UDC 517.9

G. Nowak (The Great Poland Univ. Soc. and Econ. in Sroda Wielkopolska, Poland),
Vijay Gupta (School Appl. Sci., Netaji Subhas Inst. Technology, New Delhi, India)

THE RATE OF POINTWISE APPROXIMATION OF POSITIVE LINEAR OPERATORS BASED ON q-INTEGER

ШВИДКІСТЬ ПОТОЧКОВОГО НАБЛИЖЕННЯ ДОДАТНИХ ЛІНІЙНИХ ОПЕРАТОРІВ, ЩО БАЗУЮТЬСЯ НА q-ЦІЛОМУ

This paper is concerned with positive linear operators based on a q-integer. The rate of covergence of these operators are established. For these operators, we give Voronovskaya-type theorems and apply them to q-Bernstein polynomials and q-Stancu operators.

Розглянуто додатні лінійні оператори, що базуються на q-цілому. Встановлено швидкість збіжності цих операторів. Теореми типу Вороновської наведено для цих операторів та застосовано до q-поліномів Бернштейна та q-операторів Станку.

1. Introduction. First formulate in what we know about q-calculus, which was initiatd by Euler in the eighteenth century. Many remarkable results were obtained in the nineteenth century. In 1910, Jackson [9] introduced the notion of the definite q-integral. He also was the first to develop q-calculus in a systematic way. In the second half of the twentieth century there was a significant increase of activity in the area of the q-calculus due to applications of the q-calculus in mathematics and physics.

We now present definitions and facts from the q-calculus necessary for understanding of this paper. We follow the terminology and notations from the recent book [10] (see also [11]).

Definition 1.1. *For an arbitrary function $f(x)$, the q-differential is defined by $(d_q f)(t) := f(qt) - f(t)$. In particular $d_q t = (q-1)t$. The q-derivative of a function f is defined by*

$$D_q f(t) := \frac{(d_q f)(t)}{d_q t} = \frac{f(qt) - f(t)}{(q-1)t}, \quad t \neq 0,$$

$$D_q f(0) = \lim_{t \to 0} D_q f(t)$$

and high q-derivatives are

$$D_q^0 f := f, \qquad D_q^n f := D_q(D_q^{n-1} f), \quad n = 1, 2, 3, \ldots .$$

Clearly, if $f(x)$ is differentiable, then $\lim_{q \to 1} (D_q f)(x) = df(x)/dx$.

Definition 1.2. *Suppose $a < b$ and $0 < b$. In q-analysis, q-integral is defined as*

$$\int_0^b f(t) d_q t := (1-q) b \sum_{j=0}^{\infty} q^j f(q^j b),$$

and

$$\int_a^b f(t) d_q t := \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

Notice that $\lim_{q\to 1}\int_a^b f(t)d_qt = \int_a^b f(t)dt$.

For any fixed real number $q > 0$ and for nonnegative integer r, the q-integers of the number $[r]_q$ is defined by

$$[r]_q = (1 - q^r)/(1 - q), \quad \text{for} \quad q \neq 1, \qquad [r]_q = r, \quad \text{for} \quad q = 1.$$

Also $[0]_q = 0$.

The q-factorial $[r]_q!$, for $r \in N_0 = \{0, 1, 2, \ldots\}$ is defined in the following

$$[r]_q! = [1]_q[2]_q \ldots [r]_q, \quad r = 1, 2, \ldots, \qquad [0]_q! = 1.$$

For the integers n, k, $n \geq k \geq 0$, the q-binomial or the Gaussian coefficients is defined by (see [10, p. 12])

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

For $f \in C[0; 1]$, $\delta > 0$, we define the modulus of continuity $\omega(f; \delta)$ and the second q-modulus of smoothness $\omega_{q,2}(f; \delta)$ as follows:

$$\omega(f;\delta) := \sup_{|h|\leq\delta}\left\{\max_{x\in[0;1-|h|]}|f(x+h)-f(x)|\right\},$$

$$\omega_{q,2}(f;\delta) := \sup_{|h|\leq\delta}\left\{\max_{x\in[0;1-[2]_q|h|]}|f(x+[2]_qh)-[2]_qf(x+h)+qf(x)|\right\}.$$

It is clear that, if $f \in C[0; 1]$, then

$$\omega_{q,2}(f;\delta) \to 0 \quad \text{for} \quad \delta \to 0$$

and

$$\lim_{q\to 1}\omega_{q,2}(f;\delta) = \omega_2(f;\delta),$$

where $\omega_2(f; \delta)$ is a second modulus of smoothness. In the note, we obtain the estimates for the rate of convergence for q-Bernstein – Stancu polynomials for $0 < q < 1$, $\alpha \geq 0$ in terms of $\omega(f; \cdot)$ and $\omega_{q,2}(f; \cdot)$. Results are also new theorems for q-Bernstein polynomials. In theorems and proofs we are using the q-differential and the q-integral.

