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A GENERALIZED MIXED TYPE OF QUARTIC, CUBIC, QUADRATIC AND ADDITIVE FUNCTIONAL EQUATION

УЗАГАЛЬНЕНИЙ МІШАНИЙ ТИП КВАРТИЧНОГО, КУБІЧНОГО, КВАДРАТИЧНОГО ТА ДОДАТКОВОГО ФУНКЦІОНАЛЬНОГО РІВНЯННЯ

We determine the general solution of the functional equation $f(x+ky)+f(x-ky)=g(x+y)+g(x-y)+ + h(x)+\tilde{h}(y)$ for fixed integers k with $k\neq 0,\pm 1$ without assuming any regularity condition on the unknown functions $f,\ g,\ h,\ \tilde{h}$. The method used for solving these functional equations is elementary but exploits an important result due to Hosszú. The solution of this functional equation can also be determined in certain type of groups using two important results due to Székelyhidi.

Визначено загальний розв'язок функціонального рівняння $f(x+ky)+f(x-ky)=g(x+y)+ + g(x-y)+h(x)+\tilde{h}(y)$ для фіксованих цілих k при $k\neq 0,\pm 1$ без припущення наявності будь-якої умови регулярності для невідомих функцій $f,\ g,\ h,\ \tilde{h}$. Метод, що використано для розв'язку цих функціональних рівнянь, елементарний, але базується на важливому результаті Хозу. Розв'язок цього функціонального рівняння може бути визначений у певному типі груп з використанням двох важливих результатів Чекеліхіді.

1. Introduction and preliminaries. J. M. Rassias [11] (in 2001) introduced the first cubic functional equation

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y) \tag{1.1}$$

and established the solution of the Ulam – Hyers stability problem for this cubic functional equation. Since the function $f(x)=x^3$ satisfies the functional equation (1.1), this equation is called cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function.

J. M. Rassias [12] (in 1999) introduced the first quartic functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4[f(x+y)-f(x-y)+6f(y)]. \tag{1.2}$$

and established the solution of the Ulam – Hyers stability problem for the quartic functional equation. Since the function $f(x)=x^4$ satisfies the functional equation (1.2), this equation is called quartic functional equation. J. K. Chung and P. K. Sahoo [2] determined the general solution of the functional equation (1.2).

M. Eshaghi Gordji and H. Khodaei [5] (in 2009) introduced the following generalized mixed type of cubic, quadratic and additive functional equation

$$f(x+ky)+f(x-ky)=k^2f(x+y)+k^2f(x-y)+2(1-k^2)f(y) \tag{1.3}$$

and established the general solution and the generalized Ulam – Hyers stability for the functional equation (1.3). They proved that a function f with $f(0)=0$ between real

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vector spaces X and Y is a solution of (1.3) if and only if there exist functions $C\colon X^3 \to Y$ and $B\colon X^2 \to Y$ and $A\colon X \to Y$, such that $f(x) = C(x,x,x) + B(x,x) + A(x)$ for all $x \in X$, where the function C is symmetric for each fixed variable and is additive for fixed two variables and B is symmetric bi-additive and A is additive. In this paper, we determine the general solution of the functional equation (1.3) using an elementary technique but without assuming $f(0) = 0$.

M. Eshaghi Gordji, S. Kaboli-Gharetapeh, C. Park, and S. Zolfaghari [7] introduced an additive-cubic-quartic functional equation

$$11[f(x+2y) + f(x-2y)] =$$

$$= 44[f(x+y) + f(x-y)] + 12f(3y) - 48f(2y) + 60f(y) - 66f(x) \qquad (1.4)$$

and established the general solution and the generalized Ulam – Hyers stability for the functional equation (1.4). In [8], M. Eshaghi Gordji, H. Khodaei and Th. M. Rassias introduced a generalized mixed type of quartic, cubic, quadratic and additive functional equation

$$f(x+ky) + f(x-ky) = k^2 f(x+y) + k^2 f(x-y) +$$

$$+2(1-k^2)f(x) + \frac{k^2(k^2-1)}{12}[\tilde{f}(2y) - 4\tilde{f}(y)], \qquad (1.5)$$

where $\tilde{f}(y) = f(y) + f(-y)$ for all $y \in X$. They proved that a function f between real vector spaces X and Y is a solution of (1.5) if and only if there exist a symmetric multi-additive function $M\colon X^4 \to Y$, a function $C\colon X^3 \to Y$, a symmetric bi-additive function $B\colon X^2 \to Y$ and an additive function $A\colon X \to Y$, such that $f(x) = M(x,x,x,x) + C(x,x,x) + B(x,x) + A(x)$ for all $x \in X$, where the function C is symmetric for each fixed variable and is additive for two fixed variables. In this paper, we also determine the general solution of the functional equation (1.5) using an elementary technique.

Let k be a fixed integer with $k \neq 0, \pm 1$, X and Y are real vector spaces. Equations (1.2) – (1.5) can be generalized to

$$f(x+ky) + f(x-ky) = g(x+y) + g(x-y) + h(x) + \tilde{h}(y) \qquad (1.6)$$

for all $x,\, y \in X$, where $f,\, g,\, h,\, \tilde{h}\colon X \to Y$ are unknown functions to be determined. In this paper, we determine the general solution of the functional equation (1.6) and some other related functional equations. We will first solve these functional equations using an elementary technique [2, 14, 15, 22, 23] but without using any regularity condition on the unknown functions. The motivation for studying these functional equations came from the fact that recently polynomial equations have found applications in approximate checking, self-testing, and self-correcting of computer programs that compute polynomials. The interested reader should refer to [4] and [13] and references therein (see also [17 – 21]).

A function $A\colon X \to Y$ is said to be additive if $A(x+y) = A(x) + A(y)$ for all $x, y \in X$. It is easy to see that $A(rx) = rA(x)$ for all $x \in X$ and all $r \in \mathbb{Q}$ (the set of rational numbers).

Let $n \in \mathbb{N}$ (the set of natural numbers). A function $A_n\colon X^n \to Y$ is called n-additive if it is additive in each of its variables. A function A_n is called symmetric if

$A_n(x_1, x_2, \ldots, x_n) = A_n(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$ for every permutation $\{\pi(1), \pi(2), \ldots, \pi(n)\}$ of $\{1, 2, \ldots, n\}$. If $A_n(x_1, x_2, \ldots, x_n)$ is an n-additive symmetric map, then $A^n(x)$ will denote the diagonal $A_n(x, x, \ldots, x)$ for $x \in X$ and note that $A^n(rx) = r^n A^n(x)$ whenever $x \in X$ and $r \in \mathbb{Q}$. Such a function $A^n(x)$ will be called a monomial function of degree n (assuming $A^n \not\equiv 0$). Furthermore the resulting function after substitution $x_1 = x_2 = \ldots = x_l = x$ and $x_{l+1} = x_{l+2} = \ldots = x_n = y$ in $A_n(x_1, x_2, \ldots, x_n)$ will be denoted by $A^{l,n-l}(x, y)$.

