UDC 517.9

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ON SOME IMBEDDING RELATIONS BETWEEN CERTAIN SEQUENCE SPACES*

ПРО ДЕЯКІ СПІВВІДНОШЕННЯ ВКЛАДЕННЯ МІЖ ПЕВНИМИ ПРОСТОРАМИ ПОСЛІДОВНОСТЕЙ

In the present paper, we introduce the sequence space ℓ_p^λ of non-absolute type which is a p-normed space and a BK-space in the cases of $0 < p < 1$ and $1 \leq p < \infty$, respectively. Further, we derive some imbedding relations and construct the basis for the space ℓ_p^λ, where $1 \leq p < \infty$.

Введено поняття простору послідовностей ℓ_p^λ неабсолютного типу, який є p-нормованим простором і BK-простором у випадках $0 < p < 1$ і $1 \leq p < \infty$ відповідно. Крім того, отримано деякі співвідношення вкладення та побудовано базис для простору ℓ_p^λ, де $1 \leq p < \infty$.

1. Introduction. By w, we denote the space of all complex valued sequences. Any vector subspace of w is called a sequence space.

A sequence space E with a linear topology is called a K-space provided each of the maps $p_i\colon E \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$; where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A K-space E is called an FK-space provided E is a complete linear metric space. An FK-space whose topology is normable is called a BK-space [2, p. 1451], that is, a BK-space is a Banach sequence space with continuous coordinates [11, p. 187].

We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively, which are BK-spaces with the usual sup-norm defined by

$$\|x\|_{\ell_\infty} = \sup_k |x_k|,$$

where, here and in the sequel, the supremum $\sup_k$ is taken over all $k \in \mathbb{N}$. Also, by ℓ_p, $0 < p < \infty$, we denote the sequence space of all p-absolutely convergent series. It is known that the space ℓ_p is a complete p-normed space and a BK-space in the cases of $0 < p < 1$ and $1 \leq p < \infty$, respectively, with respect to the usual p-norm and ℓ_p-norm defined by

$$\|x\|_{\ell_p} = \sum_k |x_k|^p, \quad 0 < p < 1,$$

and

$$\|x\|_{\ell_p} = \left(\sum_k |x_k|^p\right)^{1/p}, \quad 1 \leq p < \infty,$$

respectively. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞.

Let X and Y be sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, we say that A defines a matrix mapping from X into Y, and we denote it by writing $A\colon X \to Y$, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{A_n(x)\}$, the A-transform of x, exists and is in Y, where

*Research of the second author was supported by Department of Mathematics, Faculty of Education and Science, Al Baida University, Yemen.

$$A_n(x) = \sum_k a_{nk} x_k, \quad n \in \mathbb{N}. \tag{1.1}$$

By $(X : Y)$, we denote the class of all infinite matrices $A = (a_{nk})$ such that $A \colon X \to Y$. Thus, $A \in (X : Y)$ if and only if the series on the right-hand side of (1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and $Ax \in Y$ for all $x \in X$. A sequence x is said to be A-summable to $l \in \mathbb{C}$ if Ax coverges to l which is called the A-limit of x.

For a sequence space X, the matrix domain of an infinite matrix A in X is defined by

$$X_A = \{x \in w \colon Ax \in X\} \tag{1.2}$$

which is a sequence space.

We shall write $e^{(k)}$ for the sequence whose only non-zero term is a 1 in the kth place for each $k \in \mathbb{N}$.

The approach constructing a new sequence space by means of the matrix domain of a particular limitation method has recently been employed by several authors in many research papers (see, for example, [1 – 7,12 – 15,17, 18]). The main purpose of this paper is to introduce the sequence space ℓ_p^λ of non-absolute type and is to derive some related results. Further, we establish some imbedding relations concerning the space ℓ_p^λ, $0 < p < \infty$. Finally, we construct the basis for the space ℓ_p^λ, where $1 \le p < \infty$.

2. The sequence space ℓ_p^λ of non-absolute type. Throughout this paper, let $\lambda = (\lambda_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of positive reals tending to ∞, that is

$$0 < \lambda_0 < \lambda_1 < \ldots \text{ and } \lambda_k \to \infty \text{ as } k \to \infty. \tag{2.1}$$

By using the convention that any term with a negative subscript is equal to naught, we define the infinite matrix $\Lambda = (\lambda_{nk})$ by

$$\lambda_{nk} = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \tag{2.2}$$

for all $n, k \in \mathbb{N}$. Then, it is obvious by (2.2) that the matrix $\Lambda = (\lambda_{nk})$ is a triangle, that is $\lambda_{nn} \ne 0$ and $\lambda_{nk} = 0$ for all $k > n$, $n \in \mathbb{N}$. Further, by using (1.1), we have for every $x = (x_k) \in w$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k, \quad n \in \mathbb{N}. \tag{2.3}$$

Recently, Mursaleen and Noman [14] introduced the sequence spaces c_0^λ, c^λ and ℓ_∞^λ as follows:

$$c_0^\lambda = \left\{x \in w \colon \lim_n \Lambda_n(x) = 0\right\},$$

$$c^\lambda = \left\{x \in w \colon \lim_n \Lambda_n(x) \text{ exists}\right\}$$

and

$$\ell_\infty^\lambda = \left\{x \in w \colon \sup_n |\Lambda_n(x)| < \infty\right\}.$$

Moreover, it has been shown that the inclusions $c_0 \subset c_0^{\lambda}$, $c \subset c^{\lambda}$ and $\ell_{\infty} \subset \ell_{\infty}^{\lambda}$ hold. We refer the reader to [14] for relevant terminology.

