A be a ring such that $B = A/\operatorname{rad} A$ is an orthogonally finite ring* (*i.e., does not contain an infinite direct sum of nonzero right ideals*)*. Then A is Dedekind-finite.*

From this corollary and Proposition 2.2 we immediately obtain the following fact.

***Corollary* 2.3.** *A ring A is Dedekind-finite if and only if $B = A/\operatorname{rad} A$ is Dedekind-finite.*

The next statement can be easily obtained from the above corollaries and Proposition 2.3 and gives further examples of Dedekind-finite rings.

Proposition 2.4. *A ring A is Dedekind-finite if it satisfies any of the following conditions*:

1. *A has finite Goldie dimension.*
2. *A is right Noetherian.*
3. *A satisfies a.c.c. on right* (*left*) *direct summands.*
4. *A satisfies a.c.c. on right* (*left*) *annihilators.*
5. *A is a local ring.*

6. *A is a semilocal ring.*
7. *A is a semiperfect ring.*
8. *A is a perfect ring.*
9. *A is an orthogonally finite ring.*

Proposition 2.5. (i) *Let N be a nilpotent ideal of a ring A. Then A is Dedekind-finite if and only if $B = A/N$ is Dedekind-finite.*

(ii) *Let A be a Dedekind-finite ring. Then for any idempotent $e \in A$ the ring eAe is also Dedekind-finite.*

(iii) *Let A, B be rings and X be an A-B-bimodule. Then the ring*

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

is Dedekind-finite if and only if A and B are both Dedekind-finite.

Proof. (i) The "only if" part follows from Proposition 2.2(3).

Suppose that A/N is Dedekind-finite and $\varphi : A \to A/N$ is the natural epimorphism. Suppose $ab = 1$ for $a, b \in A$. Since A/N is Dedekind-finite, $\varphi(ba) = \varphi(b)\varphi(a) = \varphi(a)\varphi(b) = \varphi(ab) = \bar{1}$. Therefore $ba = 1 + x$, where $x \in I$. So it is invertible in A, i.e., there exists $u \in U(A)$ such that $bau = 1$, whence $a = abau = au$. Thus $ba = 1$, as required.

(ii) Suppose that $ab = e$ for $a, b \in eAe$. Put $f = 1 - e$. Since e, f are orthogonal idempotents, $(a+f)(b+f) = ab+f = e+f = 1$. Taking into account that $a+f$, $b+f \in A$, we obtain that $(b + f)(a + f) = 1$, whence $ba = 1 - f = e$, as desired.

(iii) The proof follows immediately from (i) and (ii).

Consider the next important class of rings with a finiteness condition.

Definition 2.4 ([24], Chapter 2)**.** *A ring A is called an* **FDI-ring** *if there exists a decomposition of the identity $1 \in A$ into a finite sum $1 = e_1 + e_2 + \ldots + e_n$ of pairwise orthogonal primitive idempotents e_i. In this case the regular A-module A_A can be decomposed into a finite direct sum of indecomposable modules e_iA.*

Note that the decomposition of $1 \in A$, given in the definition of an FDI-ring, may not be unique.

The following are important examples of FDI-rings:

1. Division rings.
2. Finite direct sums of FDI-rings.
3. Rings which are finite dimensional vector spaces over a division ring.
4. The rings of matrices $M_n(D)$ over division rings D.
5. Semisimple rings.
6. Artinian rings.
7. Noetherian rings.
8. Semiperfect ring.
9. Rings with finite Goldie dimension.
10. Orthogonally finite rings.
11. Perfect rings.

The following statement is obvious.

Proposition 2.6. *Let A, B be rings and X be A-B-bimodule. If A and B are both FDI-rings then the ring*

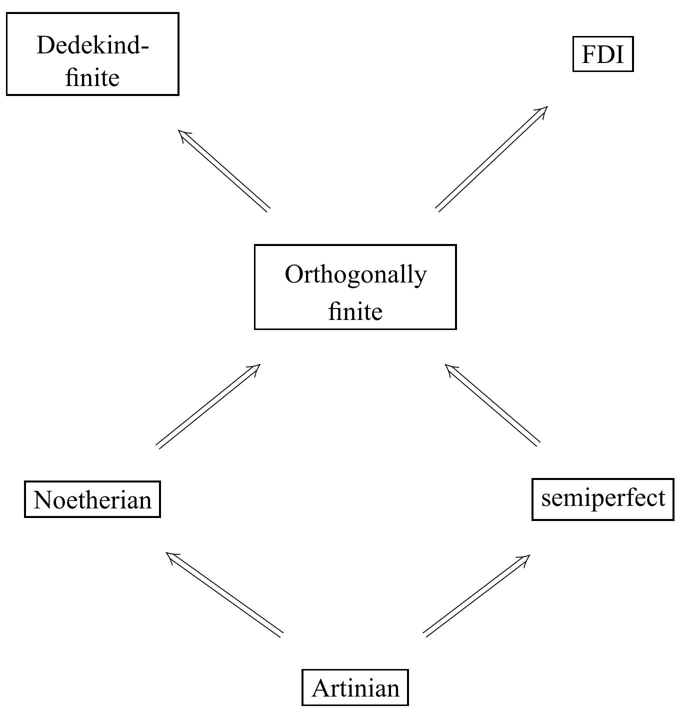
$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

is also FDI.

Proposition 2.7 ([24], Chapter 2)**.** (i) *Let A be an FDI-ring. Then the identity of A can be written as a sum of a finite number of orthogonal centrally primitive idempotents*[3].

(ii) *Any FDI-ring can be uniquely decomposed into a direct product of a finite number of indecomposable rings.*

From the above we obtain the following diagram of a relationships between the main classes of rings with finiteness conditions. Arrows in the diagram below mean containments of classes of rings.



Next we show that there exist FDI-rings which are not Dedekind-finite.

Theorem 2.3 (J. C. Shepherdson [39])**.** *There exists a ring R with 1 and no divisors of zero over which there exist 2×2-matrices A, B, X such that $AB = I$, $AX = 0$, $X \neq 0$, where I is the unit 2×2-matrix.*

It follows from this theorem that there exist a ring R with 1 and matrices $A, B \in M_2(R)$ such that $AB = I$ and $BA \neq I$. Indeed, otherwise if $BA = I$, then $BAX = 0$ and $IX = X = 0$. Therefore the ring $M_2(R)$, where R as in Theorem 2.3,

[3] Recall that a central idempotent is called centrally primitive if it can not be decomposed into a sum of orthogonal central non-zero idempotents.

is not Dedekind-finite. At the same time it is an FDI-ring, since $1 = e_{11} + e_{22}$, and, moreover, $e_{11}M_2(R)e_{11} \simeq e_{22}M_2(R)e_{22} \simeq R$, where R is a domain.

Thus there are FDI-rings, which are non-Dedekind-finite. On the other hand the following theorem holds.