A function $p\colon X \to Y$ is called a generalized polynomial (GP) function of degree $n \in \mathbb{N}$ provided that there exist $A^0(x) = A^0 \in Y$ and i-additive symmetric functions $A_i\colon X^i \to Y$ (for $1 \leq i \leq n$) such that

$$p(x) = \sum_{i=0}^{n} A^i(x), \quad \text{for all} \quad x \in X$$

and $A^n \not\equiv 0$.

For $f\colon X \to Y$, let $\triangle_h$ be the difference operator defined as follows:

$$\triangle_h f(x) = f(x + h) - f(x)$$

for $h \in X$. Furthermore, let $\triangle_h^0 f(x) = f(x)$, $\triangle_h^1 f(x) = \triangle_h f(x)$ and $\triangle_h \circ \triangle_h^n f(x) = \triangle_h^{n+1} f(x)$ for all $n \in \mathbb{N}$ and all $h \in X$. Here $\triangle_h \circ \triangle_h^n$ denotes the composition of the operators $\triangle_h$ and $\triangle_h^n$. For any given $n \in \mathbb{N}$, the functional equation $\triangle_h^{n+1} f(x) = 0$ for all $x, h \in X$ is well studied. In explicit form the last functional equation can be written as

$$\triangle_h^{n+1} f(x) = \sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} f(x + jh) = 0.$$

The following theorem was proved by Mazur and Orlicz, and in greater generality by Djoković (see [3]).

Theorem 1.1. *Let X and Y be real vector spaces, $n \in \mathbb{N}$ and $f\colon X \to Y$, then the following are equivalent*:

(1) $\triangle_h^{n+1} f(x) = 0$ *for all* x, $h \in X$.

(2) $\triangle_{x_1, \ldots, x_{n+1}} f(x_0) = 0$ *for all* $x_0, x_1, \ldots, x_{n+1} \in X$.

(3) $f(x) = A^n(x) + A^{n-1}(x) + A^2(x) + A^1(x) + A^0(x)$ *for all* $x \in X$, *where* $A^0(x) = A^0$ *is an arbitrary element of* Y *and* $A^i(x)$, $i = 1, 2, \ldots, n$, *is the diagonal of an* i*-additive symmetric function* $A_i\colon X^i \to Y$.

2. Solution of equation (1.6) on real vector spaces. In this section, we determine the general solution of the functional equation (1.6) and some other related equations without assuming any regularity condition on the unknown functions.

Theorem 2.1. *Let X and Y be real vector spaces. If the functions* f, g, h, $\tilde{h}\colon X \to Y$ *satisfy the functional equation*

$$f(x + ky) + f(x - ky) = g(x + y) + g(x - y) + h(x) + \tilde{h}(y), \quad \textit{for all} \quad x, y \in X \tag{2.1}$$

for fixed integers k *with* $k \neq 0, \pm 1$, *then* f *is a solution of the Fréchet functional equation* $\triangle_{x_1, x_2, x_3, x_4, x_5} f(x_0) = 0$ *for all* $x_0, x_1, x_2, x_3, x_4, x_5 \in X$.

Proof. Replacing $x+ky$ by x_0 and $x-ky$ by y_1 $\left(\text{that is, } x=\frac{1}{2}x_0+\frac{1}{2}y_1 \text{ and } y=\frac{1}{2k}x_0-\frac{1}{2k}y_1\right)$ in (2.1), respectively, we get

$$f(x_0)+f(y_1)=g\left(\frac{k+1}{2k}x_0+\frac{k-1}{2k}y_1\right)+$$

$$+g\left(\frac{k-1}{2k}x_0+\frac{k+1}{2k}y_1\right)+h\left(\frac{1}{2}x_0+\frac{1}{2}y_1\right)+\tilde{h}\left(\frac{1}{2k}x_0-\frac{1}{2k}y_1\right). \tag{2.2}$$

Replacing x_0 by x_0+x_1 in (2.2), we have

$$f(x_0+x_1)+f(y_1)=g\left(\frac{k+1}{2k}(x_0+x_1)+\frac{k-1}{2k}y_1\right)+$$

$$+g\left(\frac{k-1}{2k}(x_0+x_1)+\frac{k+1}{2k}y_1\right)+h\left(\frac{1}{2}(x_0+x_1)+\frac{1}{2}y_1\right)+$$

$$+\tilde{h}\left(\frac{1}{2k}(x_0+x_1)-\frac{1}{2k}y_1\right). \tag{2.3}$$

Subtracting (2.2) from (2.3), we get

$$f(x_0+x_1)-f(x_0)=g\left(\frac{k+1}{2k}(x_0+x_1)+\frac{k-1}{2k}y_1\right)+$$

$$+g\left(\frac{k-1}{2k}(x_0+x_1)+\frac{k+1}{2k}y_1\right)-g\left(\frac{k+1}{2k}x_0+\frac{k-1}{2k}y_1\right)-$$

$$-g\left(\frac{k-1}{2k}x_0+\frac{k+1}{2k}y_1\right)+h\left(\frac{1}{2}(x_0+x_1)+\frac{1}{2}y_1\right)-h\left(\frac{1}{2}x_0+\frac{1}{2}y_1\right)+$$

$$+\tilde{h}\left(\frac{1}{2k}(x_0+x_1)-\frac{1}{2k}y_1\right)-\tilde{h}\left(\frac{1}{2k}x_0-\frac{1}{2k}y_1\right). \tag{2.4}$$