Now, as a natural continuation of the above spaces, we define ℓ_p^{λ} as the set of all sequences whose Λ-transforms are in the space ℓ_p, $0 < p < \infty$; that is

$$\ell_p^{\lambda} = \left\{ x \in w \colon \sum_n |\Lambda_n(x)|^p < \infty \right\}, \quad 0 < p < \infty.$$

With the notation of (1.2), we may redefine the space ℓ_p^{λ}, $0 < p < \infty$ as the matrix domain of the triangle Λ in the space ℓ_p. This can be written as follows:

$$\ell_p^{\lambda} = (\ell_p)_{\Lambda}, \quad 0 < p < \infty. \tag{2.4}$$

It is trivial that ℓ_p^{λ}, $0 < p < \infty$, is a linear space with the coordinatewise addition and scalar multiplication. Further, it follows by (2.4) that the space ℓ_p^{λ}, $0 < p < 1$, becomes a p-normed space with the following p-norm:

$$\|x\|_{\ell_p^{\lambda}} = \|\Lambda(x)\|_{\ell_p} = \sum_n |\Lambda_n(x)|^p, \quad 0 < p < 1.$$

Moreover, since the matrix Λ is a triangle, we have the following result which is essential in the text.

Theorem 2.1. *The sequence space ℓ_p^{λ}, $1 \leq p < \infty$, is a BK-space with the norm given by*

$$\|x\|_{\ell_p^{\lambda}} = \|\Lambda(x)\|_{\ell_p} = \left(\sum_n |\Lambda_n(x)|^p \right)^{1/p}, \quad 1 \leq p < \infty. \tag{2.5}$$

Proof. Since (2.4) holds and ℓ_p, $1 \leq p < \infty$, is a BK-space with the ℓ_p-norm (see [10, p. 218]), this result is immediate by Theorem 4.3.12 of Wilansky [19, p. 63].

***Remark* 2.1.** One can easily check that the absolute property does not hold on the space ℓ_p^{λ}, $0 < p < \infty$, that is $\|x\|_{\ell_p^{\lambda}} \neq \||x|\|_{\ell_p^{\lambda}}$ for at least one sequence $x \in \ell_p^{\lambda}$. This tells us that ℓ_p^{λ} is a sequence space of non-absolute type, where $|x| = (|x_k|)$.

Theorem 2.2. *The sequence space ℓ_p^{λ} of non-absolute type is linear isometric to the space ℓ_p, where $0 < p < \infty$.*

Proof. To prove this, we should show the existence of a linear isometry between the spaces ℓ_p^{λ} and ℓ_p, where $0 < p < \infty$. For this, let us consider the transformation T defined, with the notation of (2.3), from ℓ_p^{λ} to ℓ_p by $x \longmapsto \Lambda(x) = Tx$. Then $Tx = \Lambda(x) \in \ell_p$ for every $x \in \ell_p^{\lambda}$. Also, the linearity of T is trivial. Further, it is easiy to see that $x = 0$ whenever $Tx = 0$ and hence T is injective.

Furthermore, for any given $y = (y_k) \in \ell_p$, we define the sequence $x = (x_k)$ by

$$x_k = \frac{\lambda_k y_k - \lambda_{k-1} y_{k-1}}{\lambda_k - \lambda_{k-1}}, \quad k \in \mathbb{N}.$$

Then, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1}) = y_n.$$

This shows that $\Lambda(x) = y$ and since $y \in \ell_p$, we obtain that $\Lambda(x) \in \ell_p$. Thus, we deduce that $x \in \ell_p^\lambda$ and $Tx = y$. Hence, the operator T is surjective.

Moreover, let $x \in \ell_p^\lambda$ be given. Then, we have that

$$\|Tx\|_{\ell_p} = \|\Lambda(x)\|_{\ell_p} = \|x\|_{\ell_p^\lambda}$$

and hence T is an isometry. Consequently, the spaces ℓ_p^λ and ℓ_p are linear isometric for $0 < p < \infty$.

Theorem 2.2 is proved.

Finally, we know that the space ℓ_2 is the only Hilbert space among the Banach spaces ℓ_p, $1 \leq p < \infty$. Thus, we conclude this section with the following corollary which is immediate by Theorems 2.1 and 2.2.

***Corollary* 2.1.** *Except the case $p = 2$, the space ℓ_p^λ is not an inner product space, hence not a Hilbert space for $1 \leq p < \infty$.*

3. Some imbedding relations. In the present section, we establish some imbedding relations concerning the space ℓ_p^λ, $0 < p < \infty$. We essentially characterize the case in which the imbedding $\ell_p \subset \ell_p^\lambda$ holds for $1 \leq p < \infty$.

The notion of imbedded Banach spaces can be found in [9] (Chapter I) and it can be given as follows:

Let X and Y be Banach spaces. Then, we say that X is imbedded in Y if the following conditions are satisfied:

(i) $x \in X$ implies $x \in Y$, that is, the space Y includes X.

(ii) The space Y includes a vector space structure on X coinciding with the structure of X.

(iii) There exists a constant $C > 0$ such that $\|x\|_Y \leq C\,\|x\|_X$ for all $x \in X$.