Theorem 2.4 (T. Y. Lam [30], Proposition 6.60)**.** *Assume that a ring A does not contain an infinite direct sum of nonzero right ideals. Then*

(1) *A is an FDI-ring.*

(2) *A is a Dedekind-finite ring.*

***Remark* 2.2.** The converse statement of this theorem is not true. The example constructed by K. R. Goodearl in [20] (Example 5.15) shows that there exist unit-regular rings (and therefore, by Lemma 2.1, Dedekind-finite rings) which contain uncountable direct sums of nonzero pairwise isomorphic right ideals. Moreover, an example constructed by L. A. Skornyakov in [36] shows that there exist FDI-rings which contain an infinite set of pairwise orthogonal idempotents, and thus contain an infinite direct set of principal right (left) ideals generated by idempotents.

For the proof of a generalization of the Krull – Remak – Schmidt – Azumaya theorem P. Crawley and B. Jónsson [14] introduced the following definition.

Definition 2.5. *Given a cardinal $\mathfrak{N}$, an A-module M is said to have the $\mathfrak{N}$-* **exchange property** *if for any A-module X and any two decompositions of X* :

$$X = M_1 \oplus N = \bigoplus_{i \in I} X_i$$

with $M_1 \simeq M$ and $|I| \leq \mathfrak{N}$, there are submodules $Y_i \subseteq X_i$ such that

$$X = M_1 \oplus \left(\bigoplus_{i \in I} Y_i \right).$$

A module M has the **exchange property** *if M has the $\mathfrak{N}$-exchange property for any cardinal $\mathfrak{N}$.*

In [46] R. B. Warfield, Jr. introduced an **exchange ring** A as a ring whose left regular module ${}_AA$ has the exchange property, and he also proved that this definition is left-right symmetric. This class of rings includes (von Neumann) regular rings, local rings and semiperfect rings. He also proved the following statement.

Proposition 2.8 (R. B. Warfield, Jr. [45])**.** *An indecomposable module has the exchange property if and only if its endomorphism ring is local.*

This proposition immediately implies:

Proposition 2.9 (V. Camillo, H.-P. Yu [11])**.** *Let A be an exchange ring. Then the following statements are equivalent*:

(1) *A is an orthogonally finite ring*;

(2) *A is an FDI-ring*;

(3) *A is a semiperfect ring.*

3. Semiprime FDI-rings. In this section we prove the main theorem which gives a criterion for a semiprime FDI-ring to be decomposable into a finite direct product of prime rings. This theorem can be considered as a generalization of the following theorem proved by V. V. Kirichenko and M. Khibina [28] (see also [24], Theorem 14.4.3):

Theorem 3.1. *A semiprime semiperfect ring is a finite direct product of indecomposable rings. An indecomposable semiprime semiperfect ring is either a simple Artinian ring or an indecomposable semiprime semiperfect ring such that all its principal endomorphism rings are non-Artinian.*

The following proposition is well known (see [24], Proposition 11.2.4).

Proposition 3.1. *Let A be a ring. The following conditions are equivalent*:

1. *A has no nonzero nilpotent ideal.*
2. *A has no nonzero nilpotent right ideal.*
3. *A has no nonzero nilpotent left ideal.*
4. *The prime radical of A is equal to zero.*

A ring which satisfies any of the equivalent conditions of the above proposition is called **semiprime**.

Lemma 3.1. *Let J be an ideal of an FDI-ring A with a decomposition of the identity $1 \in A$ into a finite sum $1 = e_1 + e_2 + \ldots + e_n$ of pairwise orthogonal primitive idempotents e_i. Then J is nilpotent if and only if $e_k J e_k$ is a nilpotent ideal in $e_k A e_k$ for any $k = 1, 2, \ldots, n$.*

The proof of this lemma is exactly the same as that of Theorem 11.4.1 in [24]. It also can be obtained by induction using the following lemma:

Lemma 3.2 (L. H. Rowen [33], Lemma 2.7.13)**.** *Let J be an ideal of a ring A. Then J is nilpotent if and only if eJe and $(1-e)J(1-e)$ are nilpotent for any idempotent $e \in A$.*

Lemma 3.3. *Let A be a semiprime ring. Suppose $1 = g_1 + g_2$ is a sum of two idempotents, $A = \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix}$, where $A_1 = g_1 A g_1$, $A_2 = g_2 A g_2$, $X = g_1 A g_2$ and $Y = g_2 A g_1$. Let $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ be an ideal in A and $M_{12} \neq 0$. Then $M_{12} M_{21} \neq 0$, $M_{21} \neq 0$, $M_{21} M_{12} \neq 0$.*

Proof. Suppose $M_{12} \neq 0$. We shall show that $M_{12} M_{21} \neq 0$.

Assume $M_{12} M_{21} = 0$. Denote by $I(M_{12})$ the two-sided ideal in A generated by M_{12}. Since

$$\begin{pmatrix} 0 & M_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix} = \begin{pmatrix} M_{12} Y & M_{12} \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix} \begin{pmatrix} M_{12} Y & M_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} M_{12} Y & M_{12} \\ Y M_{12} Y & Y M_{12} \end{pmatrix},$$

we obtain that $I(M_{12}) = \begin{pmatrix} M_{12} Y & M_{12} \\ Y M_{12} Y & Y M_{12} \end{pmatrix}$.

Obviously, $M_{12} Y M_{12} \subseteq M_{12}$ and $Y M_{12} Y \subseteq M_{21}$. Therefore

$$(M_{12} Y)^3 = (M_{12} Y M_{12})(Y M_{12} Y) \subseteq M_{12} M_{21} = 0.$$

Consequently, $M_{12} Y$ is nilpotent. In addition $Y M_{12} Y M_{12} Y M_{12} \subseteq Y M_{12} M_{21} Y = 0$. Therefore $Y M_{12}$ is also nilpotent and, by Lemma 3.2, $I(M_{12})$ is a nilpotent ideal.

The rest of the statement is verified analogously.

Lemma 3.3 is proved.

The following lemma is obvious.

Lemma 3.4. *Let A be as in Lemma 3.3. Then* $\begin{pmatrix} XY & X \\ Y & YX \end{pmatrix}$ *is a two-sided ideal in a ring A.*

Theorem 3.2. *Let A be a semiprime FDI-ring, and suppose that $1 \in A$ have the following decomposition into a sum of pairwise orthogonal primitive idempotents: $1 = e_1 + \ldots + e_n$. A ring A is a finite direct product of prime rings if and only if all rings $e_i A e_i$ are prime.*

Proof. Let A be a semiprime ring, and let $1 = e_1 + \ldots + e_n$ be a decomposition of the identity of A into a sum of pairwise orthogonal primitive idempotents. Obviously it suffices to prove only the converse statement.

Suppose that all $e_i A e_i$ are prime rings. We use induction on n.