Letting $y_2=\frac{1}{2}x_0+\frac{1}{2}y_1$ (that is, $y_1=2y_2-x_0$) in (2.4), we have

$$f(x_0+x_1)-f(x_0)=g\left(\frac{1}{k}x_0+\frac{k+1}{2k}x_1+\frac{k-1}{k}y_2\right)+$$

$$+g\left(-\frac{1}{k}x_0+\frac{k-1}{2k}x_1+\frac{k+1}{k}y_2\right)-g\left(\frac{1}{k}x_0+\frac{k-1}{k}y_2\right)-$$

$$-g\left(-\frac{1}{k}x_0+\frac{k+1}{k}y_2\right)+h\left(\frac{1}{2}x_1+y_2\right)-$$

$$-h(y_2)+\tilde{h}\left(\frac{1}{k}x_0+\frac{1}{2k}x_1-\frac{1}{k}y_2\right)-\tilde{h}\left(\frac{1}{k}x_0-\frac{1}{k}y_2\right). \tag{2.5}$$

Replacing x_0 by $x_0 + x_2$ in (2.5), we get

$$f(x_0 + x_1) - f(x_0) = g\left(\frac{1}{k}(x_0 + x_2) + \frac{k+1}{2k}x_1 + \frac{k-1}{k}y_2\right) +$$
$$+g\left(-\frac{1}{k}(x_0 + x_2) + \frac{k-1}{2k}x_1 + \frac{k+1}{k}y_2\right) -$$
$$-g\left(\frac{1}{k}(x_0 + x_2) + \frac{k-1}{k}y_2\right) - g\left(-\frac{1}{k}(x_0 + x_2) + \frac{k+1}{k}y_2\right) + h\left(\frac{1}{2}x_1 + y_2\right) -$$
$$-h(y_2) + \tilde{h}\left(\frac{1}{k}(x_0 + x_2) + \frac{1}{2k}x_1 - \frac{1}{k}y_2\right) - \tilde{h}\left(\frac{1}{k}(x_0 + x_2) - \frac{1}{k}y_2\right). \tag{2.6}$$

Subtracting (2.5) from (2.6), we get

$$f(x_0 + x_1 + x_2) - f(x_0 + x_1) - f(x_0 + x_2) + f(x_0) =$$
$$= g\left(\frac{1}{k}x_0 + \frac{k+1}{2k}x_1 + \frac{1}{k}x_2 + \frac{k-1}{k}y_2\right) +$$
$$+g\left(-\frac{1}{k}x_0 + \frac{k-1}{2k}x_1 - \frac{1}{k}x_2 + \frac{k+1}{k}y_2\right) -$$
$$-g\left(\frac{1}{k}x_0 + \frac{1}{k}x_2 + \frac{k-1}{k}y_2\right) - g\left(-\frac{1}{k}x_0 - \frac{1}{k}x_2 + \frac{k+1}{k}y_2\right) -$$
$$-g\left(\frac{1}{k}x_0 + \frac{k+1}{2k}x_1 + \frac{k-1}{k}y_2\right) - g\left(-\frac{1}{k}x_0 + \frac{k-1}{2k}x_1 + \frac{k+1}{k}y_2\right) +$$
$$+g\left(\frac{1}{k}x_0 + \frac{k-1}{k}y_2\right) + g\left(-\frac{1}{k}x_0 + \frac{k+1}{k}y_2\right) +$$
$$+\tilde{h}\left(\frac{1}{k}x_0 + \frac{1}{2k}x_1 + \frac{1}{k}x_2 - \frac{1}{k}y_2\right) - \tilde{h}\left(\frac{1}{k}x_0 + \frac{1}{k}x_2 - \frac{1}{k}y_2\right) -$$
$$-\tilde{h}\left(\frac{1}{k}x_0 + \frac{1}{2k}x_1 - \frac{1}{k}y_2\right) + \tilde{h}\left(\frac{1}{k}x_0 - \frac{1}{k}y_2\right). \tag{2.7}$$

Letting $y_3 = \frac{1}{k}x_0 - \frac{1}{k}y_2$ (that is, $y_2 = x_0 - ky_3$) in (2.7), we have

$$f(x_0 + x_1 + x_2) - f(x_0 + x_1) - f(x_0 + x_2) + f(x_0) =$$
$$= g\left(x_0 + \frac{k+1}{2k}x_1 + \frac{1}{k}x_2 - (k-1)y_3\right) +$$
$$+g\left(x_0 + \frac{k-1}{2k}x_1 - \frac{1}{k}x_2 - (k+1)y_3\right) - g\left(x_0 + \frac{1}{k}x_2 - (k-1)y_3\right) -$$

$$-g\left(x_0-\frac{1}{k}x_2-(k+1)y_3\right)-g\left(x_0+\frac{k+1}{2k}x_1-(k-1)y_3\right)-$$

$$-g\left(x_0+\frac{k-1}{2k}x_1-(k+1)y_3\right)+g(x_0-(k-1)y_3)+g(x_0-(k+1)y_3)+$$

$$+\tilde{h}\left(\frac{1}{2k}x_1+\frac{1}{k}x_2+y_3\right)-\tilde{h}\left(\frac{1}{k}x_2+y_3\right)-\tilde{h}\left(\frac{1}{2k}x_1+y_3\right)+\tilde{h}(y_3). \qquad (2.8)$$

Again replacing x_0 by x_0+x_3 in (2.8) and subtracting (2.8) from the resulting expression, we get

$$f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2)-f(x_0+x_1+x_3)-$$

$$-f(x_0+x_2+x_3)+f(x_0+x_3)+f(x_0+x_1)+f(x_0+x_2)-f(x_0)=$$

$$=g\left(x_0+\frac{k+1}{2k}x_1+\frac{1}{k}x_2+x_3-(k-1)y_3\right)+$$

$$+g\left(x_0+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+x_3-(k+1)y_3\right)-$$

$$-g\left(x_0+\frac{1}{k}x_2+x_3-(k-1)y_3\right)-g\left(x_0-\frac{1}{k}x_2+x_3-(k+1)y_3\right)-$$

$$-g\left(x_0+\frac{k+1}{2k}x_1+x_3-(k-1)y_3\right)-g\left(x_0+\frac{k-1}{2k}x_1+x_3-(k+1)y_3\right)+$$

$$+g(x_0+x_3-(k-1)y_3)+g(x_0+x_3-(k+1)y_3)-$$

$$-g\left(x_0+\frac{k+1}{2k}x_1+\frac{1}{k}x_2-(k-1)y_3\right)-g\left(x_0+\frac{k-1}{2k}x_1-\frac{1}{k}x_2-(k+1)y_3\right)+$$

$$+g\left(x_0+\frac{1}{k}x_2-(k-1)y_3\right)-g\left(x_0-\frac{1}{k}x_2-(k+1)y_3\right)+$$

$$+g\left(x_0+\frac{k+1}{2k}x_1-(k-1)y_3\right)-g\left(x_0+\frac{k-1}{2k}x_1-(k+1)y_3\right)-$$