In tis paper we present a few approximation theorems concerning with positive operators based on q-integer. Typical examples these operators are: q-Bernstein operators introduced by Phillips [15], generalized q-Bernstein operators introduced by Nowak [13] and other. In third section we present these theorems for generalized q-Bernstein operators. Now we are defining these operators.

For $f \in C[0; 1]$, $q > 0$, $\alpha \geq 0$ and each positive integer n we introduce (see [13]) the following generalized q-Bernstein operators:

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n} p_{n,k}^{q,\alpha}(x) f\left(\frac{[k]_q}{[n]_q}\right), \tag{1.1}$$

where

$$p_{n,k}^{q,\alpha}(x) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{\prod_{i=0}^{k-1}(x+\alpha[i]_q)\prod_{s=0}^{n-1-k}(1-q^s x+\alpha[s]_q)}{\prod_{i=0}^{n-1}(1+\alpha[i]_q)}. \tag{1.2}$$

Note, that an empty product in (1.2) denotes 1. In the case, where $\alpha = 0$, $B_n^{q,\alpha}(f;x)$ reduces to the well-known q-Bernstein polynomials introduced by Phillips [15] in 1997

$$B_{n,q}(f;x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{i=0}^{n-k-1}(1-q^i x) f\left(\frac{[k]_q}{[n]_q}\right). \tag{1.3}$$

In the case, where $q = 1$, $B_n^{q,\alpha}(f;x)$ reduces to Bernstein – Stancu polynomials, introduced by Stancu [26] in 1968

$$S_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x+\alpha i)\prod_{s=0}^{n-k-1}(1-x+s\alpha)}{\prod_{i=0}^{n-1}(1+i\alpha)} f\left(\frac{k}{n}\right).$$

When $q = 1$ and $\alpha = 0$ we obtain the classical Bernstein polynomial defined by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

Basic facts on Bernstein polynomials, their generalizations and applications, can be found e.g. in [12, 22 – 24]. In recent years, the q-Bernstein polynomials have attracted much interest, and a great number of interesting results related to the $B_{n,q}(f)$ polynomials have been obtained (see [6, 8, 15 – 21, 27 – 30]). Some approximation properties of the Stancu operators are presented in [3 – 5, 26].

Throughout, the symbols K, K_1, K_2, ... will mean some positive absolute constants, not necessarily the same at each occurence.

2. Main result. Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators on $C[0,1]$.

We introduce the standard notation for $r \in N_0 = \{0, 1, \ldots\}$, $n \in N$,

$$\mu_{r,n}(x) = L_n f(x), \quad \text{where} \quad f(x) = (t-x)^r,$$

$$e_i := e_i(x) = x^i.$$

Theorem 2.1. *Let $(L_n)_{n\geq 1}$ be a sequence of linear positive operators, satisfying $\mu_{0,n}(x) = 1$ and $\mu_{1,n}(x) = 0$. Suppose that $f \in C[0,1]$. Then there exist $\overline{q} \in (0;1)$ such that for all $q \in (\overline{q};1)$ and all $n \in N$*

$$\left|L_n(f;x) - f(x)\right| \leq 2\delta_n \omega(D_q f; \delta_n),$$

where $\delta_n = \delta_n(x) = \sqrt{\mu_{2,n}(x)}$.