In what follows, we shall denote the imbedding of X in Y by $X \subset Y$, assuming that the symbol $\subset$ means not only the set-theoretic inclusion, but imbedding have the properties (ii) and (iii). Further, we say that the imbedding $X \subset Y$ strictly holds if the space Y strictly includes X.

Since any two sequence spaces have the same vector space structure, the condition (ii) is redundant when X and Y are BK-spaces.

Now, we may begin with the following basic result:

Theorem 3.1. *If $0 < p < s < \infty$, then the imbedding $\ell_p^\lambda \subset \ell_s^\lambda$ strictly holds.*

Proof. Since the space ℓ_s strictly includes ℓ_p, the space ℓ_s^λ strictly includes ℓ_p^λ. Therefore, this result is immediate by the fact that the topology of the space ℓ_p^λ is stronger than the topology of ℓ_s^λ, that is

$$\|x\|_{\ell_s^\lambda} = \|\Lambda(x)\|_{\ell_s} \leq \|\Lambda(x)\|_{\ell_p} = \|x\|_{\ell_p^\lambda}$$

for all $x \in \ell_p^\lambda$, where $0 < p < s < \infty$.

Theorem 3.1 is proved.

Although the imbeddings $c_0 \subset c_0^\lambda$, $c \subset c^\lambda$ and $\ell_\infty \subset \ell_\infty^\lambda$ always holds, the space ℓ_p may not be included in ℓ_p^λ for $0 < p < \infty$. This will be shown in the following lemma in which we write $\dfrac{1}{\lambda} = \left(\dfrac{1}{\lambda_k}\right)$.

Lemma 3.1. *Let $0 < p < \infty$. Then, the spaces ℓ_p and ℓ_p^λ overlap. Further, if $\dfrac{1}{\lambda} \notin \ell_p$ then neither of the spaces ℓ_p and ℓ_p^λ includes the other one.*

Proof. Obviously, the spaces ℓ_p and ℓ_p^λ always overlap, since the sequence $(\lambda_1 - \lambda_0, -\lambda_0, 0, 0, \ldots)$ belongs to both spaces ℓ_p and ℓ_p^λ for $0 < p < \infty$.

Suppose now that $\dfrac{1}{\lambda} \notin \ell_p$, $0 < p < \infty$, and consider the sequence $x = e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$. Then, by using (2.3), we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})e_k^{(0)} = \frac{\lambda_0}{\lambda_n}.$$

Thus, we obtain that

$$\sum_n |\Lambda_n(x)|^p = \lambda_0^p \sum_n \frac{1}{\lambda_n^p}$$

which shows that $\Lambda(x) \notin \ell_p$ and hence $x \notin \ell_p^\lambda$. Thus, the sequence x is in ℓ_p but not in ℓ_p^λ. Hence, the space ℓ_p^λ does not include ℓ_p when $\dfrac{1}{\lambda} \notin \ell_p$, where $0 < p < \infty$.

On the other hand, let $1 \le p < \infty$ and define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} \dfrac{1}{\lambda_k}, & k \text{ is even}, \\[2ex] -\dfrac{1}{\lambda_{k-1}}\Big(\dfrac{\lambda_{k-1} - \lambda_{k-2}}{\lambda_k - \lambda_{k-1}}\Big), & k \text{ is odd}, \end{cases} \qquad k \in \mathbb{N}.$$

Then $y \notin \ell_p$, since $\dfrac{1}{\lambda} \notin \ell_p$. Besides, we have for every $n \in \mathbb{N}$ that

$$\Lambda_n(y) = \begin{cases} \dfrac{1}{\lambda_n}\Big(\dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n}\Big), & n \text{ is even}, \\[2ex] 0, & n \text{ is odd}. \end{cases}$$

Thus, we obtain that

$$\sum_n |\Lambda_n(y)|^p = \sum_n |\Lambda_{2n}(y)|^p = \sum_n \frac{1}{\lambda_{2n}^p}\left(\frac{\lambda_{2n} - \lambda_{2n-1}}{\lambda_{2n}}\right)^p \le$$

$$\le \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty}\frac{1}{\lambda_{2n-2}^p}\left(\frac{\lambda_{2n} - \lambda_{2n-2}}{\lambda_{2n}}\right)^p \le$$

$$\le \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty}\frac{1}{\lambda_{2n-2}^p}\left(\frac{\lambda_{2n}^p - \lambda_{2n-2}^p}{\lambda_{2n}^p}\right) =$$

$$= \frac{1}{\lambda_0^p} + \sum_{n=1}^{\infty}\Big(\frac{1}{\lambda_{2n-2}^p} - \frac{1}{\lambda_{2n}^p}\Big) = \frac{2}{\lambda_0^p} < \infty.$$

This shows that $\Lambda(y) \in \ell_p$ and hence $y \in \ell_p^\lambda$. Thus, the sequence y is in ℓ_p^λ but not in ℓ_p, where $1 \le p < \infty$.

Similarly, one can construct a sequence belonging to the set $\ell_p^\lambda \setminus \ell_p$ for $0 < p < 1$. Therefore, the space ℓ_p also does not include ℓ_p^λ when $\dfrac{1}{\lambda} \notin \ell_p$ for $0 < p < \infty$.

Lemma 3.1 is proved.