$$-g(x_0-(k-1)y_3)+g(x_0-(k+1)y_3). \qquad (2.9)$$

Putting $y_4=x_0-(k-1)y_3$ $\left(\text{that is, } y_3=\frac{1}{k-1}x_0-\frac{1}{k-1}y_4\right)$ in (2.9), we get

$$f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2)-$$

$$-f(x_0+x_1+x_3)-f(x_0+x_2+x_3)+$$

$$+f(x_0+x_3)+f(x_0+x_1)+f(x_0+x_2)-f(x_0)=$$

$$= g\left(\frac{k+1}{2k}x_1 + \frac{1}{k}x_2 + x_3 + y_4\right) +$$

$$+g\left(\frac{-2}{k-1}x_0 + \frac{k-1}{2k}x_1 - \frac{1}{k}x_2 + x_3 + \frac{k+1}{k-1}y_4\right) -$$

$$-g\left(\frac{1}{k}x_2 + x_3 + y_4\right) - g\left(\frac{-2}{k-1}x_0 - \frac{1}{k}x_2 + x_3 + \frac{k+1}{k-1}y_4\right) -$$

$$-g\left(\frac{k+1}{2k}x_1 + x_3 + y_4\right) - g\left(\frac{-2}{k-1}x_0 + \frac{k-1}{2k}x_1 + x_3 + \frac{k+1}{k-1}y_4\right) +$$

$$+g(x_3 + y_4) + g\left(\frac{-2}{k-1}x_0 + x_3 + \frac{k+1}{k-1}y_4\right) -$$

$$-g\left(\frac{k+1}{2k}x_1 + \frac{1}{k}x_2 + y_4\right) - g\left(\frac{-2}{k-1}x_0 + \frac{k-1}{2k}x_1 - \frac{1}{k}x_2 + \frac{k+1}{k-1}y_4\right) +$$

$$+g\left(\frac{1}{k}x_2 + y_4\right) - g\left(\frac{-2}{k-1}x_0 - \frac{1}{k}x_2 + \frac{k+1}{k-1}y_4\right) + g\left(\frac{k+1}{2k}x_1 + y_4\right) -$$

$$-g\left(\frac{-2}{k-1}x_0 + \frac{k-1}{2k}x_1 + \frac{k+1}{k-1}y_4\right) - g(y_4) + g\left(\frac{-2}{k-1}x_0 + \frac{k+1}{k-1}y_4\right). \quad (2.10)$$

Replacing x_0 by $x_0 + x_4$ in (2.10) to get

$$f(x_0 + x_1 + x_2 + x_3 + x_4) - f(x_0 + x_1 + x_2 + x_4) -$$

$$-f(x_0 + x_1 + x_3 + x_4) - f(x_0 + x_4) - f(x_0 + x_2 + x_3 + x_4) +$$

$$+f(x_0 + x_3 + x_4) + f(x_0 + x_1 + x_4) + f(x_0 + x_2 + x_4) =$$

$$= g\left(\frac{k+1}{2k}x_1 + \frac{1}{k}x_2 + x_3 + y_4\right) +$$

$$+g\left(\frac{-2}{k-1}(x_0 + x_4) + \frac{k-1}{2k}x_1 - \frac{1}{k}x_2 + x_3 + \frac{k+1}{k-1}y_4\right) -$$

$$-g\left(\frac{1}{k}x_2 + x_3 + y_4\right) - g\left(\frac{-2}{k-1}(x_0 + x_4) - \frac{1}{k}x_2 + x_3 + \frac{k+1}{k-1}y_4\right) -$$

$$-g\left(\frac{k+1}{2k}x_1 + x_3 + y_4\right) - g\left(\frac{-2}{k-1}(x_0 + x_4) + \frac{k-1}{2k}x_1 + x_3 + \frac{k+1}{k-1}y_4\right) +$$

$$+g(x_3 + y_4) + g\left(\frac{-2}{k-1}(x_0 + x_4) + x_3 + \frac{k+1}{k-1}y_4\right) - g\left(\frac{k+1}{2k}x_1 + \frac{1}{k}x_2 + y_4\right) -$$

$$-g\left(\frac{-2}{k-1}(x_0 + x_4) + \frac{k-1}{2k}x_1 - \frac{1}{k}x_2 + \frac{k+1}{k-1}y_4\right) +$$

$$+g\left(\frac{1}{k}x_2+y_4\right)-g\left(\frac{-2}{k-1}(x_0+x_4)-\frac{1}{k}x_2+\frac{k+1}{k-1}y_4\right)+$$

$$+g\left(\frac{k+1}{2k}x_1+y_4\right)-g\left(\frac{-2}{k-1}(x_0+x_4)+\frac{k-1}{2k}x_1+\frac{k+1}{k-1}y_4\right)-$$

$$-g(y_4)+g\left(\frac{-2}{k-1}(x_0+x_4)+\frac{k+1}{k-1}y_4\right). \qquad (2.11)$$

Subtract (2.10) from (2.11), we get

$$f(x_0+x_1+x_2+x_3+x_4)-f(x_0+x_1+x_2+x_3)-$$

$$-f(x_0+x_1+x_2+x_4)-f(x_0+x_1+x_3+x_4)-$$

$$-f(x_0+x_2+x_3+x_4)+f(x_0+x_1+x_2)+f(x_0+x_1+x_3)+$$

$$+f(x_0+x_2+x_3)+f(x_0+x_3+x_4)+f(x_0+x_1+x_4)+f(x_0+x_2+x_4)-$$

$$-f(x_0+x_1)-f(x_0+x_2)-f(x_0+x_3)-f(x_0+x_4)+f(x_0)=$$

$$=g\left(\frac{-2}{k-1}(x_0+x_4)+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+x_3+\frac{k+1}{k-1}y_4\right)-$$

$$-g\left(\frac{-2}{k-1}(x_0+x_4)-\frac{1}{k}x_2+x_3+\frac{k+1}{k-1}y_4\right)-$$

$$-g\left(\frac{-2}{k-1}(x_0+x_4)+\frac{k-1}{2k}x_1+x_3+\frac{k+1}{k-1}y_4\right)+$$

$$+g\left(\frac{-2}{k-1}(x_0+x_4)+x_3+\frac{k+1}{k-1}y_4\right)-$$

$$-g\left(\frac{-2}{k-1}(x_0+x_4)+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+\frac{k+1}{k-1}y_4\right)-$$