Proof. By the q-mean value theorem [25], there exists $\overline{q} \in (0;1)$ such that $\forall q \in$ $\in (\overline{q};1)$ $\exists \xi \in (t;x)$:

$$f(t) - f(x) = (t-x) D_q f(\xi).$$

As the operators L_n are linear and positive and on the fact that $\mu_{0,n}(x) = 1$, $\mu_{1,n}(x) = 0$ it follows immediately the equality

$$L_n(f;x) - f(x) = L_n((e_1 - x) D_q f(\xi_{t,x}); x) =$$

$$= L_n((e_1 - x)(D_q f(\xi_{t,x}) - D_q f(x)); x),$$

where $\xi_{t,x} \in (u, v)$, $u = \min\{x; t\}$, $v = \max\{x; t\}$. Using the property of modulus of continuity (see [12]), we get

$$\left|D_q f(\xi_{t,x}) - D_q f(x)\right| \leq \omega(D_q f; |\xi_{t,x} - x|) \leq$$

$$\leq (1 + \delta_n^{-1}|\xi_{t,x} - x|)\omega(D_q f; \delta_n) \leq (1 + \delta_n^{-1}|t - x|)\omega(D_q f; \delta_n),$$

where $\delta_n > 0$. Consequently,

$$|L_n(f; x) - f(x)| \leq \omega(D_q f; \delta_n) L_n(|e_1 - x|(1 + \delta_n^{-1}|e_1 - x|)).$$

The known Schwarz inequality and $\delta_n = \delta_n(x) = \sqrt{\mu_{2,n}(x)}$ lead to theorem.

Theorem 2.2. *Let $f \in C[0; 1]$ and $L_n(f; x)$ be as in above theorem. Then there exist $\overline{q} \in (0; 1)$ such that for all $q \in (\overline{q}; 1)$ the following inequality holds*:

$$\left|L_n(f; x) - f(x)\right| \leq K\sqrt{\mu_{2,n}(x)}\|D_q^2 f\|, \tag{2.1}$$

for all $n \in N$, where $K = K(n, x, q) = \dfrac{1}{[2]_q}\left(\sqrt{\mu_{2,n}(x)} + (1 - q)\right)$.

Proof. Using the q-Taylor formula [2] we write

$$f(t) = f(x) + D_q f(x)(t - x) + R_{f,q,x}(t),$$

where

$$R_{f,q,x}(t) = \int_x^t (t - qv) D_q^2 f(v) d_q v.$$

It is shown in [25] that there exist $\overline{q} \in (0; 1)$ such that for all $q \in (\overline{q}; 1)$, $\xi_{t,x} \in (u, v)$, $u = \min\{x; t\}$, $v = \max\{x; t\}$, which satisfies

$$R_{f,q,x}(t) = \frac{D_q^2 f(\xi_{t,x})}{[2]_q!}(t - x)(t - qx) = \frac{D_q^2 f(\xi_{t,x})}{[2]_q!}((t - x)^2 + (1 - q)(t - x)).$$

Schwarz's inequality yields inequality (2.1) immediately.

Theorem 2.3. *Let $(L_n)_{n \geq 1}$ be a sequence of linear positive operators satisfying $\mu_{0,n}(x) = 1$, $\mu_{1,n}(x) = 0$ and $\mu_{2,n}(x) \neq 0$. Suppose that $f \in C[0, 1]$. Then there exist $\overline{q} \in (0; 1)$ such that for all $q \in (\overline{q}; 1)$ and all $n \in N$*

$$\left|L_n(f; x) - f(x) - \frac{D_q^2 f(x)}{[2]!}\mu_{2,n}(x)\right| \leq K(n, x, q)\mu_{2,n}(x)\omega(D_q^2 f; \delta_n), \tag{2.2}$$

where

$$\delta_n = \delta_n(q, x) = \max\left\{\sup_x\left(\sqrt{\frac{\mu_{4,n}(x)}{\mu_{2,n}(x)}}\right); \frac{1}{[n]_q}\right\}$$

and

$$K(n, x, q) = \frac{1}{[2]_q}\left(3 + \frac{x}{[n]_q\sqrt{\mu_{2,n}(x)}}\right).$$

Proof. By the q-Taylor formula [25], there exists $\overline{q} \in (0;1)$ such that $\forall q \in (\overline{q};1)$ $\exists \xi_{t,x} \in (t;x)$, such that

$$f(t) = f(x) + D_q f(x)(t-x) + \frac{D_q^2 f(x)}{[2]_q!}(t-x)(t-qx) + r_{f,q,x}(t),$$

where

$$r_{f,q,x}(t) = \frac{D_q^2 f(\xi_{t,x}) - D_q^2 f(x)}{[2]_q!}(t-x)(t-qx).$$

The property $\mu_{0,n}(x) = 1$ and $\mu_{1,n}(x) = 0$, give us

$$L_n(f;x) - f(x) - \frac{D_q^2 f(x)}{[2]!}\mu_{2,n}(x) = L_n(r_{f,q,x};x).$$

Using the property of modulus of continuity (see [12]), we get

$$\left|D_q^2 f(\xi_t) - D_q^2 f(x)\right| \leq (1 + \delta^{-1}|t-x|)\omega(D_q^2 f;\delta),$$