As an immediate consequence of Lemma 3.1, we have the following lemma.

Lemma 3.2. *If the imbedding $\ell_p \subset \ell_p^\lambda$ holds, then $\dfrac{1}{\lambda} \in \ell_p$, where $0 < p < \infty$.*

Proof. Suppose that the imbedding $\ell_p \subset \ell_p^\lambda$ holds, where $0 < p < \infty$, and consider the sequence $x = e^{(0)} = (1, 0, 0, \ldots) \in \ell_p$. Then $x \in \ell_p^\lambda$ and hence $\Lambda(x) \in \ell_p$. Thus, we obtain that

$$\lambda_0^p \sum_n \left(\frac{1}{\lambda_n}\right)^p = \sum_n |\Lambda_n(x)|^p < \infty$$

which shows that $\dfrac{1}{\lambda} \in \ell_p$.

Lemma 3.2 is proved.

We shall later show that the condition $\dfrac{1}{\lambda} \in \ell_p$ is not only necessary but also sufficient for the imbedding $\ell_p \subset \ell_p^\lambda$ to be held, where $1 \le p < \infty$. Before that, by taking into account the definition of the sequence $\lambda = (\lambda_k)$ given by (2.1), we find that

$$0 < \frac{\lambda_k - \lambda_{k-1}}{\lambda_n} < 1, \quad 0 \le k \le n,$$

for all $n, k \in \mathbb{N}$ with $n + k > 0$. Furthermore, if $\dfrac{1}{\lambda} \in \ell_1$ then we have the following lemma which is easy to prove.

Lemma 3.3. *If $\dfrac{1}{\lambda} \in \ell_1$, then*

$$\sup_k (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} < \infty.$$

Now, we prove the following:

Theorem 3.2. *The imbedding $\ell_1 \subset \ell_1^\lambda$ holds if and only if $\dfrac{1}{\lambda} \in \ell_1$.*

Proof. The necessity is immediate by Lemma 3.2.

Conversely, suppose that $\dfrac{1}{\lambda} \in \ell_1$. Then, it follows by Lemma 3.3 that

$$M = \sup_k (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} < \infty.$$

Therefore, we have for every $x = (x_k) \in \ell_1$ that

$$\|x\|_{\ell_1^\lambda} = \sum_n |\Lambda_n(x)| \le \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})|x_k| =$$

$$= \sum_{k=0}^{\infty} |x_k| (\lambda_k - \lambda_{k-1}) \sum_{n=k}^{\infty} \frac{1}{\lambda_n} \le M \sum_{k=0}^{\infty} |x_k| = M\|x\|_{\ell_1}.$$

This also shows that the space ℓ_1^λ includes ℓ_1. Hence, the imbedding $\ell_1 \subset \ell_1^\lambda$ holds which concludes the proof.

***Corollary* 3.1.** *If $\dfrac{1}{\lambda} \in \ell_1$, then the imbedding $\ell_p \subset \ell_p^\lambda$ holds for $1 \le p < \infty$.*

Proof. The imbedding trivially holds for $p = 1$ by Theorem 3.2, above. Thus, let $1 < p < \infty$ and take any $x \in \ell_p$. Then $|x|^p \in \ell_1$ and hence $|x|^p \in \ell_1^\lambda$ by Theorem 3.2 which implies that $x \in \ell_p^\lambda$. This shows that the space ℓ_p is included in ℓ_p^λ.

Further, let $x = (x_k) \in \ell_p$ be given. Then, for every $n \in \mathbb{N}$, we obtain by applying the Hölder's inequality that

$$|\Lambda_n(x)|^p \le \left[\sum_{k=0}^{n}\left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}\right)|x_k|\right]^p \le$$

$$\le \left[\sum_{k=0}^{n}\left(\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}\right)|x_k|^p\right]\left[\sum_{k=0}^{n}\frac{\lambda_k - \lambda_{k-1}}{\lambda_n}\right]^{p-1} = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k|^p.$$

Thus, we derive that

$$\|x\|_{\ell_p^\lambda}^p = \sum_n |\Lambda_n(x)|^p \le \sum_{n=0}^{\infty}\frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})|x_k|^p =$$

$$= \sum_{k=0}^{\infty}|x_k|^p(\lambda_k - \lambda_{k-1})\sum_{n=k}^{\infty}\frac{1}{\lambda_n} \le M\sum_{k=0}^{\infty}|x_k|^p = M\|x\|_{\ell_p}^p,$$

where $M = \sup_k\left[(\lambda_k - \lambda_{k-1})\sum_{n=k}^{\infty}\dfrac{1}{\lambda_n}\right] < \infty$ by Lemma 3.3. Hence, the imbedding $\ell_p \subset \ell_p^\lambda$ also holds for $1 < p < \infty$.

Corollary 3.1 is proved.

Now, as a generalization of Theorem 3.2, the following theorem shows the necessity and sufficiency of the condition $\dfrac{1}{\lambda} \in \ell_p$ for the imbedding $\ell_p \subset \ell_p^\lambda$ to be held, where $1 \le p < \infty$.