$$-g\left(\frac{-2}{k-1}(x_0+x_4)-\frac{1}{k}x_2+\frac{k+1}{k-1}y_4\right)-$$

$$-g\left(\frac{-2}{k-1}(x_0+x_4)+\frac{k-1}{2k}x_1+\frac{k+1}{k-1}y_4\right)+g\left(\frac{-2}{k-1}(x_0+x_4)+\frac{k+1}{k-1}y_4\right)-$$

$$-g\left(\frac{-2}{k-1}x_0+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+x_3+\frac{k+1}{k-1}y_4\right)+$$

$$+g\left(\frac{-2}{k-1}x_0-\frac{1}{k}x_2+x_3+\frac{k+1}{k-1}y_4\right)+$$

$$+g\left(\frac{-2}{k-1}x_0+\frac{k-1}{2k}x_1+x_3+\frac{k+1}{k-1}y_4\right)-g\left(\frac{-2}{k-1}x_0+x_3+\frac{k+1}{k-1}y_4\right)+$$

$$+g\left(\frac{-2}{k-1}x_0+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+\frac{k+1}{k-1}y_4\right)+g\left(\frac{-2}{k-1}x_0-\frac{1}{k}x_2+\frac{k+1}{k-1}y_4\right)+$$

$$+g\left(\frac{-2}{k-1}x_0+\frac{k-1}{2k}x_1+\frac{k+1}{k-1}y_4\right)-g\left(\frac{-2}{k-1}x_0+\frac{k+1}{k-1}y_4\right). \tag{2.12}$$

Setting $y_5=\dfrac{-2}{k-1}x_0+\dfrac{k+1}{k-1}y_4$ $\left(\text{that is, } y_4=\dfrac{2}{k+1}x_0+\dfrac{k-1}{k+1}y_5\right)$ in (2.12), we have

$$f(x_0+x_1+x_2+x_3+x_4)-f(x_0+x_1+x_2+x_3)-$$

$$-f(x_0+x_1+x_2+x_4)-f(x_0+x_1+x_3+x_4)-f(x_0+x_2+x_3+x_4)+$$

$$+f(x_0+x_1+x_2)+f(x_0+x_1+x_3)+f(x_0+x_2+x_3)+f(x_0+x_3+x_4)+$$

$$+f(x_0+x_1+x_4)+f(x_0+x_2+x_4)-f(x_0+x_1)-$$

$$-f(x_0+x_2)-f(x_0+x_3)-f(x_0+x_4)+f(x_0)=$$

$$=g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+x_3+y_5\right)-g\left(\frac{-2}{k-1}x_4-\frac{1}{k}x_2+x_3+y_5\right)-$$

$$-g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1+x_3+y_5\right)+g\left(\frac{-2}{k-1}x_4+x_3+y_5\right)-$$

$$-g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+y_5\right)-g\left(\frac{-2}{k-1}x_4-\frac{1}{k}x_2+y_5\right)-$$

$$-g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1+y_5\right)+g\left(\frac{-2}{k-1}x_4+y_5\right)+g\left(-\frac{1}{k}x_2+x_3+y_5\right)-$$

$$-g\left(\frac{k-1}{2k}x_1-\frac{1}{k}x_2+x_3+y_5\right)+g\left(\frac{k-1}{2k}x_1+x_3+y_5)-g(x_3+y_5\right)+$$

$$+g\left(\frac{k-1}{2k}x_1-\frac{1}{k}x_2+y_5\right)+g\left(-\frac{1}{k}x_2+y_5\right)+g\left(\frac{k-1}{2k}x_1+y_5\right)-g(y_5). \tag{2.13}$$

Replacing x_0 by x_0+x_5 in (2.13) to get

$$f(x_0+x_1+x_2+x_3+x_4+x_5)-f(x_0+x_1+x_2+x_3+x_5)-$$

$$-f(x_0+x_1+x_2+x_4+x_5)-f(x_0+x_1+x_3+x_4+x_5)-$$

$$-f(x_0+x_2+x_3+x_4+x_5)+f(x_0+x_1+x_2+x_5)+$$

$$+f(x_0+x_1+x_3+x_5)+f(x_0+x_2+x_3+x_5)+f(x_0+x_3+x_4+x_5)+$$

$$+f(x_0+x_1+x_4+x_5)+f(x_0+x_2+x_4+x_5)-f(x_0+x_1+x_5)-$$

$$-f(x_0+x_2+x_5)-f(x_0+x_3+x_5)-f(x_0+x_4+x_5)+f(x_0+x_5)=$$

$$=g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+x_3+y_5\right)-g\left(\frac{-2}{k-1}x_4-\frac{1}{k}x_2+x_3+y_5\right)-$$

$$-g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1+x_3+y_5\right)+g\left(\frac{-2}{k-1}x_4+x_3+y_5\right)-$$

$$-g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1-\frac{1}{k}x_2+y_5\right)-g\left(\frac{-2}{k-1}x_4-\frac{1}{k}x_2+y_5\right)-$$

$$-g\left(\frac{-2}{k-1}x_4+\frac{k-1}{2k}x_1+y_5\right)+g\left(\frac{-2}{k-1}x_4+y_5\right)+g\left(-\frac{1}{k}x_2+x_3+y_5\right)-$$

$$-g\left(\frac{k-1}{2k}x_1-\frac{1}{k}x_2+x_3+y_5\right)+g\left(\frac{k-1}{2k}x_1+x_3+y_5\right)-g(x_3+y_5)+$$

$$+g\left(\frac{k-1}{2k}x_1-\frac{1}{k}x_2+y_5\right)+g\left(-\frac{1}{k}x_2+y_5\right)+g\left(\frac{k-1}{2k}x_1+y_5\right)-g(y_5). \tag{2.14}$$