1. If $n = 1$ the theorem is obvious.

2. Let $n = 2$. Suppose $1 = e_1 + e_2$ and $A = \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix}$. One can assume that A is an indecomposable ring. Suppose $X = 0$. If $Y \neq 0$ then $\begin{pmatrix} 0 & 0 \\ Y & 0 \end{pmatrix}$ is a nonzero nilpotent ideal. This is impossible, since A is semiprime. So $Y = 0$. But this is also impossible, since A is indecomposable. Analogously one can prove that $Y = 0$ implies $X = 0$. Thus if A is indecomposable then $X \neq 0$ and $Y \neq 0$.

Next we prove that A is prime.

Let $L = \begin{pmatrix} L_1 & L_{12} \\ L_{21} & L_2 \end{pmatrix}$ and $M = \begin{pmatrix} M_1 & M_{12} \\ M_{21} & M_2 \end{pmatrix}$ be two-sided ideals of A, and $LM = 0$. Then $L_1 M_1 = 0$ and $L_2 M_2 = 0$. Since A_1 and A_2 are prime, it follows that $L_1 = 0$ or $M_1 = 0$, and $L_2 = 0$ or $M_2 = 0$.

Assume that $L \neq 0$. If $L_1 = 0$ and $L_2 = 0$, then L is a nonzero nilpotent ideal by Lemma 3.2. Therefore $L_1 \neq 0$ or $L_2 \neq 0$. Without loss of generality one can assume that $L_1 = 0$ and $L_2 \neq 0$. In this case $M_2 = 0$ and $M = \begin{pmatrix} 0 & M_{12} \\ M_{21} & M_2 \end{pmatrix}$ is a two-sided ideal in A. Hence $M^2 \subseteq M$ and thus we obtain that $M_{12} M_{21} = 0$. Then, by Lemma 3.3, $M_{12} = 0$ and $M_{21} = 0$. Consequently, $M = \begin{pmatrix} 0 & 0 \\ 0 & M_2 \end{pmatrix}$. If $M_2 = 0$, then $M = 0$ and so A is prime.

Suppose $M_2 \neq 0$. Since M is a two-sided ideal in A, $MA = M$ and from the equality

$$\begin{pmatrix} 0 & 0 \\ 0 & M_2 \end{pmatrix} \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ M_2 Y & M_2 \end{pmatrix}$$

it follows that $M_2 Y = 0$. So, $M_2 Y X = 0$ in A_2, which implies that $YX = 0$, since A_2 is prime and $M_2 \neq 0$. But the equality $YX = 0$ contradicts to Lemma 3.3. Therefore, $M_2 = 0$ and if $L \neq 0$ then $M = 0$.

3. Let $n \geq 3$. Denote $g_1 = e_1 + \ldots + e_{n-1}$, $g_2 = e_n$. By the induction hypothesis one can assume that the ring $A_1 = g_1 A g_1$ is a finite direct product of prime rings, and $A_2 = g_2 A g_2$ is prime.

***Case* a.** Assume that $A_1 = g_1 A g_1$ is prime. If A is a decomposable ring, then the statement is proved. If $A = \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix}$ is indecomposable, then one can apply a proof similar to that in the previous case.

***Case* b.** Let $A_1 = C_1 \times \ldots \times C_t$, where $t \geq 2$ and each C_i is prime, for $i = 1, \ldots, t$. If A is indecomposable, then there exists a 3×3-minor eAe, where $e = f_1 + f_2 + f_3$,

$f_1,\ f_2 \in \{e_1, \ldots, e_{n-1}\}$ and $f_3 = e_n$,

$$B = eAe = \begin{pmatrix} B_1 & 0 & B_{13} \\ 0 & B_2 & B_{23} \\ B_{31} & B_{32} & B_3 \end{pmatrix},$$

with $B_1 = f_1Af_1,\ B_2 = f_2Af_2,\ B_3 = f_3Af_3,\ B_{13} = f_1Af_3,\ B_{23} = f_2Af_3,\ B_{31} = f_3Af_1,\ B_{32} = f_3Af_2$ and all B_{ij} are nonzero. Since each B_1, B_2 is one of the C_i, $i = 1, 2, \ldots, t$, and $B_3 = A_2$, all B_i are prime rings.

Let $I(B_{13})$ be the ideal generated by $B_{13} \neq 0$ and $I(B_{23})$ the ideal generated by $B_{23} \neq 0$. Then

$$I(B_{13}) = \begin{pmatrix} B_{13}B_{31} & 0 & B_{13} \\ 0 & 0 & 0 \\ B_{31}B_{13}B_{31} & 0 & B_{31}B_{13} \end{pmatrix}$$

and

$$I(B_{23}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & B_{23}B_{32} & B_{23} \\ 0 & B_{32}B_{23}B_{32} & B_{32}B_{23} \end{pmatrix}.$$

Obviously, $I(B_{13})I(B_{23}) = 0$. Therefore, $(B_{31}B_{13})(B_{32}B_{23}) = 0$. Since B_3 is prime, $B_{31}B_{13} \neq 0$ implies $B_{32}B_{23} = 0$. But $B_{23} \neq 0$, therefore it follows from Lemma 3.3 that $B_{32}B_{23} \neq 0$. This contradiction shows that either A_1 is indecomposable, or A is decomposable and $A = C_1 \times \ldots \times C_t \times A_2$, where $A_2,\ C_i,\ i = 1, 2, \ldots, t$, are prime.

Remark **3.1.** The proof of this theorem was inspired by Corollary 2.3.12 given in [33], which states that a semiprime FDI-ring A is a semisimple Artinian. This statement is not correct as shown by the obvious example of a discrete valuation ring $\mathcal{O}$.

Proposition 3.2. *A right uniserial semiprime ring $\mathcal{O}$ is prime.*

Proof. Let L and N be nonzero two-sided ideals of $\mathcal{O}$ and $LN = 0$. One can assume that $N \subseteq L$. Therefore $N^2 = 0$, which implies that $N = 0$, since $\mathcal{O}$ is semiprime. Consequently, $\mathcal{O}$ is a prime ring.

Definition 3.1. *A ring A is said to be a* **right** (*resp.* **left**) **Bézout ring** *if every finitely generated right* (*left*) *ideal of A is principal. A ring which is both right and left Bézout is called a Bézout ring. A* **Bézout domain** *is a (commutative) integral domain in which every finitely generated ideal is principal.*

Any principal ideal ring is obviously Bézout. In a certain sense, a Bézout ring is a non-Noetherian analogue of a principal ideal ring.

The main examples of commutative Bézout domains which are neither principal ideal domains nor Noetherian rings are as follows.

1. The ring of all functions in a single complex variable holomorphic in a domain of the complex plane **C**.
2. The ring of holomorphic functions given on the entire complex plane **C**.
3. The ring of all algebraic integers.

Corollary **3.1.** *A right Bézout semiprime semiperfect ring is a finite direct product of prime rings.*

Proof. The proof follows from Proposition 3.2 and [15] (Theorem 3.6).

Recall that a module M is called **distributive** if $K \cap (L + N) = K \cap L + K \cap N$ for all submodules K, L, N. A module is called **semidistributive** if it is a direct sum of distributive modules. A ring A is called **right** (**left**) **semidistributive** if the right (left) regular module A_A (${}_AA$) is semidistributive. A right and left semidistributive ring is called **semidistributive**.