where $\delta > 0$. By te inequality $1/(1-q) \leq 1/[n]_q$ and simple calculation, we have

$$|L_n(r_{f,q,x};x)| \leq$$

$$\leq \frac{\omega(D_q^2 f;\delta)}{[2]_q}\mu_{2,n}(x)\left\{1 + \frac{x}{[n]_q\sqrt{\mu_{2,n}(x)}} + \delta^{-1}\left(\sqrt{\frac{\mu_{4,n}(x)}{\mu_{2,n}(x)}} + \frac{x}{[n]_q}\right)\right\}.$$

Choosing $\delta_n = \max\left\{\sup_x\left(\sqrt{\frac{\mu_{4,n}(x)}{\mu_{2,n}(x)}}\right); \frac{1}{[n]_q}\right\}$ we get estimate (2.2) immediately.

Theorem 2.4. *Let* $(L_n)_{n\geq 1}$ *be a sequence of linear positive operators satisfying* $\mu_{0,n}(x) = 0$ *and* $\mu_{1,n}(x) = 0$*. Suppose that* $f \in C[0,1]$*. Then there exist* $\overline{q} \in (0;1)$ *such that for all* $q \in (\overline{q};1)$ *the following inequality holds*:

$$\|L_n f - f\| \leq \frac{6}{q}\omega_{q,2}(f;\delta_n)$$

for all $n \in N$*, where* $\delta_n = \delta_n(q,x) = \sqrt{\mu_{2,n}(x)}\max\left\{\sqrt{\mu_{2,n}(x)}; 1-q\right\}$.

Proof. For $0 < h \leq \frac{1}{[2]_q}\min\{x; 1-x\}$ we define

$$g_{q,h}(x) = \frac{1}{qh^2}\int\limits_{-h/2}^{h/2}\int\limits_{-h/2}^{h/2}\left\{[2]_q f(x+t_1+t_2) - f(x+[2]_q t_1 + [2]_q t_2)\right\}d_q t_1 d_q t_2.$$

It is easy to check that

$$\int\limits_a^b D_q f(t) d_q t = f(b) - f(a).$$

Consequently

$$\left|D_q^2(g_{q,h}(x))\right| = \frac{1}{qh^2}\Big|(f(x+[2]_q h) - [2]_q f(x+h) + q f(x)) +$$

$$+(f(x-[2]_q h)-[2]_q f(x-h)+qf(x))\Big| \le \frac{2}{q\delta^2}\omega_{q,2}(f;\delta).$$

By definition of q-integral $\left|\int_{-a}^{a} h(t)d_qt\right| \le 2a\sup_{u\in\langle -a;a\rangle}|h(u)|$. Therefore

$$\big|f(x)-g_{q,h}(x)\big| =$$

$$= \frac{1}{qh^2}\left|\int\limits_{-h/2}^{h/2}\int\limits_{-h/2}^{h/2}(f(x+[2]_q(t_1+t_2))-[2]_q f(x+t_1+t_2)+qf(x))d_qt_1d_qt_2\right| \le$$

$$\le \frac{1}{q}\omega_{q,2}(f;\delta).$$

Using these inequalities and (2.1) we have

$$\|L_nf-f\| \le \|L_n(f-g_{q,h})\| + \|L_ng_{q,h}-g_{q,h}\| + \|f-g_{q,h}\| \le$$

$$\le 2\|f-g_{q,h}\| + (\mu_{2,n}(x)+(1-q)\sqrt{\mu_{2,n}(x)})\|D_q^2g_{q,h}\|.$$

Choosing $\delta_n = \sqrt{\mu_{2,n}(x)}\max\{\sqrt{\mu_{2,n}(x)};1-q\}$ we get the desired estimate.