Subtract (2.13) from (2.14), we get

$$f(x_0+x_1+x_2+x_3+x_4+x_5)-f(x_0+x_1+x_2+x_3+x_4)-$$

$$-f(x_0+x_1+x_2+x_3+x_5)-f(x_0+x_1+x_2+x_4+x_5)-$$

$$-f(x_0+x_1+x_3+x_4+x_5)-f(x_0+x_2+x_3+x_4+x_5)+$$

$$+f(x_0+x_1+x_2+x_3)+f(x_0+x_1+x_2+x_4)+f(x_0+x_1+x_3+x_4)+$$

$$+f(x_0+x_2+x_3+x_4)+f(x_0+x_1+x_2+x_5)+f(x_0+x_1+x_3+x_5)+$$

$$+f(x_0+x_2+x_3+x_5)+f(x_0+x_3+x_4+x_5)+f(x_0+x_1+x_4+x_5)+$$

$$+f(x_0+x_2+x_4+x_5)-f(x_0+x_1+x_5)-f(x_0+x_2+x_5)-$$

$$-f(x_0+x_3+x_5)-f(x_0+x_4+x_5)-f(x_0+x_1+x_2)-$$

$$-f(x_0+x_1+x_3)-f(x_0+x_2+x_3)-f(x_0+x_3+x_4)-$$

$$-f(x_0+x_1+x_4)-f(x_0+x_2+x_4)+f(x_0+x_5)+$$

$$+f(x_0+x_1)+f(x_0+x_2)+f(x_0+x_3)+f(x_0+x_4)-f(x_0)=0$$

which is $\triangle_{x_1,x_2,x_3,x_4,x_5}f(x_0)=0$ for all $x_0,x_1,x_2,x_3,x_4,x_5\in X$.

Theorem 2.1 is proved.

As an application of Theorem 2.1, we can get the following theorem which is a further improvement of the Theorem 2.3 in [5].

Theorem 2.2. *Let X and Y be real vector spaces, then the function $f\colon X \to Y$ satisfies the functional equation* (1.3) *for all $x, y \in X$ if and only if f is of the form $f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x)$ $(\forall x \in X)$, where $A^0(x) = A^0$ is an arbitrary element of Y, $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon X^i \to Y$ for $i = 1, 2, 3$.*

Proof. By Theorems 2.1 and 1.1, we have

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \quad x \in X, \tag{2.15}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon X^i \to Y$ for $i = 1, 2, 3, 4$. Putting (2.15) into (1.3), and noting that

$$A^4(x+y) + A^4(x-y) = 2A^4(x) + 2A^4(y) + 12A^{2,2}(x,y),$$

$$A^3(x+y) + A^3(x-y) = 2A^3(x) + 6A^{1,2}(x,y),$$

$$A^2(x+y) + A^2(x-y) = 2A^2(x) + 2A^2(y),$$

and $A^{2,2}(x, ky) = k^2A^{2,2}(x,y)$, $A^{1,2}(x, ky) = k^2A^{1,2}(x,y)$, we conclude that $A^4(y) = 0$ for all $y \in X$. Therefore

$$f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x)$$

for all $x \in X$. The converse is easily verified.

Theorem 2.2 is proved.

Using the techniques are similar to that of Theorem 2.2, we have the following results (Theorems 2.3 – 2.6).

Theorem 2.3 ([6], Theorem 2.1)**.** *Let X and Y be real vector spaces, then the function $f\colon X \to Y$ satisfies the functional equation*

$$3[f(x+2y) + f(x-2y)] =$$

$$= 12[f(x+y) + f(x-y)] + 4f(3y) - 18f(2y) + 36f(y) - 18f(x) \tag{2.16}$$

for all $x, y \in X$ if and only if f is of the form $f(x) = A^4(x) + A^3(x) + A^2(x)$ $(\forall x \in X)$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon X^i \to Y$ for $i = 2, 3, 4$.

Theorem 2.4 ([7], Theorem 2.4)**.** *Let X and Y be real vector spaces, then the function $f\colon X \to Y$ satisfies the functional equation* (1.4) *for all $x, y \in X$ if and only if f is of the form $f(x) = A^4(x) + A^3(x) + A^1(x)$ $(\forall x \in X)$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon X^i \to Y$ for $i = 1, 3, 4$.*

Theorem 2.5 ([2], Theorem 3.1)**.** *Let X and Y be real vector spaces, then the function $f\colon X \to Y$ satisfies the functional equation* (1.2) *for all $x, y \in X$ if and only if f is of the form $f(x) = A^4(x)$ $(\forall x \in X)$, where $A^4(x)$ is the diagonal of the 4-additive symmetric map $A_4 : X^4 \to Y$.*

Theorem 2.6 ([8], Theorem 2.3)**.** *Let X and Y be real vector spaces, then the function $f\colon X \to Y$ satisfies the functional equation* (1.5) *for all $y \in X$ if and only if f is of the form $f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x)$ $(\forall x \in X)$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon X^i \to Y$ for $i = 1, 2, 3, 4$.*

Theorem 2.7. *Let X and Y be real vector spaces, then the functions f, g, h, $\tilde{h}$: $X \to Y$ satisfy the functional equation* (1.6) *for all $x, y \in X$ if and only if*

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x),$$

$$g(x) = k^2A^4(x) + k^2A^3(x) + B^2(x) + B^0(x) + C^1(x) + D^0(x),$$

$$h(x) = (2 - 2k^2)A^4(x) + (2 - 2k^2)A^3(x) + 2A^2(x) + 2A^1(x) + 2A^0(x)- \tag{2.17}$$

$$-2B^2(x) - 2C^1(x) - 2B^0(x),$$

$$\tilde{h}(x) = (2k^4 - 2k^2)A^4(x) + 2k^2A^2(x) - 2B^2(x) - 2D^0(x),$$

where $A^0(x) = A^0$, $B^0(x) = B^0$ and $D^0(x) = D^0$ are arbitrary elements of Y, and $A^i(x), B^i(x), C^i(x)$ are the diagonal of the i-additive symmetric maps A_i, B_i, $C_i\colon X^i \to$ $\to Y$, respectively, for $i = 1, 2, 3, 4$.

Proof. Assume that f, g, h, $\tilde{h}$ satisfy the functional equation (1.6). By Theorem 2.1 we see that f is a solution of the Fréchet functional equation $\triangle_{x_1,x_2,x_3,x_4,x_5} f(x_0) = 0$ for all x_0, x_1, x_2, x_3, x_4, $x_5 \in X$. Hence from Theorem 1.1 we have