Theorem 3.3. *The imbedding $\ell_p \subset \ell_p^\lambda$ holds if and only if $\dfrac{1}{\lambda} \in \ell_p$, where $1 \le p < < \infty$.*

Proof. The necessity is trivial by Lemma 3.2. Thus, we turn to the sufficiency. For this, suppose that $\dfrac{1}{\lambda} \in \ell_p$, where $1 \le p < \infty$. Then $\dfrac{1}{\lambda^p} = \left(\dfrac{1}{\lambda_k^p}\right) \in \ell_1$. Therefore, it follows by Lemma 3.3 that

$$\sup_k (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty}\frac{1}{\lambda_n^p} \le \sup_k (\lambda_k^p - \lambda_{k-1}^p)\sum_{n=k}^{\infty}\frac{1}{\lambda_n^p} < \infty.$$

Further, we have for every fixed $k \in \mathbb{N}$ that

$$\Lambda_n\big(e^{(k)}\big) = \begin{cases} \dfrac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & 0 \le k \le n, \\ 0, & k > n, \end{cases} \qquad n \in \mathbb{N}.$$

Thus, we obtain that

$$\|e^{(k)}\|_{\ell_p^\lambda}^p = (\lambda_k - \lambda_{k-1})^p \sum_{n=k}^{\infty}\frac{1}{\lambda_n^p} < \infty, \quad k \in \mathbb{N},$$

which yields that $e^{(k)} \in \ell_p^\lambda$ for every $k \in \mathbb{N}$, i.e., every basis element of the space ℓ_p is in ℓ_p^λ. This shows that the space ℓ_p^λ contains the Schauder basis for the space ℓ_p such that

$$\sup_k \|e^{(k)}\|_{\ell_p^\lambda} < \infty.$$

Therefore, we deduce that the space ℓ_p^λ includes ℓ_p. Moreover, by using the same technique used in the proof of Corollary 3.1, it can similarly be shown that the topology of the space ℓ_p is stronger than the topology of ℓ_p^λ. Hence, the imbedding $\ell_p \subset \ell_p^\lambda$ holds, where $1 \leq p < \infty$.

Theorem 3.3 is proved.

Now, in the following example, we give an important particular case of the space ℓ_p^λ, where $1 \leq p < \infty$.

***Example* 3.1.** Consider the particular case of the sequence $\lambda = (\lambda_k)$ given by $\lambda_k = k + 1$ for all $k \in \mathbb{N}$. Then $\dfrac{1}{\lambda} \notin \ell_1$ and hence ℓ_1 is not included in ℓ_1^λ by Lemma 3.1.

On the other hand, we have $\dfrac{1}{\lambda} \in \ell_p$ for $1 < p < \infty$ and so ℓ_p is included in ℓ_p^λ. Further, by applying the well-known inequality (see [8, p. 239])

$$\sum_n \left(\sum_{k=0}^{n} \frac{|x_k|}{n+1} \right)^p < \left(\frac{p}{p-1} \right)^p \sum_n |x_n|^p, \quad 1 < p < \infty,$$

we immediately obtain that

$$\|x\|_{\ell_p^\lambda} < \frac{p}{p-1} \|x\|_{\ell_p}, \quad 1 < p < \infty,$$

for all $x \in \ell_p$. This shows that the imbedding $\ell_p \subset \ell_p^\lambda$ holds for $1 < p < \infty$. Moreover, this imbedding is strict. For example, the sequence $y = \{(-1)^k\}_{k \in \mathbb{N}}$ is not in ℓ_p but in ℓ_p^λ, since

$$\sum_n |\Lambda_n(y)|^p = \sum_n \left| \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \right|^p = \sum_n \frac{1}{(2n+1)^p} < \infty, \quad 1 < p < \infty.$$

***Remark* 3.1.** In the special case $\lambda_k = k + 1$ $(k \in \mathbb{N})$ given in Example 3.1, we may note that the space ℓ_p^λ is reduced to the Cesàro sequence space X_p of non-absolute type, where $1 \leq p < \infty$ (see [16, 17]).

Now, let $x = (x_k)$ be a null sequence of positive reals, that is

$$x_k > 0 \quad \text{for all } k \in \mathbb{N} \text{ and } x_k \to 0 \quad \text{as } k \to \infty.$$

Then, as is easy to see, for every positive integer m there is a subsequence $\{x_{k_r}\}_{r \in \mathbb{N}}$ of the sequence x such that

$$x_{k_r} = O\left(\frac{1}{(r+1)^{m+1}} \right).$$

Further, this subsequence can be chosen such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$.

In general, if $x = (x_k)$ is a sequence of positive reals such that $\liminf x_k = 0$, then there is a subsequence $x' = \{x_{k'_r}\}_{r \in \mathbb{N}}$ of the sequence x such that $\lim_r x_{k'_r} = 0$. Thus x' is a null sequence of positive reals. Hence, as we have seen above, for every positive integer m there is a subsequence $\{x_{k_r}\}_{r \in \mathbb{N}}$ of the sequence x' and hence of the sequence x such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and

$$x_{k_r} = O\left(\frac{1}{(r+1)^{m+1}}\right),$$

where $k_r = k'_{\theta(r)}$ and $\theta\colon \mathbb{N} \to \mathbb{N}$ is a suitable increasing function. Thus, we obtain that

$$(r+1)x_{k_r} = O\left(\frac{1}{(r+1)^{m}}\right).$$

Now, let $0 < p < \infty$. Then, we can choose a positive integer m such that $mp > 1$. In this situation, the sequence $\{(r+1)x_{k_r}\}_{r\in\mathbb{N}}$ is in the space ℓ_p.