3. *q*-Stancu polynomial. The generalized q-Bernstein polynomial defined by (1.1) can be simply expressed in terms of q-differences. For any functions f we define

$$\Delta_q^0 f_j = f_j$$

for $j=0,1,\ldots,n$ and, recursively,

$$\Delta_q^{k+1}f_j = \Delta_q^k f_{j+1} - q^k\Delta_q^k f_j,$$

for $k=0,1,\ldots,n-j-1$, where $f_j = f([j]/[n])$. It is easily established by induction that q-differences satisfy the relation

$$\Delta_q^k f_j = \sum_{i=0}^{k}(-1)^i q^{i(i-1)/2}\begin{bmatrix}k\\ i\end{bmatrix}_q f_{j+k-i}. \tag{3.1}$$

Lemma 3.1 [13]. *Let $0<q<1$, $\alpha\ge 0$. The generalized q-Bernstein polynomial may be expressed in the form*

$$B_n^{q,\alpha}(f;x) = \sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_q \Delta_q^k f_0\prod_{i=0}^{k-1}\frac{x+\alpha[i]_q}{1+\alpha[i]_q}, \tag{3.2}$$

for all $n\in N$ and $x\in[0;1]$.

Let $0<q<1$, $\alpha\ge 0$. From (3.1) and (3.2) by a simple calculation [13], we have

$$B_n^{q,\alpha}(1;x) = 1, \tag{3.3}$$

$$B_n^{q,\alpha}(t;x) = x, \tag{3.4}$$

$$B_n^{q,\alpha}(t^2;x)=\frac{1}{1+\alpha}\left(x(x+\alpha)+\frac{x(1-x)}{[n]_q}\right), \tag{3.5}$$

$$\mu_{1,n}^{q,\alpha}(x)=0, \tag{3.6}$$

$$\mu_{2,n}^{q,\alpha}(x)=\frac{x(1-x)}{1+\alpha}\left(\alpha+\frac{1}{[n]_q}\right), \tag{3.7}$$

for all $n\in N$ and $x\in[0;1]$.

Lemma 3.2. *Let* $0<q<1$, $\alpha\geq 0$. *Then*

$$B_n^{q,\alpha}(t^3;x)=\frac{1}{\prod_{i=0}^{2}(1+[i]_q\alpha)}\sum_{k=0}^{2}\frac{1}{[n]_q^k}\overline{W}_k(q,\alpha,x), \tag{3.8}$$

$$B_n^{q,\alpha}(t^4;x)=\frac{1}{\prod_{i=0}^{3}(1+[i]_q\alpha)}\sum_{k=0}^{3}\frac{1}{[n]_q^k}W_k(q,\alpha,x), \tag{3.9}$$

where

$$\overline{W}_0(q,\alpha,x)=x(x+\alpha)(x+[2]_q\alpha),$$

$$\overline{W}_1(q,\alpha,x)=x(1-x)(x+\alpha)(2+q),$$

$$\overline{W}_2(q,\alpha,x)=x(1-x)(1-[2]_qx),$$

$$W_0(q,\alpha,x)=x(x+\alpha)(x+[2]_q\alpha)(x+[3]_q\alpha),$$

$$W_1(q,\alpha,x)=x(1-x)(x+\alpha)(x+[2]_q\alpha)(q_2+2q+3),$$

$$W_2(q,\alpha,x)=x(1-x)(x+\alpha)\times$$

$$\times\Big\{(q^2+3q+3)x(x+\alpha)-[2]_q^2(x+\alpha-[2]_qx(1+[3]_q\alpha)\Big\},$$

$$W_3(q,\alpha,x)=x(1-x)\{[2]_qx([3]_qx-q-2)+1-q\alpha\}$$

for all $n\in N$ *and* $x\in[0;1]$.

Proof. The q-difference of the monomial x^k of order greater than k are zero. Consequently, we see from (3.2) that for all $n\geq k$, $B_n^{q,\alpha}(x^k;x)$ is a polynomial of degree k.

We deduce from (3.2) that

$$B_n^{q,\alpha}(t^4;x)=\sum_{k=0}^{4}\begin{bmatrix}n\\k\end{bmatrix}_q\Delta_q^kf_0\prod_{i=0}^{k-1}\frac{x+\alpha[i]_q}{1+\alpha[i]_q}.$$

From (3.1), for $f(x)=x^4$, we have

$$\Delta_q^0f_0=f_0=0,\qquad \Delta_q^1f_0=f_1-f_0=1/[n]_q^4,$$

$$\Delta_q^2 f_0 = f_2 - (1+q) f_1 + q f_0 = \frac{[2]_q q}{[n]_q^4}(q^2+3q+3),$$

$$\Delta_q^3 f_0 = f_3 - [3]_q f_2 + q[3]_q f_1 - q^3 f_0 = \frac{[2]_q [3]_q q^3}{[n]_q^4}(q^2+2q+3),$$

$$\Delta_q^4 f_0 = f_4 - [4]_q f_3 + q(1+q^2)[3]_q f_2 - q^3[4]_q f_1 + q^6 f_0 = \frac{[2]_q[3]_q[4]_q q^6}{[n]_q^4}.$$