The class of distributive (resp. semidistributive) modules contains the class of uniserial (resp. serial) modules.

We write an $SPSDR$**-ring** ($SPSDL$**-ring**) for a semiperfect right (left) semidistributive ring and an $SPSD$**-ring** for a semiperfect semidistributive ring.

Theorem 3.3 ([19, 40, 41], Proposition 11.43)**.** *A semiperfect ring A is right* (*left*) *semidistributive if and only if for any local idempotents e and f of A the set eAf is a uniserial right fAf-module* (*uniserial left eAe-module*)*. In particular, if e is a local idempotent of a semiperfect right* (*left*) *semidistributive ring A then eAe is a uniserial ring.*

***Corollary* 3.2.** *Every semiprime $SPSDR$-ring is isomorphic to a finite direct product of prime $SPSDR$-rings.*

Proof. The proof follows from Proposition 3.2 and Theorem 3.3.

Since any right serial ring is an SPSDR-ring we obtain:

***Corollary* 3.3.** *Every right serial semiprime ring is isomorphic to a finite direct product of prime serial rings.*

Note that the last corollary is proved in [31] (Proposition 3.4).

4. Generalized matrix rings. Generalized matrix rings form one of the largest classes of matrix rings and studied extensively in various contexts.

Let A, B be rings, M an A-B-bimodule, and N an B-A-bimodule. Assume that there are two bimodule homomorphisms $f\colon M \otimes_B N \to A$ and $g\colon N \otimes_A M \to B$, satisfying the associativity conditions:

$$f(m \otimes n)m_1 = mg(n \otimes m_1) \quad \text{and} \quad g(n \otimes m)n_1 = nf(m \otimes n_1)$$

for all $m, m_1 \in M$ and $n, n_1 \in N$.

Consider a set

$$Q = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

of all matrices $\begin{pmatrix} a & m \\ n & b \end{pmatrix}$, where $a \in A$, $b \in B$, $m \in M$ and $n \in N$. Addition in this set is defined componentwise and multiplication by the rule:

$$\begin{pmatrix} a & m \\ n & b \end{pmatrix} \begin{pmatrix} a_1 & m_1 \\ n_1 & b_1 \end{pmatrix} = \begin{pmatrix} aa_1 + f(m \otimes n_1) & am_1 + mb_1 \\ na_1 + bn_1 & g(n \otimes m_1) + bb_1 \end{pmatrix}.$$

With respect to these operations Q becomes a ring, which is called a **generalized matrix ring** (of order 2).

Note, that if we put $m \cdot n = f(m \otimes n)$ and $n \cdot m = g(n \otimes m)$ we obtain the operation of multiplication as in the usual matrix ring.

In a similar way we can define a generalized matrix ring of any order $n > 2$. Such a ring is naturally isomorphic to a generalized matrix ring of order 2 (this isomorphism is obtained by partitioning every matrix ring into four blocks).

***Remark* 4.1.** If a ring A has a finite set of pairwise orthogonal idempotents $e_1, e_2, \ldots, e_n$ such that $1 = e_1 + e_2 + \ldots + e_n$, then its two-sided Peirce decomposition in a usual way can be considered as a generalized matrix ring.

Lemma 4.1. *Let* $A = \begin{pmatrix} A_1 & X \\ Y & A_2 \end{pmatrix}$ *be a generalized matrix ring, where* X *is an* A_1*-*A_2*-bimodule and* Y *is an* A_2*-*A_1*-bimodule. Suppose that* $XY \subseteq \mathrm{rad}(A_1)$ *and* $YX \subset \mathrm{rad}(A_2)$. *Then* $\mathrm{rad}(A) = \begin{pmatrix} \mathcal{R}_1 & X \\ Y & \mathcal{R}_2 \end{pmatrix}$, *where* $\mathcal{R}_1 = \mathrm{rad}(A_1)$ *and* $\mathcal{R}_2 = \mathrm{rad}(A_1)$.

Proof. Write $1 = e_1 + e_2$, with $e_iAe_i = A_i$, $e_1Ae_2 = X$, $e_2Ae_1 = Y$, and $\mathcal{R} = \mathrm{rad}(A)$. By [24] (Proposition 3.4.8.), $\mathcal{R}_i = e_i\mathcal{R}e_i$ for $i = 1, 2$. Write also

$$L = \begin{pmatrix} \mathcal{R}_1 & X \\ Y & \mathcal{R}_2 \end{pmatrix}.$$

Since, by assumption, $XY \subseteq \mathcal{R}_1$ and $YX \subseteq \mathcal{R}_2$, L is a two-sided ideal in A.

We shall show that $1 - u$ is invertible for any $u \in L$. Write

$$1 - u = \begin{pmatrix} 1 - r_1 & x \\ y & 1 - r_2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 - r_1 & x \\ y & 1 - r_2 \end{pmatrix} \begin{pmatrix} 1 & -(1 - r_1)^{-1}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - r_1 & 0 \\ y & 1 - r_2' \end{pmatrix},$$

where $r_2' = r_2 + y(1 - r_1)^{-1}x \in \mathcal{R}_2$. Furthermore,

$$\begin{pmatrix} 1 & 0 \\ -(1 - r_1)^{-1}y & 1 \end{pmatrix} \begin{pmatrix} 1 - r_1 & 0 \\ y & 1 - r_2' \end{pmatrix} = \begin{pmatrix} 1 - r_1 & 0 \\ 0 & 1 - r_2' \end{pmatrix}.$$

Since the latter matrix is clearly invertible, so too is $1 - u$.

By [24] (Proposition 3.4.6), $L \subseteq \mathrm{rad}\, A$, and by [24] (Proposition 3.4.8), $\mathrm{rad}\, A \subseteq L$. Thus, $\mathrm{rad}\, A = L$, as required.

Lemma 4.1 is proved.

Theorem 4.1. *Let*

$$A = \begin{pmatrix} A_1 & X_{12} & \ldots & X_{1n} \\ X_{21} & A_2 & \ldots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \ldots & A_n \end{pmatrix}$$

be a generalized matrix ring, where X_{ij} *is an* A_i*-*A_j*-bimodule* $i = 1, 2, \ldots, n$. *Suppose that* $X_{ij}X_{ji} \subseteq \mathrm{rad}(A_i)$ *for* $i = 1, 2, \ldots, n$, $i \neq j$. *Then*

$$\mathrm{rad}(A) = \begin{pmatrix} R_1 & X_{12} & \ldots & X_{1n} \\ X_{21} & R_2 & \ldots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \ldots & R_n \end{pmatrix},$$

where $\mathcal{R}_i = \mathrm{rad}(A_i)$ *for* $i = 1, 2, \ldots, n$.

Proof. The theorem follows by induction on n using Lemma 4.1.