Obviously, we observe that the subsequence $\{x_{k_r}\}_{r\in\mathbb{N}}$ depends on the positive integer m which is, in turn, depending on p. Thus, our subsequence depends on p.

Hence, from the above discussion, we conclude the following result:

Lemma 3.4. *Let $x = (x_k)$ be a sequence of positive reals such that* $\liminf x_k = 0$. *Then, for every positive number $p \in (0,\infty)$ there is a subsequence $x^{(p)} = \{x_{k_r}\}_{r\in\mathbb{N}}$ of x, depending on p, such that $k_{r+1} - k_r \geq 2$ for all $r \in \mathbb{N}$ and $\sum_r |(r+1)x_{k_r}|^p < \infty$.*

Moreover, we have the following two lemmas (see [14]) which are needed in the sequel.

Lemma 3.5. *For any sequence $x = (x_k) \in w$, the equalities*

$$S_n(x) = x_n - \Lambda_n(x), \quad n \in \mathbb{N}, \tag{3.1}$$

and

$$S_n(x) = \frac{\lambda_{n-1}}{\lambda_n - \lambda_{n-1}}\Big[\Lambda_n(x) - \Lambda_{n-1}(x)\Big], \quad n \in \mathbb{N}, \tag{3.2}$$

hold, where the sequence $S(x) = \{S_n(x)\}$ is defined by

$$S_0(x) = 0 \quad \textit{and} \quad S_n(x) = \frac{1}{\lambda_n}\sum_{k=1}^{n}\lambda_{k-1}(x_k - x_{k-1}) \quad \textit{for} \quad n \geq 1.$$

Lemma 3.6. *For any sequence $\lambda = (\lambda_k)$ satisfying* (2.1), *the sequence $\left\{\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right\}_{k\in\mathbb{N}}$ is bounded if and only if* $\liminf \dfrac{\lambda_{k+1}}{\lambda_k} > 1$, *and is unbounded if and only if* $\liminf \dfrac{\lambda_{k+1}}{\lambda_k} = 1$.

Now, we know by Theorem 3.3 that the imbedding $\ell_p \subset \ell_p^\lambda$ holds whenever $\dfrac{1}{\lambda} \in \ell_p$, $1 \leq p < \infty$. More precisely, the following theorem gives the necessary and sufficient conditions for this imbedding to be strict.

Theorem 3.4. *Let $1 \leq p < \infty$. Then, the imbedding $\ell_p \subset \ell_p^\lambda$ strictly holds if and only if $\dfrac{1}{\lambda} \in \ell_p$ and* $\liminf \dfrac{\lambda_{n+1}}{\lambda_n} = 1$.

Proof. Suppose that the imbedding $\ell_p \subset \ell_p^\lambda$ is strict, where $1 \leq p < \infty$. Then, the necessity of the first condition is immediate by Theorem 3.3. Further, since ℓ_p^λ strictly includes ℓ_p, there is a sequence $x \in \ell_p^\lambda$ such that $x \notin \ell_p$, that is $\Lambda(x) \in \ell_p$ while $x \notin \ell_p$. Thus, we obtain by (3.1) of Lemma 3.5 that $S(x) = \{S_n(x)\} \notin \ell_p$. Moreover, since $\Lambda(x) \in \ell_p$, we have $\sum_n |\Lambda_n(x)|^p < \infty$ and hence $\sum_n |\Lambda_n(x) - \Lambda_{n-1}(x)|^p < \infty$ by applying the Minkowski's inequality. This means that $\{\Lambda_n(x) - \Lambda_{n-1}(x)\} \in \ell_p$. Thus, by combining this with the fact that $\{S_n(x)\} \notin \ell_p$, it follows by (3.2) that the sequence $\left\{\dfrac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\right\}$ is unbounded and hence $\left\{\dfrac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right\}\notin\ell_\infty$. This leads us with Lemma 3.6 to the necessity of the second condition.

Conversely, since $\dfrac{1}{\lambda}\in\ell_p$, we have by Theorem 3.3 that the imbedding $\ell_p\subset\ell_p^\lambda$ holds. Further, since $\liminf\dfrac{\lambda_{k+1}}{\lambda_k}=1$, we obtain by Lemma 3.6 that

$$\liminf\frac{\lambda_k-\lambda_{k-1}}{\lambda_k}=0.$$

Thus, it follows by Lemma 3.4 that there is a subsequence $\lambda^{(p)}=\{\lambda_{k_r}\}_{r\in\mathbb{N}}$ of the sequence $\lambda=(\lambda_k)$, depending on p, such that $k_{r+1}-k_r\geq 2$ for all $r\in\mathbb{N}$ and

$$\sum_r\left|(r+1)\Big(\frac{\lambda_{k_r}-\lambda_{k_r-1}}{\lambda_{k_r}}\Big)\right|^p<\infty. \tag{3.3}$$

Let us now define the sequence $y=(y_k)$ for every $k\in\mathbb{N}$ by

$$y_k=\begin{cases} r+1, & k=k_r, \\ -(r+1)\left(\dfrac{\lambda_{k-1}-\lambda_{k-2}}{\lambda_k-\lambda_{k-1}}\right), & k=k_r+1, \qquad r\in\mathbb{N}, \\ 0, & \text{otherwise}. \end{cases}$$

Then, it is clear that $y\notin\ell_p$. Moreover, we have for every $n\in\mathbb{N}$ that

$$\Lambda_n(y)=\begin{cases} (r+1)\left(\dfrac{\lambda_n-\lambda_{n-1}}{\lambda_n}\right), & n=k_r, \\ & \qquad\qquad r\in\mathbb{N}. \\ 0, & n\neq k_r, \end{cases}$$

This and (3.3) imply that $\Lambda(y)\in\ell_p$ and hence $y\in\ell_p^\lambda$. Thus, the sequence y is in ℓ_p^λ but not in ℓ_p. Therefore, the imbedding $\ell_p\subset\ell_p^\lambda$ strictly holds, where $1\leq p<\infty$.