Thanks to obvious equality $q^r[n-r]_q = [n]_q - [r]_q$, for $r = 1,2,3$, we have

$$B_n^{q,\alpha}(t^4;x) = \frac{x}{[n]_q^3} + \frac{[n]_q - 1}{[n]_q^3}(q^2+3q+3)\frac{x(x+\alpha)}{1+\alpha} +$$

$$+\frac{([n]_q-1)([n]_q-[2]_2)}{[n]_q^3}(q^2+2q+3)\frac{x(x+\alpha)(x+[2]_q\alpha)}{(1+\alpha)(1+[2]_q\alpha)} +$$

$$+\frac{([n]_q-1)([n]_q-[2]_q)([n]_q-[3]_q)}{[n]_q^3}\prod_{i=0}^{3}\frac{x+[i]_q\alpha}{1+[i]_q\alpha}.$$

By simple calulation we obtain (3.9). On the same way we can prove (3.8).

Lemma 3.3. *Let $q \in (0;1)$, $\alpha \geq 0$. Then for every $n \in N$ and $x \in [0;1]$*

$$\mu_{4,n}^{q,\alpha}(x) \leq \frac{Kx(1-x)}{1+\alpha}\frac{1}{[n]_q}\left(\frac{1}{[n]_q}+\alpha\right).$$

Proof. In view of (3.3) – (3.9)

$$\mu_{4,n}^{q,\alpha}(x) = B_n^{q,\alpha}(t^4;x) - 4xB_n^{q,\alpha}(t^3;x) + 6x^2 B_n^{q,\alpha}(t^2;x) - 3x^4 =$$

$$= \sum_{i=0}^{3}\frac{1}{[n]_q^i}V_i(q,\alpha,x),$$

where

$$V_0(q,\alpha,x) = \frac{x(x+\alpha)(x+[2]_q\alpha)(x+[3]_q\alpha)}{(1+\alpha)(1+[2]_q\alpha)(1+[3]_q\alpha)} -$$

$$-\frac{4x_2(x+\alpha)(x+[2]_q\alpha)}{(1+\alpha)(1+[2]_q\alpha)} + \frac{6x^3(x+\alpha)}{1+\alpha} - 3x^4,$$

$$V_1(q,\alpha,x) = \frac{x(1-x)(x+\alpha)(x+[2]_q\alpha)}{(1+\alpha)(1+[2]_q\alpha)(1+[3]_q\alpha)}(q^2+2q+3) -$$

$$-\frac{4x^2(1-x)(x+\alpha)}{(1+\alpha)(1+[2]_q\alpha)}(2+q) + \frac{6x^3(1-x)}{1+\alpha},$$

$$V_2(q,\alpha,x) = \frac{x(1-x)(x+\alpha)}{(1+\alpha)(1+[2]_q\alpha)(1+[3]_q\alpha)} \times$$

$$\times\Big\{(q^2+3q+3)(x+\alpha)x-[2]_q^2(x+\alpha)-[2]_qx(1+[3]_q\alpha)\Big\}-$$

$$-\frac{4x^2(1-x)(1-[2]_qx)}{(1+\alpha)(1+[2]_q\alpha)},$$

$$V_3(q,\alpha,x)=\frac{x(1-x)}{(1+\alpha)(1+[2]_q\alpha)(1+[3]_q\alpha)}\Big\{[2]_qx([3]_qx-q-2)+1-q\alpha\Big\}.$$

For $0\le x\le 1$, $0<q<1$, $\alpha>0$, we have $\alpha/(1+[2]_q\alpha)<1$, $1-q<1/[n]_q$. Thanks to that it is easy to see that

$$V_0(q,\alpha,x)=\frac{x(1-x)\alpha(1-q)^2}{(1+\alpha)(1+[2]_q\alpha)(1+[3]_q\alpha)}\le\frac{x(1-x)}{1+\alpha}\frac{1}{[n]_q^2},$$

$$|V_1(q,\alpha,x)|\le K_1\frac{x(1-x)}{1+\alpha}\left(\frac{1}{[n]_q^2}+\alpha\right),$$

$$|V_2(q,\alpha,x)|\le K_2\frac{x(1-x)}{1+\alpha},$$

$$|V_3(q,\alpha,x)|\le K_3\frac{x(1-x)}{1+\alpha}.$$

Collecting the results we get estimate (3.6) immediately.