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x), \quad \text{for all} \quad x \in X, \tag{2.18}$$

where $A^0(x) = A^0$ is an arbitrary element of Y, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon X^i \to Y$ for $i = 1, 2, 3, 4$. Putting (2.18) into (1.6), and noting that

$$A^4(x + y) + A^4(x - y) = 2A^4(x) + 2A^4(y) + 12A^{2,2}(x, y),$$

$$A^3(x + y) + A^3(x - y) = 2A^3(x) + 6A^{1,2}(x, y),$$

$$A^2(x + y) + A^2(x - y) = 2A^2(x) + 2A^2(y),$$

and $A^{2,2}(x, ky) = k^2A^{2,2}(x, y)$, $A^{1,2}(x, ky) = k^2A^{1,2}(x, y)$, we conclude that

$$g(x + y) + g(x - y) + h(x) + \tilde{h}(y) =$$

$$= 2A^4(x) + 2k^4A^4(y) + 12k^2A^{2,2}(x, y) + 2A^3(x) + 6k^2A^{1,2}(x, y)+$$

$$+2A^2(x) + 2k^2A^2(y) + 2A^1(x) + 2A^0.$$

Therefore

$$g(x + y) + g(x - y) + h(x) + \tilde{h}(y) =$$

$$= k^2A^4(x + y) + k^2A^4(x - y) + k^2A^3(x + y) + k^2A^3(x - y)+$$

$$+(2-2k^2)A^4(x)+(2-2k^2)A^3(x)+$$

$$+2A^2(x)+2A^1(x)+2A^0+(2k^4-2k^2)A^4(y)+2k^2A^2(y). \tag{2.19}$$

Letting

$$G(x)=g(x)-k^2A^4(x)-k^2A^3(x),$$
$$\tilde{H}(x)=-\tilde{h}(x)+(2k^4-2k^2)A^4(x)+2k^2A^2(x) \tag{2.20}$$

and

$$H(x)=-h(x)+(2-2k^2)A^4(x)+(2-2k^2)A^3(x)+2A^2(x)+2A^1(x)+2A^0. \tag{2.21}$$

From (2.19) we have

$$G(x+y)+G(x-y)=H(x)+\tilde{H}(y). \tag{2.22}$$

Let G satisfies (2.22). We decompose G into the even part and odd part by putting

$$G_e(x)=\frac{1}{2}(G(x)+G(-x)), \qquad G_o(x)=\frac{1}{2}(G(x)-G(-x))$$

for all $x\in X$. It is clear that $G(x)=G_e(x)+G_o(x)$ for all $x\in X$. Similarly, we have $H(x)=H_e(x)+H_o(x)$ and $\tilde{H}(x)=\tilde{H}_e(x)+\tilde{H}_o(x)$. Thus

$$G_e(x+y)+G_e(x-y)=H_e(x)+\tilde{H}_e(y), \tag{2.23}$$

and

$$G_o(x+y)+G_o(x-y)=H_o(x)+\tilde{H}_o(y). \tag{2.24}$$

Letting $y=0$ in (2.23), we have $H_e(x)=2G_e(x)-\tilde{H}_e(0)$. Setting $x=0$ in (2.23) to get $\tilde{H}_e(y)=2G_e(y)-H_e(0)$. Hence

$$G_e(x+y)+G_e(x-y)=2G_e(x)+2G_e(y)-2G_e(0) \tag{2.25}$$

for all $x,y\in X$. Setting $M(x)=G_e(x)-G_e(0)$, we get

$$M(x+y)+M(x-y)=2M(x)+2M(y) \tag{2.26}$$

which is the quadratic functional equation and its solution is given by $M(x)=B^2(x)$ for all $x\in X$, where $B^2(x)$ is the diagonal of the 2-additive symmetric map $B_2\colon X^2\to Y$. In this case, we obtain

$$G_e(x)=B^2(x)+G(0), \qquad H_e(x)=2B^2(x)+H_e(0),$$
$$\tilde{H}_e(x)=2B^2(x)+\tilde{H}_e(0). \tag{2.27}$$

Similarly, letting $y=0$ in (2.24), we have $H_o(x)=2G_o(x)$. Setting $x=0$ in (2.24) to get $\tilde{H}_o(y)=0$. Then from (2.24) we have

$$G_o(x+y)+G_o(x-y)=2G_o(x), \tag{2.28}$$

which is the Jensen functional equation and its solution is given by $G_o(x) = C^1(x)$, where $C^1\colon X \to Y$ is an additive function. Thus

$$G(x) = G_e(x) + G_o(x) = B^2(x) + B^0(x) + D^0(x) + C^1(x),$$

$$H(x) = H_e(x) + H_o(x) = 2B^2(x) + 2C^1(x) + 2B^0(x), \tag{2.29}$$

$$\tilde{H}(x) = \tilde{H}_e(x) + \tilde{H}_o(x) = 2B^2(x) + 2D^0(x),$$

where $B^0(x) = B^0$ (that is, $H_e(0) = 2B^0(x) = 2B^0$) and $D^0(x) = D^0$ (that is, $\tilde{H}_e(0) = 2D^0(x) = 2D^0$) are arbitrary elements of Y. Therefore from (2.20), (2.21), (2.29), we obtain the asserted solution (2.17). The converse is easily verified.

Theorem 2.7 is proved.

3. Solution of equation (1.6) on commutative groups. In this section, we solve the functional equation (1.6) on commutative groups with some additional requirements.

A group G is said to be divisible if for every element $b \in G$ and every $n \in \mathbb{N}$, there exists an element $a \in G$ such that $na = b$. If this element a is unique, then G is said to be uniquely divisible. In a uniquely divisible group, this unique element a is denoted by $\dfrac{b}{n}$. That the equation $na = b$ has a solution is equivalent to saying that the multiplication by n is surjective. Similarly, that the equation $na = b$ has a unique solution is equivalent to saying that the multiplication by n is bijective. Hence the notions of n-divisibility and n-unique divisibility refer, respectively, to surjectivity and bijectivity of the multiplication by n.

Lemma 3.1 (Hosszú [9]). *Let $n \geq 0$ be an integer, G and S be abelian groups. Furthermore let S be uniquely divisible. The map F from G into S satisfies the functional equation $\triangle_{x_1,\ldots,x_{n+1}} F(x_0) = 0$ for all $x_0, x_1, \ldots, x_{n+1} \in G$ if and only if F is given by $F(x) = A^n(x) + \ldots + A^1(x) + A^0(x)$ for all $x \in G$, where $A^0(x) = A^0$ is an arbitrary element of S and $A^n(x)$ is the diagonal of an n-additive symmetric function $A_n : G^n \to S$.*

Using Lemma 3.1, one can prove the similar results (Theorems 2.2 – 2.7) for unknown functions map a commutative uniquely divisible group into another one. Also, Theorems 2.2 – 2.7 can be further strengthened using two important results due to Székelyhidi [16]. By the use of the two important results, the proofs become even shorter but not so elementary any more. The results needed for this improvement are the following (see [16]).