Theorem 3.4 is proved.

As an immediate consequence of Theorem 3.4, we have the following result:

Theorem 3.5. *The equality* $\ell_p^\lambda=\ell_p$ *holds if and only if* $\liminf\dfrac{\lambda_{n+1}}{\lambda_n}>1$, *where* $1\leq p<\infty$.

Proof. The necessity is immediate by Theorems 3.3 and 3.4. For, if the equality holds then ℓ_p is imbedded in ℓ_p^λ and hence $\dfrac{1}{\lambda}\in\ell_p$ by Theorem 3.3. Further, since the imbedding $\ell_p\subset\ell_p^\lambda$ cannot be strict, we have by Theorem 3.4 that $\liminf\dfrac{\lambda_{n+1}}{\lambda_n}\neq 1$ and hence $\liminf\dfrac{\lambda_{n+1}}{\lambda_n}>1$.

Conversely, suppose that $\liminf\dfrac{\lambda_{n+1}}{\lambda_n}>1$. Then, there exists a constant $a>1$ such that $\dfrac{\lambda_{n+1}}{\lambda_n}\geq a$ and hence $\lambda_n\geq\lambda_0 a^n$ for all $n\in\mathbb{N}$. This shows that $\dfrac{1}{\lambda}\in\ell_1$ which leads us with Corollary 3.1 to the consequence that the imbedding $\ell_p\subset\ell_p^\lambda$ holds and hence ℓ_p is included in ℓ_p^λ, where $1\leq p<\infty$.

On the other hand, by using Lemma 3.6, we have the bounded sequence $\left\{\dfrac{\lambda_n}{\lambda_n-\lambda_{n-1}}\right\}$ and hence $\left\{\dfrac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}\right\}\in\ell_\infty$.

Now, let $x\in\ell_p^\lambda$. Then $\Lambda(x)=\{\Lambda_n(x)\}\in\ell_p$ and hence $\{\Lambda_n(x)-\Lambda_{n-1}(x)\}\in\ell_p$. Thus, we obtain by (3.2) that $S(x)=\{S_n(x)\}\in\ell_p$. Therefore, it follows by (3.1) that $x=S(x)+\Lambda(x)\in\ell_p$. This shows that $x\in\ell_p$ for all $x\in\ell_p^\lambda$ and hence ℓ_p^λ is also included in ℓ_p. Consequently, the equality $\ell_p^\lambda=\ell_p$ holds, where $1\le p<\infty$.

Theorem 3.5 is proved.

Finally, we conclude this section by the following corollary:

***Corollary* 3.2.** *Although the spaces ℓ_p^λ, c_0, c and ℓ_∞ overlap, the space ℓ_p^λ does not include any of the other spaces. Further, if* $\liminf\dfrac{\lambda_{n+1}}{\lambda_n}=1$ *then none of the spaces c_0, c or ℓ_∞ includes the space ℓ_p^λ, where $0<p<\infty$.*

Proof. Let $0<p<\infty$. Then, it is obvious by Lemma 3.1 that the spaces ℓ_p^λ, c_0, c and ℓ_∞ overlap.

Further, the space ℓ_p^λ does not include the space c_0. To show this, we define the sequence $x=(x_k)\in c_0$ by

$$x_k=\frac{1}{(k+1)^{1/p}},\qquad k\in\mathbb{N}.$$

Then, we have for every $n\in\mathbb{N}$ that

$$|\Lambda_n(x)|=\frac{1}{\lambda_n}\sum_{k=0}^{n}\frac{\lambda_k-\lambda_{k-1}}{(k+1)^{1/p}}\ge$$

$$\ge\frac{1}{\lambda_n(n+1)^{1/p}}\sum_{k=0}^{n}(\lambda_k-\lambda_k-1)=\frac{1}{(n+1)^{1/p}}$$

which shows that $\Lambda(x)\notin\ell_p$ and hence $x\notin\ell_p^\lambda$. Thus, the sequence x is in c_0 but not in ℓ_p^λ. Hence, the space ℓ_p^λ does not include any of the spaces c_0, c or ℓ_∞.

Moreover, if $\liminf\dfrac{\lambda_{n+1}}{\lambda_n}=1$ then the space ℓ_∞ does not include the space ℓ_p^λ. To see this, let $0<p<\infty$. Then, Lemma 3.4 implies that the sequence y, defined in the proof of Theorem 3.4, is in ℓ_p^λ but not in ℓ_∞. Therefore, none of the spaces c_0, c or ℓ_∞ includes the space ℓ_p^λ when $\liminf\dfrac{\lambda_{n+1}}{\lambda_n}=1$, where $0<p<\infty$.

Corollary 3.2 is proved.