***Corollary* 4.1.** *Let*

$$A = \begin{pmatrix} A_1 & X_{12} & \dots & X_{1n} \\ 0 & A_2 & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{pmatrix}$$

be a triangular matrix ring, where X_{ij} is an A_i-A_j-bimodule, $i = 1, 2, \dots, n$. Then

$$\mathrm{rad}(A) = \begin{pmatrix} R_1 & X_{12} & \dots & X_{1n} \\ 0 & R_2 & \dots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_n \end{pmatrix},$$

where $\mathcal{R}_i = \mathrm{rad}(A_i)$ for $i = 1, 2, \dots, n$.

Definition 4.1. *Let A be an FDI-ring. A decomposition of the identity $1 = f_1 + f_2 + \dots + f_m \in A$, where f_i are idempotents, is called* **triangular** *if $f_i A f_j = 0$ for all $i > j$. Such a decomposition is called* **prime** *if $f_i A f_i$ is a prime ring for any $i = 1, \dots, m$.*

Definition 4.2. *An FDI-ring A is called* **primely triangular** *if there exists a prime triangular decomposition of the identity of A. In this case the two-sided Peirce decomposition of A can be represented in the following triangular form*:

$$A = \begin{pmatrix} A_1 & A_{12} & \dots & A_{1,n-1} & A_{1n} \\ 0 & A_2 & \dots & A_{2,n-1} & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & A_{n-1} & A_{n-1,n} \\ 0 & 0 & \dots & 0 & A_n \end{pmatrix}, \qquad (1)$$

where all A_i are prime rings.

***Remark* 4.2.** Note that the term “triangular ring” was first introduced by S. U. Chase [12] in 1961 for semiprimary rings. According to S. U. Chase a semiprimary ring A with Jacobson radical R is triangular if there exists a complete set $e_1, e_2, \dots, e_k$ of mutually orthogonal primitive idempotents of A such that $e_i A e_j = 0$ for all $i > j$. This notion was used by L. W. Small in [38] for arbitrary right Noetherian rings. M. Harada [23] in 1966 introduced the term “generalized triangular matrix rings” for rings with triangular decomposition of the identity where $A_i = e_i A e_i$ are arbitrary rings. He studied the properties of such rings when all A_i are semiprimary rings. It is obvious that in this case the generalized triangular matrix rings are also semiprimary. Yu. A. Drozd [17] in 1980 used the term “triangular ring” for rings with a triangular prime decomposition of the identity.

The following simple fact is well known.

Proposition 4.1 (see [24], Proposition 9.2.13)**.** *Let A be a prime* (*semiprime*) *ring with an idempotent $e^2 = e \in A$. Then eAe is also a prime* (*semiprime*) *ring.*

Theorem 3.2 and the above proposition immediately give the next:

***Corollary* 4.2.** *Any semiprime primely triangular FDI-ring is isomorphic to a finite direct product of prime rings.*

Recall that a ring A is a **piecewise domain** (or simply PWD) with respect to a complete set of pairwise orthogonal idempotents $\{e_1, e_2, \ldots, e_n\}$ if $xy = 0$ implies $x = 0$ or $y = 0$ with $x \in e_iAe_j$ and $y \in e_jAe_k$ for $1 \le i, j, k \le n$.

Note that [24] (Proposition 10.7.8) implies that the condition in the above definition is equivalent to the following one: every nonzero homomorphism $e_iA \to e_jA$ is a monomorphism for $1 \le i, j \le n$.

Piecewise domains were first introduced and studied by R. Gordon and L. Small in [22]. Given a piecewise domain A, each e_iAe_i is a domain, $i = 1, \ldots, m$, and hence the set of $\{e_1, e_2, \ldots, e_m\}$ is a complete set of primitive orthogonal idempotents. Consequently, any piecewise domain is an FDI-ring. The structure of piecewise domains was obtained by R. Gordon and L. Small in [22]. The next fact follows from their description.

Proposition 4.2. *A piecewise domain is a primely triangular ring.*

***Remark* 4.3.** Note that a piecewise domain can be considered as a generalization of an l-hereditary ring. Recall that a ring A is called **l-hereditary** if it is Artinian and satisfies the following condition: given any pair of indecomposable projective left A-modules P and Q and any non-zero A-homomorphism $\varphi : P \to Q$, either $\varphi = 0$ or φ is a monomorphism. There is the following equivalent condition: any local submodule of an indecomposable projective module is projective again (recall that a **local module** is a module with a unique maximal submodule). Note that such algebras were first considered by R. Bautista [4] and the term l-hereditary was introduced by R. Bautista and D. Simson in [5] in their study of representations of incidence algebras of posets.

Assume that $1 = e_1 + e_2 + \ldots + e_n$ is a decomposition of the identity of a piecewise domain A. If e is a sum of different idempotents from the set $\{e_1, e_2, \ldots, e_n\}$, then the ring eAe is again a piecewise domain. Therefore the above proposition and Theorem 3.2 give the next fact.

***Corollary* 4.3** (R. Gordon and L. Small [22])**.** *Every semiprime piecewise domain is isomorphic to a finite direct product of prime piecewise domains.*

Since any right hereditary (semihereditary) FDI-ring A is a piecewise domain and eAe is a hereditary (semihereditary) ring for any idempotent $e \in A$ (see [34], Corollary 1) we obtain the following:

***Corollary* 4.4.** *Every semiprime right hereditary* (*semihereditary*) *FDI-ring is isomorphic to a finite direct product of right hereditary* (*semihereditary*) *prime rings.*

Since any right Noetherian right hereditary (semihereditary) ring A is a piecewise domain and eAe is a right Noetherian right hereditary (semihereditary) ring for any idempotent $e \in A$, we have the next:

***Corollary* 4.5.** *Every semiprime right Noetherian right hereditary* (*semihereditary*) *ring is isomorphic to a finite direct product of right Noetherian right hereditary* (*semihereditary*) *prime rings.*

5. An application of the decomposition of the identity to the unit group. In this section we give an improvement on a fact on Sylow subgroups of general linear groups over finite local rings given in Corollary 7.6 in [16]. Elements of an FDI-ring A can be represented by generalized matrices which makes it convenient to perform calculations with them and, in particular, to obtain results on the group of units $\mathcal{U}(A)$ of A (see, for example, [7 – 10]).

Our goal is to produce a Sylow p-subgroup in the unit group of a semiperfect ring A assuming that the characteristic of A is a power of p and the Jacobson radical of A is a nil ideal, and comment conjugacy. Recall that a semiperfect ring whose Jacobson radical is nilpotent is called **semiprimary**. We shall say that a ring A is **almost perfect** if A is semiperfect and its Jacobson radical is nil. Note that since any left (or right) T-nilpotent ideal is nil, it follows that any left (or right) perfect ring is almost perfect (see [3], Theorem P). In particular, examples from [3, p. 475, 476] show that there exist almost perfect rings which are not semiprimary.