In next theorems we assume that (q_n) and (α_n) denotes a sequence such that $0<$ $<q_n\le 1$, $\alpha_n\ge 0$ and $B_n^{q,\alpha}(f;x)$ is defined by (1.1) with $q=q_n$ and $\alpha=\alpha_n$.

Theorem 3.1. *Suppose that* $f\in C[0,1]$. *Then there exist* $\overline{q}\in(0;1)$ *such that for all* $q\in(\overline{q};1)$

$$|B_n^{q_n,\alpha_n}(f;x)-f(x)|\le 2\sqrt{\frac{x(1-x)}{1+\alpha_n}}\delta_n\omega(D_qf;\delta_n),$$

where $\delta_n=(1/[n]_{q_n}+\alpha_n)^{1/2}$ *and* $n\in N$.

Theorem 3.2. *Let* $f\colon[0,1]\to R$. *Then there exist* $\overline{q}\in(0;1)$ *such that for all* $q\in(\overline{q};1)$ *the following inequality holds*:

$$\left|B_n^{q_n,\alpha_n}(f;x)-f(x)\right|\le\frac{x\sqrt{(1-x)(1-q_nx)}}{[2]_q(1+\alpha_n)}\left(\alpha_n+\frac{1}{[n]_{q_n}}\right)\|D_q^2f\|$$

for all $n\in N$.

Theorem 3.3. *At assumptions from the previous theorem we have*

$$\left|B_n^{q_n,\alpha_n}(f;x)-f(x)-\frac{D_q^2f(x)}{[2]_q(1+\alpha_n)}x(1-x)\left(\alpha_n+\frac{1}{[n]_{q_n}}\right)\right|\le$$

$$\le Kx\sqrt{1-x}\left(\alpha_n+\frac{1}{[n]_{q_n}}\right)\omega\left(D_q^2f;\delta_n\right),$$

where $\delta_n=\max\left\{\frac{1}{[n]_{q_n}^{1/2}};\frac{1}{[n]_q}\right\}$.

It is easy to check that $\lim_{q\to 1}(D_q^2 f)(x) = df''(x)/dx$. Therefore, we have the following corollary.

***Corollary* 3.1.** *If* $f \in C^{(2)}[0;1]$ *and* $q_n \to 1$, $a_n \to 0$ *as* $n \to \infty$, *then*

$$\lim_{n\to\infty}\left(\frac{1}{[n]_{q_n}} + \alpha_n\right)^{-1}(1+\alpha_n)\{B_n^{q_n,\alpha_n}(f;x) - f(x)\} = \frac{f''(x)}{2}x(1-x)$$

uniformly on $[0;1]$.

It will be noted that, for $\alpha = 0$, thit is corollary obtain by Videnskii [27] for q-Bernstein polynomial defined by (1.3).

***Remark* 3.1.** For the function $f(t) = t^2$ takes place the exact equality

$$\left(\frac{1}{[n]_{q_n}} + \alpha_n\right)^{-1}(1+\alpha_n)\left\{B_n^{q_n,\alpha_n}(t^2;x) - x^2\right\} = \frac{(x^2)''}{2}x(1-x).$$

without passing to the limit.

Theorem 3.4. *If* f *is continuous on* $[0,1]$, *then there exist* $\overline{q} \in (0;1)$ *such that for all* $q_n \in (\overline{q};1)$ *the following inequality holds*:

$$\|B_n^{q_n,\alpha_n} f - f\| \le \frac{9}{4q}\omega_{q,2}(f;\delta_n),$$

where $\delta_n = \max\left\{(\alpha_n + 1/[n]_{q_n})^{1/2}; 1-q\right\}$.