4. The basis for the space ℓ_p^λ. In the present section, we give a sequence of points of the space ℓ_p^λ which forms a basis for ℓ_p^λ, where $1\le p<\infty$.

If a normed sequence space X contains a sequence (b_n) with the property that for every $x\in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_n\|x-(\alpha_0b_0+\alpha_1b_1+\ldots+\alpha_nb_n)\|=0,$$

then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum_k\alpha_kb_k$ which has the sum x is then called the expansion of x with respect to (b_n), and written as $x=\sum_k\alpha_kb_k$.

Now, because of the transformation T defined from ℓ_p^λ to ℓ_p in the proof of Theorem 2.2 is onto, the inverse image of the basis $\{e^{(k)}\}_{k\in\mathbb{N}}$ of the space ℓ_p is the basis for the new space ℓ_p^λ, where $1 \le p < \infty$. Therefore, we have the following:

Theorem 4.1. *Let $1 \le p < \infty$ and define the sequence $e^{(k)}(\lambda) = \{e_n^{(k)}(\lambda)\}_{n\in\mathbb{N}}$ of the elements of the space ℓ_p^λ for every fixed $k \in \mathbb{N}$ by*

$$e_n^{(k)}(\lambda) = \begin{cases} (-1)^{n-k}\dfrac{\lambda_k}{\lambda_n - \lambda_{n-1}}, & k \le n \le k+1, \\ 0, & n < k \ \text{ or } \ n > k+1, \end{cases} \qquad n \in \mathbb{N}. \tag{4.1}$$

Then, the sequence $\{e^{(k)}(\lambda)\}_{k\in\mathbb{N}}$ is a basis for the space ℓ_p^λ and every $x \in \ell_p^\lambda$ has a unique representation of the form

$$x = \sum_k \alpha_k(\lambda) e^{(k)}(\lambda), \tag{4.2}$$

where $\alpha_k(\lambda) = \Lambda_k(x)$ for all $k \in \mathbb{N}$,

Proof. Let $1 \le p < \infty$. Then, it is clear by (4.1) that

$$\Lambda\big(e^{(k)}(\lambda)\big) = e^{(k)} \in \ell_p, \quad k \in \mathbb{N},$$

and hence $e^{(k)}(\lambda) \in \ell_p^\lambda$ for all $k \in \mathbb{N}$.

Further, let $x \in \ell_p^\lambda$ be given. For every non-negative integer m, we put

$$x^{(m)} = \sum_{k=0}^{m} \alpha_k(\lambda) e^{(k)}(\lambda).$$

Then, we have that

$$\Lambda(x^{(m)}) = \sum_{k=0}^{m} \alpha_k(\lambda)\Lambda\big(e^{(k)}(\lambda)\big) = \sum_{k=0}^{m} \Lambda_k(x) e^{(k)}$$

and hence

$$\Lambda_n(x - x^{(m)}) = \begin{cases} 0, & 0 \le n \le m, \\ \Lambda_n(x), & n > m, \end{cases} \qquad n, m \in \mathbb{N}.$$

Now, for any given $\varepsilon > 0$, there is a non-negative integer m_0 such that

$$\sum_{n=m_0+1}^{\infty} |\Lambda_n(x)|^p \le \left(\frac{\varepsilon}{2}\right)^p.$$

Hence, we have for every $m \ge m_0$ that

$$\|x - x^{(m)}\|_{\ell_p^\lambda} = \left(\sum_{n=m+1}^{\infty} |\Lambda_n(x)|^p\right)^{1/p} \le \left(\sum_{n=m_0+1}^{\infty} |\Lambda_n(x)|^p\right)^{1/p} \le \frac{\varepsilon}{2} < \varepsilon.$$

Thus, we obtain that

$$\lim_m \|x - x^{(m)}\|_{\ell_p^\lambda} = 0$$

which shows that $x \in \ell_p^\lambda$ is represented as in (4.2).

Finally, let us show the uniqueness of the representation (4.2) of $x \in \ell_p^\lambda$. For this, suppose on the contrary that there exists another representation $x = \sum_k \beta_k(\lambda) e^{(k)}(\lambda)$. Since the linear transformation T defined from ℓ_p^λ to ℓ_p in the proof of Theorem 2.2 is continuous [19] (Theorem 4.2.8), we have that

$$\Lambda_n(x) = \sum_k \beta_k(\lambda)\Lambda_n(e^{(k)}(\lambda)) = \sum_k \beta_k(\lambda) e_n^{(k)} = \beta_n(\lambda), \quad n \in \mathbb{N},$$

which contradicts the fact that $\Lambda_n(x) = \alpha_n(\lambda)$ for all $n \in \mathbb{N}$. Hence, the representation (4.2) of $x \in \ell_p^\lambda$ is unique.

Theorem 4.1 is proved.

Now, it is known by Theorem 2.1 that ℓ_p^λ, $1 \leq p < \infty$, is a Banach space with its natural norm. This leads us together with Theorem 4.1 to the following corollary:

***Corollary* 4.1.** *The sequence space ℓ_p^λ of non-absolute type is separable for $1 \leq$ $\leq p < \infty$.*

Finally, we conclude our work by expressing from now on that the aim of the next paper is to determine the α-, β- and γ-duals of the space ℓ_p^λ and is to characterize some matrix classes concerning the space ℓ_p^λ, where $1 \leq p < \infty$.

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Received 10.08.09,
after revision — 18.06.10