If D is a division ring and $m > 0$, we denote by $\mathrm{UT}_m(D)$ the upper unitriangular group, i.e., the group of those $m \times m$-upper triangular matrices over D whose diagonal entries are all equal to 1. Notice that $\mathrm{UT}_m(D)$ contains all upper-triangular transvections, i.e., the matrices of the form $1 + e_{ij}(x)$, where $i < j$; $i, j \in \{1, \ldots, m\}$, $x \in D$, and $e_{ij}(x)$ stands for the $m \times m$-matrix whose unique non-zero entry is placed in the (i, j)-position and is equal to x[4]. Multiplying any element $g = (g_{ij})$ of the general linear group $\mathrm{GL}_m(D)$ by $1 + e_{ij}(x)$ from the left we add to the i-row of g the j-one multiplied by any element $x \in D$ from the left, and, similarly, multiplying g by $1 + e_{ij}(x)$ from the right we add to the j-column of g the i-one multiplied by x from the right.

The next fact is known. It follows from [35, p. 17 – 19].

Lemma 5.1. *Let D be a division ring of characteristic $p > 0$ which is locally finite-dimensional over its center. Then any Sylow p-subgroup of the general linear group $\mathrm{GL}_m(D)$ is conjugate in $\mathrm{GL}_m(D)$ to $\mathrm{UT}_m(D)$.*

The fact that $\mathrm{UT}_m(D)$ is a Sylow p-subgroup of $\mathrm{GL}_m(D)$ when D is any division ring with characteristic $p > 0$, is rather elementary. For the sake of completeness we give a proof below.

Lemma 5.2. *Let D be a division ring whose characteristic is $p > 0$ and let m be a positive integer. Then $\mathrm{UT}_m(D)$ is a Sylow p-subgroup of the general linear group $\mathrm{GL}_m(D)$.*

Proof. Suppose on the contrary that $\tilde{G} \subseteq GL(n, D)$ is a p-subgroup which contains properly $\mathrm{UT}_m(D)$, and take $g = (g_{ij}) \in \tilde{G} \setminus \mathrm{UT}_m(D)$. It is easily seen that g can be chosen to have all entries in the first row equal to 0 except the $(1, 1)$-position. Indeed, if $g_{11} = 0$ then some $g_{k1} \neq 0$, as otherwise g can not be invertible. Adding to the first row the k-one we come to the case $g_{11} \neq 0$. Now for any $i > 1$, adding to the i-column the first one multiplied by $-g_{11}^{-1} \cdot g_{1i}$ from the right, we produce 0 in the $(1, i)$-entry. Since i was arbitrary, we may suppose that $g_{11} \neq 0$ and $g_{1i} = 0$ for all $i > 1$, as desired.

Let $\overline{G}$ be the subgroup of $\tilde{G}$ formed by those matrices, all whose entries in the first row, except the $(1, 1)$-one, are zero. Then the $(1, 1)$-entries of the elements from $\overline{G}$ form a p-subgroup of units in D, and since D has no p-elements, we have that

$$h_{11} = 1 \quad \forall h = (h_{ij}) \in \overline{G}, \tag{2}$$

in particular, $g_{11} = 1$. Furthermore, the lower-right $(m - 1) \times (m - 1)$ blocks of the matrices of $\overline{G}$ form a p-subgroup in $GL(m - 1, K)$ which contains $\mathrm{UT}_{m-1}(D)$. By

[4] In fact, $\mathrm{UT}_m(D)$ is generated by the upper triangular transvections.

induction on m, it coincides with $\mathrm{UT}_{m-1}(D)$. Then since $g \notin \mathrm{UT}_m(D)$, g must have a non-zero non-diagonal entry in the first column, say $g_{1l} \neq 0$, $l > 1$. Adding to the first row of g the l-one multiplied by xg_{1l}^{-1} from the left, we obtain an element $\tilde{g} \in \tilde{G}$ with arbitrary $x \in D$ in the $(1,1)$-position. Some non-diagonal entries of the first row of $\tilde{g}$ are not 0 now. But they can be made equal to 0 as above using the non-zero $(1,1)$-position. Thus we come to an element $h \in \bar{G}$ whose $(1,1)$-entry x is an arbitrary non-zero element of D. If $|D| > 2$ this is a contradiction. The lemma for the case $|D| = 2$ is a very well-known example from basic group theory.

Lemma 5.2 is proved.

Note that the structure and conjugacy of the Sylow subgroups of $\mathrm{GL}_m(D)$, where D is a division ring which is finite-dimensional over its center, were extensively studied by A. E. Zalesski in [48].

Let A be a semiperfect ring with a fixed decomposition $1 = f_1 + \ldots + f_s$ of the identity into pairwise orthogonal idempotents such that

$$A = f_1A + \ldots + f_sA = P_1^{n_1} \oplus \ldots \oplus P_s^{n_s},$$

where $P_1, \ldots P_s$ are the pairwise non-isomorphic principal right A-modules. Note that by [24] (Proposition 2.1.3) we have that each f_iAf_i is isomorphic to the full matrix ring over the endomorphism ring of P_i:

$$f_iAf_i \cong M_{n_i}(\mathrm{End}_A(P_i)), \quad i = 1, \ldots, s,$$

and, moreover, each $\mathrm{End}_A(P_i) = \mathcal{O}_i$ is a local ring whose (unique) maximal ideal shall be denoted by $\mathcal{M}_i$. Then the two-sided Peirce decomposition of A has the following form:

$$A = \begin{pmatrix} M_{n_1}(\mathcal{O}_1) & A_{12} & \ldots & A_{1s} \\ A_{21} & M_{n_2}(\mathcal{O}_2) & \ldots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \ldots & M_{n_s}(\mathcal{O}_s) \end{pmatrix}, \tag{3}$$

where $A_{ij} = f_iAf_j$, $i, j = 1, \ldots, s$. Since $\mathrm{rad}\,(M_{n_1}(\mathcal{O}_1)) = M_{n_1}(\mathcal{M}_1)$, we obtain by Proposition 11.1.1 from [24] that the Jacobson radical $\mathcal{R} = \mathrm{rad}\,A$ of A has the following two-sided Peirce decomposition:

$$\mathcal{R} = \begin{pmatrix} M_{n_1}(\mathcal{M}_1) & A_{12} & \ldots & A_{1s} \\ A_{21} & M_{n_2}(\mathcal{M}_2) & \ldots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \ldots & M_{n_s}(\mathcal{M}_s) \end{pmatrix}. \tag{4}$$

Notice that $A_{ij}A_{ji} \subseteq M_{n_s}(\mathcal{M}_s)$, $i = 1, \ldots, s$, $i \neq j$, as $\mathcal{R}$ is a two-sided ideal in A.