1. *Cheney E. W.* Introduction to approximation theory. – New York: Chelsea Publ. Comp., 1982.
2. *Ernst T.* A new notation for q-calculus and a new q-Taylor formula // Upsala Univ. Rept Depart. Math. – 1999. – P. 62 – 64.
3. *Finta Z.* Direct and converse results for Stancu operator // Period. math. hung. – 2002. – **44**, № 1. – P. 1 – 16.
4. *Finta Z.* On approximation properties of Stancu's operators // Stud. Univ. BABE-SBOLYAI. Math. – 2002. – **47**, № 4. – P. 47 – 55.
5. *Gonska H. H., Meier J.* Quantitative theorems on approximation by Bernstein – Stancu operators // Calcolo. – 1984. – **21**, № 4. – P. 317 – 335.
6. *Goodman T. N. T., Oruc H., Phillips G. M.* Convexity and generalized Bernstein polynomials // Proc. Edinburgh Math. Soc. – 1999. – **42**. – P. 179 – 190.
7. *Hoschek J., Lasser D.* Fundamentals of computer aided geometric design. – Wellesley, Mass.; A. K. Peters, 1993.
8. *Il'inskii A., Ostrovska S.* Convergence of generalized Bernstein polynomials // J. Approxim. Theory. – 2002. – **116**, № 1. – P. 100 – 112.
9. *Jackson F. H.* On q-definite integrals // Quart. J. Pure and Appl. Math. – 1910. – **41**. – P. 193 – 203.
10. *Kac V., Cheung P.* Quantum calculus. – New York: Springer, 2002.
11. *Kim T.* q-Bernoulli numbers and polynomials associated with Gaussian binomial coefficients // Russ. J. Math. Phys. – 2008. – **15**. – P. 51 – 57.
12. *Lorentz G. G.* Bernstein polynomials // Math. Expo. – Toronto: Univ. Toronto Press, 1953.
13. *Nowak G.* Approximation properties for generalized q-Bernstein polynomials // J. Math. Anal. and Appl. – 2009. – **350**, № 1. – P. 50 – 55.
14. *Nowak G.* A de Casteljau algorithm for q-Bernstein – Stancu polynomials // Abstrs and Appl. Anal. – 2011. – **211**. – Article ID 609431.
15. *Phillips G. M.* Bernstein polynomials based on the q-integers // Ann. Numer. Math. – 1997. – **4**. – P. 511 – 518.
16. *Phillips G. M.* A de Casteljau algorithm for generalized Bernstein polynomials // BIT. – 1996. – **36**. – P. 232 – 236.
17. *Oruc H., Phillips G. M.* A generalization of the Bernstein polynomials // Proc. Edinburgh Math. Soc. – 1999. – **42**. – P. 403 – 413.

18. *Oruc H., Phillips G. M.* Explicit factorization of the Vandermonde matrix // Linear Algebra and Its Appl. – 2000. – **315**. – P. 113 – 123.
19. *Ostrovska S.* q-Bernstein polynomials and their iterates // J. Approxim. Theory. – 2003. – **123**, № 2. – P. 232 – 255.
20. *Ostrovska S.* On the limit q-Bernstein operators // Math. Balkan. – 2004. – **18**. – P. 165 – 172.
21. *Ostrovska S.* On the improvement of analytic properties under the limit q-Bernstein operators // J. Approxim. Theory. – 2006. – **138**, № 1. – P. 37 – 53.
22. *Pych-Taberska P.* Some approximation properties of Bernstein and Kantorovic polynomials // Funct. Approxim. – 1978. – **6**. – P. 57 – 67.
23. *Pych-Taberska P.* On the rate of pointwise convergence of Bernstein and Kantorovic polynomials // Ibid. – 1988. – **16**. – P. 63 – 76.
24. *Pych-Taberska P.* Rate of pointwise convergence of bernstein polynomials for some absolutely continuous functions // J. Math. Anal. and Appl. – 1997. – **212**. – P. 9 – 19.
25. *Rajkovic P. M., Stankovic M. S., Marinkovic S. D.* Mean value theorems in q-calculus // Math. Vesnic. – 2002. – **54**. – P. 171 – 178.
26. *Stancu D. D.* Approximation of functions by a new class of linear polynomial operators // Rev. roum. math. pures et appl. – 1968. – **13**, № 8. – P. 1173 – 1194.
27. *Videnskii V. S.* On some classes of q-parametric positive operators // Operator Theory: Adv. Appl. – 2005. – **158**. – P. 213 – 222.
28. *Wang H., Meng F.* The rate of convergence of q-Bernstein polynomials for $0 < q < 1$ // J. Approxim. Theory. – 2005. – **136**, № 2. – P. 151 – 158.
29. *Wang H.* Korovkin-type theorem and application // Ibid. – 2005. – **132**, № 2. – P. 258 – 264.
30. *Wang H.* Voronovskaya-type formulas and saturation of convergence for q-Bernstein polynomials for $0 < q < 1$ // Ibid. – 2007. – **145**. – P. 182 – 195.

Received 20.02.10