Theorem 3.1. *Let G be a commutative semigroup with identity, S a commutative group and n a nonnegative integer. Let the multiplication by $n!$ be bijective in S. The function $f\colon G \to S$ is a solution of Fréchet functional equation*

$$\triangle_{x_1,\ldots,x_{n+1}} f(x_0) = 0 \tag{3.1}$$

for all $x_0, x_1, \ldots, x_{n+1} \in G$ if and only if f is a polynomial of degree at most n.

Theorem 3.2. *Let G and S be commutative groups, n a nonnegative integer, φ_i, ψ_i additive functions from G into G and $\varphi_i(G) \subseteq \psi_i(G)$, $i = 1, 2, \ldots, n+1$. If the functions $f, f_i\colon G \to S$, $i = 1, 2, \ldots, n+1$, satisfy*

$$f(x) + \sum_{i=1}^{n+1} f_i(\varphi_i(x) + \psi_i(y)) = 0, \tag{3.2}$$

then f satisfies Fréchet functional equation $\triangle_{x_1,\ldots,x_{n+1}} f(x_0) = 0$.

Using these two theorems, Theorems 2.2 – 2.7 can be further improved.

Theorem 3.3. *Let G and S be commutative groups. Let the multiplication by* 6 *and* $2(k^2 - 1)$ *be bijective in S, respectively. Then the function $f\colon G \to S$ satisfies the functional equation* (1.3) *for all $x, y \in G$ if and only if f is of the form $f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x)$ for all $x \in G$, where $A^0(x) = A^0$ is an arbitrary element of S, $A^i(x)$ is the diagonal of the i-additive symmetric map $G_i\colon X^i \to S$ for $i = 1, 2, 3$.*

Proof. Assume that f satisfies the functional equation (1.3). Using the unique divisibility of S by $2(k^2 - 1)$, we can rewrite the functional equation (1.3) in the form

$$f(x) + \frac{1}{2(k^2-1)} f(kx+y) + \frac{1}{2(k^2-1)} f(-kx-y) -$$

$$- \frac{k^2}{2(k^2-1)} f(x+y) - \frac{k^2}{2(k^2-1)} f(-x+y) = 0.$$

Thus by Theorem 3.2, f satisfies the Fréchet functional equation (3.1). By Theorem 3.1, f is a generalized polynomial function of degree at most 3, that is f is of the form $f(x) = A^3(x) + A^2(x) + A^1(x) + A^0(x)$, where $A^0(x) = A^0$ is an arbitrary element of S, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon G^i \to S$ for $i = 1, 2, 3$. The remaining assertion goes through by the similar way to corresponding part of Theorem 2.2.

Theorem 3.3 is proved.

Using the techniques are similar to that of Theorem 3.3, we have the following results.

Theorem 3.4. *Let G and S be commutative groups. Let the multiplication by* 2 *be surjective in G and let the multiplication by* 24 *and* 18 *be bijective in S, respectively. Then the function $f\colon G \to S$ satisfies the functional equation* (2.16) *for all $x, y \in G$ if and only if f is of the form $f(x) = A^4(x) + A^3(x) + A^2(x)$ for all $x \in G$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon G^i \to S$ for $i = 2, 3, 4$.*

Theorem 3.5. *Let G and S be commutative groups. Let the multiplication by* 2 *be surjective in G and let the multiplication by* 24 *and* 66 *be bijective in S, respectively. Then the function $f\colon G \to S$ satisfies the functional equation* (1.4) *for all $x, y \in G$ if and only if f is of the form $f(x) = A^4(x) + A^3(x) + A^1(x)$ for all $x \in G$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon G^i \to S$ for $i = 1, 3, 4$.*

Theorem 3.6. *Let G and S be commutative groups. Let the multiplication by* 2 *be surjective in G and let the multiplication by* 24 *be bijective in S. Then the function $f\colon G \to S$ satisfies the functional equation* (1.2) *for all $x, y \in G$ if and only if f is of the form $f(x) = A^4(x)$ for all $x \in G$, where $A^4(x)$ is the diagonal of the* 4*-additive symmetric map $A_4 : G^4 \to S$.*

Theorem 3.7. *Let G and S be commutative groups. Let the multiplication by k be surjective in G and let the multiplication by* 24 *and* $12(k^2 - 1)$ *be bijective in S, respectively. Then the function $f\colon G \to S$ satisfies the functional equation* (1.5) *for all $x, y \in G$ if and only if f is of the form $f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x)$ for all $x \in G$, where $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon G^i \to S$ for $i = 1, 2, 3, 4$.*

Theorem 3.8. *Let G and S be commutative groups. Let the multiplication by $2k(k^2-1)$ be surjective in G and let the multiplication by 24 be bijective in S. Then the function $f\colon G \to S$ satisfies the functional equation (1.6) for all $x, y \in G$ if and only if*

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x),$$

$$g(x) = k^2 A^4(x) + k^2 A^3(x) + B^2(x) + B^0(x) + C^1(x) + D^0(x),$$

$$h(x) = (2-2k^2)A^4(x) + (2-2k^2)A^3(x) + 2A^2(x) + 2A^1(x) + 2A^0(x) -$$

$$-2B^2(x) - 2C^1(x) - 2B^0(x),$$

$$\tilde{h}(x) = (2k^4 - 2k^2)A^4(x) + 2k^2 A^2(x) - 2B^2(x) - 2D^0(x),$$

where $A^0(x) = A^0$, $B^0(x) = B^0$ and $D^0(x) = D^0$ are arbitrary elements of S, and $A^i(x), B^i(x), C^i(x)$ are the diagonal of the i-additive symmetric maps $A_i,\ B_i,\ C_i\colon G^i \to \to S$, respectively, for $i = 1, 2, 3, 4$.

Proof. Assume that f satisfies the functional equation (1.6). Using the multiplication by $2k(k^2-1)$ be surjective in G, we can rewrite the functional equation (1.6) in the form

$$f(x) + f(x-2ky) - f(x+(1-k)y) - f(x-(1+k)y) - h(x-ky) - \tilde{h}(y) = 0.$$

Thus by Theorem 3.2, f satisfies the Fréchet functional equation (3.1). By Theorem 3.1, f is a generalized polynomial function of degree at most 4, that is f is of the form $f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0(x)$, where $A^0(x) = A^0$ is an arbitrary element of S, and $A^i(x)$ is the diagonal of the i-additive symmetric map $A_i\colon G^i \to S$ for $i = 1, 2, 3, 4$. The remaining assertion goes through by the similar way to corresponding part of Theorem 2.7.

Theorem 3.8 is proved.

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