Let B be the subring of A formed by the radically upper-triangular matrices, i.e., the subring whose two-sided Peirce decomposition has the following form:

$$B = \begin{pmatrix} H_{n_1}(\mathcal{O}_1) & A_{12} & \dots & A_{1s} \\ A_{21} & H_{n_2}(\mathcal{O}_2) & \dots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \dots & H_{n_s}(\mathcal{O}_s) \end{pmatrix},$$

where

$$H_{n_i}(\mathcal{O}_i) = \begin{pmatrix} \mathcal{O}_i & \mathcal{O}_i & \dots & \mathcal{O}_i \\ \mathcal{M}_i & \mathcal{O}_i & \dots & \mathcal{O}_i \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_i & \mathcal{M}_i & \dots & \mathcal{O}_i \end{pmatrix}.$$

Since

$$A_{ij}A_{ji} \subseteq M_{n_i}(\mathcal{M}_i) \subseteq \begin{pmatrix} \mathcal{M}_i & \mathcal{O}_i & \dots & \mathcal{O}_i \\ \mathcal{M}_i & \mathcal{M}_i & \dots & \mathcal{O}_i \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_i & \mathcal{M}_i & \dots & \mathcal{M}_i \end{pmatrix} = \operatorname{rad} H_{n_i}(\mathcal{O}_i) \quad \forall i \neq j,$$

we can apply Theorem 4.1, and consequently,

$$\mathcal{R}' = \operatorname{rad} B = \begin{pmatrix} \operatorname{rad} H_{n_1}(\mathcal{O}_1) & A_{12} & \dots & A_{1s} \\ A_{21} & \operatorname{rad} H_{n_2}(\mathcal{O}_2) & \dots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \dots & \operatorname{rad} H_{n_s}(\mathcal{O}_s) \end{pmatrix}.$$

Then $G = 1 + \mathcal{R}'$ is a normal subgroup of $\mathcal{U}(B)$. Note that if $\mathcal{R}'$ is nilpotent, i.e., $(\mathcal{R}')^m = 0$ for some $m \geq 0$, then $G = 1 + \mathcal{R}'$ is a nilpotent group and

$$1 + \mathcal{R}' \supset 1 + (\mathcal{R}')^2 \supset \ldots \supset 1 + (\mathcal{R}')^{m-1} \supset 1$$

is a central series of G.

Let A be an almost perfect ring. Notice that if the characteristic of each division ring $D_i = \mathcal{O}_i/\mathcal{M}_i$ is the same prime $p > 0$ for all $i = 1, \ldots, s$, then the characteristic of A is a positive power of p. Evidently, the converse is also true. Keeping our notation we give the next:

Theorem 5.1. *Let A be an almost perfect ring whose characteristic is a power of a prime $p > 0$. Then G is a Sylow p-subgroup of $\mathcal{U}(A)$. If, in addition, each division ring $\mathcal{O}_i/\mathcal{M}_i$ is locally finite-dimensional over its center, then all Sylow p-subgroups of $\mathcal{U}(A)$ are conjugate to G.*

Proof. It is easily seen that $\mathcal{R}'$ nilpotent modulo $\mathcal{R}$ and, consequently, $\mathcal{R}'$ is a nil ideal as so too is $\mathcal{R}$. Then the binomial formula implies that G is a p-group. Suppose that $\tilde{G}$ is a p-subgroup of $\mathcal{U}(A)$ with $\tilde{G} \supseteq G$. Let $\varphi : A \to A/\mathcal{R}$ be the canonical map. By (3) and (4) we have the following block-matrix decomposition:

$$A/\mathcal{R} = \begin{pmatrix} M_{n_1}(D_1) & 0 & \dots & 0 \\ 0 & M_{n_2}(D_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_{n_s}(D_s) \end{pmatrix}. \tag{5}$$

Hence

$$\mathcal{U}(A/\mathcal{R}) \cong \mathrm{GL}_{n_1}(D_1) \times \ldots \times \mathrm{GL}_{n_s}(D_s) \tag{6}$$

and

$$\varphi(G) \cong \mathrm{UT}_{n_1}(D_1) \times \ldots \times \mathrm{UT}_{n_s}(D_s).$$

It follows by Lemma 5.2 that $\varphi(\tilde{G}) = \varphi(G)$, and, consequently, $\tilde{G} = G$, as $\mathrm{Ker}\,\varphi \subseteq G$.

Assume now that each D_i, $i = 1, \ldots, s$, is locally finite-dimensional over its center. Let φ be as above and

$$\bar{\varphi}\colon\ \mathcal{U}(A) \to \mathrm{GL}_{n_1}(D_1) \times \ldots \times \mathrm{GL}_{n_s}(D_s)$$

be the corresponding homomorphism of the unit groups. Notice that $\mathrm{Ker}\ \bar{\varphi} = 1 + \mathcal{R}$ is a normal p-subgroup in $\mathcal{U}(A)$, as $\mathcal{R}$ is a nil ideal. It follows that any Sylow p-subgroup $\tilde{G}$ of $\mathcal{U}(A)$ contains $\mathrm{Ker}\ \bar{\varphi}$. By Lemma 5.1 there exists a unit $v \in \bar{\varphi}(\mathcal{U}(A))$ such that $v^{-1}\bar{\varphi}(\tilde{G})v = \bar{\varphi}(G)$. Let $u, u_1 \in A$ be elements with $\bar{\varphi}(u) = v$ and $\bar{\varphi}(u_1) = v^{-1}$. Then $uu_1 = 1 + x \in 1 + \mathcal{R}$, and since $1 + \mathcal{R}$ is a group, it follows that $u_1(1 + x)^{-1}$ is a right inverse for u. Hence u is invertible, as A is Dedekind-finite (see Proposition 2.4). Finally, thanks to the fact that both $\tilde{G}$ and G contain $\mathrm{Ker}\ \bar{\varphi}$, we conclude that $u^{-1}\tilde{G}u = G$.

Theorem 5.1 is proved.

***Corollary* 5.1.** *Let A be a finite ring whose characteristic is a power of a prime $p > 0$. Then any Sylow p-subgroup of $\mathcal{U}(A)$ is conjugate to G.*

Observe that the conjugacy of the Sylow subgroups stated in the above corollary is not reliant on Lemma 5.1 thanks to Sylow's theorem.

***Remark* 5.1.** Let A be as in Theorem 5.1 and $\bar{\varphi}$ as above. Assume that q is a prime different from p. Since $\mathrm{Ker}\,\bar{\varphi}$ is a p-group, it follows that $H \cong \bar{\varphi}(H)$ for any q-subgroup H in $\mathcal{U}(A)$.

6. Nets in rings. The notion of a net often appears in different fields of mathematics, in particular, in algebra (see, for example, [1, 2, 7 – 9, 18, 43, 44]).

We shall consider the notion of a net in an associative ring A associated with a decomposition of 1 into a sum of pairwise orthogonal idempotents.

Let

$$1 = e_1 + \ldots + e_n \tag{7}$$

be a fixed decomposition of $1 \in A$ into a sum of pairwise orthogonal idempotents, and $A = e_1A \oplus \ldots \oplus e_nA$ the corresponding decomposition of the ring A into a direct sum of right ideals. For any element $a \in A$ there results $a = 1 \cdot a \cdot 1 = (e_1 + \ldots + e_n)a(e_1 + \ldots + e_n) = \sum_{i,j=1}^{n} e_iae_j$. It is not difficult to verify that such a decomposition defines a decomposition of the ring A into a direct sum of Abelian groups e_iAe_j, $i, j = 1, 2, \ldots, n$:

$$A = \bigoplus_{i,j=1}^{n} e_iAe_j. \tag{8}$$

The two-sided Peirce decomposition of A has the following form:

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}, \tag{9}$$

where $A_{ij} = e_i A e_j$. Note that the elements of $e_i A e_j$ are naturally identified with homomorphisms from $e_j A$ to $e_i A$.

Definition 6.1. *A direct sum*

$$N = \bigoplus_{i,j=1}^{n} N_{ij}, \tag{10}$$

where N_{ij} is an A_i-A_j-subbimodule in A_{ij} for all $i, j = 1, \dots, n$, is called a **net** *in A associated with* (7).

The notion of a net associated with the trivial decomposition of $1 \in A$ coincides with the notion of an ideal of A. Note if in the above definition we add the condition that $N_{ij} N_{jk} \subseteq N_{ik}$, for all i, j, k, then we obtain the concept of a net subring introduced in [10]

Let S be a poset and T a subset of S. An element $a \in S$ is called an **upper bound** (resp. **lower bound**) of T if $t \leq a$ (resp. $a \leq t$) for all $t \in T$. In general a poset can have several upper bounds or it can have none at all.

An element $a \in T$ is a **greatest** (resp. **least**) element of T if $t \leq a$ (resp. $a \leq t$) for all $t \in T$. Not every subset T of a poset S has a greatest (or least) element. But if T has such an element then it is unique. Indeed, let x and y be greatest elements of T. Then $x \leq y$ and $y \leq x$. Hence from the property of "antisymmetry" of a poset it follows that $x = y$. The uniqueness of a least element of T can be proved analogously. So the greatest (resp. least) element, if it does exist, is unique and is an upper (resp. lower) bound for T. If the set of upper bounds of T has a least element, then it is called the **least upper bound** (or **supremum**) of T and denoted by sup(T). If the set of lower bounds has a greatest element, it is called the **greatest lower bound** (or **infimum**) of T and denoted by inf(T). It is obvious that if a subset T has a supremum (resp. infimum), then it is uniquely determined.

Definition 6.2. *A poset S, whose every pair of elements has both a supremum and an infimum in S, is said to be a* **lattice**.

***Example* 6.1.** Let A be a ring and $\mathcal{L}(A)$ the set of all ideals of the ring A ordered by inclusion. Let $\mathcal{I}$ and $\mathcal{J}$ be ideals in A. Then their supremum in $\mathcal{L}(A)$ is the sum $\mathcal{I} + \mathcal{J}$ and their infimum in $\mathcal{L}(A)$ is the intersection $\mathcal{I} \cap \mathcal{J}$. Therefore $\mathcal{L}(A)$ is a lattice.

Let S be a lattice. Then each pair of elements $a, b \in S$ has both a supremum and an infimum. Denote

$$a \vee b = \sup\{a, b\} \quad \text{and} \quad a \wedge b = \inf\{a, b\}.$$

Then the maps $\vee$ and $\wedge$ from $S \times S$ to S defined by

$$(a, b) \mapsto a \vee b \quad \text{and} \quad (a, b) \mapsto a \wedge b$$

are binary operations on S.

Let $N(A)$ be a set of all nets in A associated with a fixed decomposition of the identity of A. We can define on $N(A)$ an ordering relation $\preceq$ by the following way: for two sets $N = \bigoplus\limits_{i,j=1}^{n} N_{ij}$ and $M = \bigoplus\limits_{i,j=1}^{n} M_{ij}$ of $N(A)$ we say that $N \preceq M$ if and only if $N_{ij} \subseteq M_{ij}$ for any pair $i, j = 1, 2, \ldots, n$. On the set $N(A)$ we can also define the operations of addition and intersection in the natural way: $N + M = \bigoplus\limits_{i,j=1}^{n} (N_{ij} + M_{ij})$ and $N \cap M = \bigoplus\limits_{i,j=1}^{n} (N_{ij} \cap M_{ij})$.

Proposition 6.1. *The set $N(A)$ of all nets associated with a fixed decomposition of the identity of a ring A is a lattice with $N_1 \vee N_2 = N_1 + N_2$ and $N_1 \wedge N_2 = N_1 \cap N_2$, where $N_1,\, N_2 \in N(A)$.*

The proof is obvious.

Definition 6.3. *A lattice $\mathcal{L}$ is called* **distributive** *if*:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y) \quad \textit{for all} \quad x,\, y,\, z \in \mathcal{L}.$$

Consider the lattice $\mathcal{L}(A)$ of all two-sided ideals I of a ring A. This lattice is distributive if $I \cap (K + L) = (I \cap K) + (I \cap L)$ for all $I,\, K,\, L \in \mathcal{L}(A)$.

Lemma 6.1 [6, p.12]**.** *Any chain is a distributive lattice and any direct product of distributive lattices is a distributive lattice.*

Let $\mathcal{I}$ be a two-sided ideal of A. Then for the decomposition (7) of the identity the two-sided Peirce decomposition of $\mathcal{I}$ has the following form:

$$\mathcal{I} = \begin{pmatrix} \mathcal{I}_{11} & \mathcal{I}_{12} & \ldots & \mathcal{I}_{1n} \\ \mathcal{I}_{21} & \mathcal{I}_{22} & \ldots & \mathcal{I}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{I}_{n1} & \mathcal{I}_{n2} & \ldots & \mathcal{I}_{nn} \end{pmatrix},$$

where each $\mathcal{I}_{ij} \subseteq A_{ij}$ is an A_i-A_j-module for $i, j = 1, 2, \ldots, n$. Therefore any ideal $\mathcal{I}$ is a net in A, and so $\mathcal{L}(A) \subset N(A)$.

Theorem 6.1. *The lattice $\mathcal{L}(A)$ of a $SPSDR$-ring A is distributive.*

Proof. Let A be a semiperfect right semidistributive ring, and let $1 = e_1 + e_2 + \ldots$ $\ldots + e_n$ be a decomposition of the identity of A into a sum of pairwise orthogonal local idempotents.

By [24] (Corollary 14.2.2), in this case any component A_{ij} of the decomposition (9) is a uniserial right A_i-module. Therefore any net $N(i, j)$, which i, j-component is equal to $N_{ij} \subseteq A_{ij}$ and all others are equal to 0, is a chain. Hence from Lemma 6.1 it follows that $N(A)$, as a direct product of distributive lattices, is also a distributive lattice. The lattice $\mathcal{L}(A)$ is distributive as a sublattice of a distributive lattice $N(A)$.

***Corollary* 6.1.** *A lattice $\mathcal{L}(A)$ of a right Bézout semiperfect ring A is distributive.*

Note that in the commutative case this corollary was proved for any Bézout ring by Chr. U. Jensen [26] (indeed he proved this statement for any Prüfer ring) and by P. M. Cohn [13].

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