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PROPERTIES OF A CERTAIN PRODUCT OF SUBMODULES

ВЛАСТИВОСТІ ПЕВНОГО ДОБУТКУ ПІДМОДУЛІВ

Let R be a commutative ring with identity, M an R-module and $K_1,\ldots,K_n$ submodules of M. In this article, we construct an algebraic object, called product of $K_1,\ldots,K_n$. We equipped this structure with appropriate operations to get an $R(M)$-module. It is shown that $R(M)$-module $M^n = M\ldots M$ and R-module M inherit some of the most important properties of each other. For example, we show that M is a projective (flat) R-module if and only if M^n is a projective (flat) $R(M)$-module.

Припустимо, що R — комутативне кільце з одиницею, M — R-модуль і $K_1,\ldots,K_n$ — підмодулі M. Побудовано алгебраїчний об'єкт, що називається добутком підмодулів $K_1,\ldots,K_n$. Цю структуру оснащено відповідними операціями для отримання $R(M)$-модуля. Показано, що $R(M)$-модуль $M^n = M\ldots M$ та R-модуль M успадковують деякі з найбільш важливих властивостей один одного. Наприклад, показано, що M є проективним (плоским) R-модулем тоді і тільки тоді, коли M^n — проективний (плоский) $R(M)$-модуль.

1. Introduction and preliminaries. In this paper, all rings are commutative with identity and all modules are unitary. Let M be an R-module; there are some attempts to define a product between submodules of M, see for example [5, p. 386]. Based on this idea, in this article, we introduce and investigate a kind of product of submodules of M and especially we study $R(M)$-module $M^n = M\ldots M$, in which $R(M)$ is idealization of M. It is worthy to mention that Nagata introduced the notion of idealization and the idea to use idealization is due to him. Idealization is useful for extending results about ideals to submodules and constructing examples of commutative rings with zero-divisors. The theme throughout is how properties of R-module M are related to those of $R(M)$-module M^n and this is the main goal of this article. For example, in Section 2, we show that M is a projective (flat) R-module if and only if M^n is a projective (flat) $R(M)$-module and in Section 3, we find primary and secondary representation for M^n by means of those of M and conversely. Now, we define the concepts that we will need. Recall that $R(M) = R(+)M$ with coordinate-wise addition and multiplication

$$(r_1,m_1)(r_2,m_2) = (r_1r_2, r_1m_2 + r_2m_1),$$

is a commutative ring with identity, called the idealization of M. Note that R naturally embeds into $R(M)$ via $r \longrightarrow r(+)0$, if N is a submodule of M, then $0(+)N$ is an ideal of $R(M)$, $0(+)M$ is a nilpotent ideal of $R(M)$ of index 2, every ideal that contains $0(+)M$ has the form $I(+)M$ for some ideal I of R, and every ideal that is contained in $0(+)N$ has the form $0(+)K$ for some submodule K of N. The purpose of idealization is to put M inside a commutative ring A so that the structure of M as an R-module is essentially the same as that of M as an A-module, that is, an ideal of A. Since $R \cong R(M)/0(+)M$, $I \longrightarrow I(+)M$ gives a one-to-one correspondence between ideals of R and ideals of $R(M)$ that contains $0(+)M$. Thus the prime (maximal) ideals of $R(M)$ have the form $P(+)M$ where P is a prime (maximal) ideal of R. Some basic results on idealization can be found in [10].

An R-module M is said to be multiplication if every submodule N of M has the form IM for some ideal I of R. Equivalently, $N = [N : M]M$. A submodule K of M is multiplication if and only if $N \cap K = [N : K]K$ for all submodules N of M. See for example [5], for more details.

Generalizing the case for ideals, an R-module M is called a cancellation (weak cancellation) module if $IM = JM$ for ideals I and J of R implies $I = J$ ($I + \operatorname{ann} M = J + \operatorname{ann} M$), see [3]. Examples of cancellation modules include invertible ideals, free modules, and finitely generated faithful multiplication modules [4] (Corollary to Theorem 9). The trace ideal of an R-module M is $\operatorname{Tr}(M) = \sum_{f \in \operatorname{Hom}(M,R)} f(M)$. If M is projective, then $M = \operatorname{Tr}(M)M$ and $\operatorname{ann}(M) = \operatorname{ann}\operatorname{Tr}(M)$ [9] (Proposition 3.30). The set of all prime ideals of ring R is denoted by $\operatorname{Spec}(R)$. Moreover, we denote by $Zdv_R(M)$ the set of all zero-divisors of module M over ring R. Also, for unexplained definitions and terminologies we refer to [6, 9].

2. The multiplication property and product. We begin with the following definition which plays an important role in this article.

Definition. *Let $K_1, K_2, \ldots, K_n$ be submodules of an R-module M. Define product of K_1, K_2,. . . ,K_n as follows*:

$$K_1 K_2 \ldots K_n = \big\{(1_R, k_1, k_2, \ldots, k_n) \mid k_i \in K_i, \ \textit{for all} \ 1 \leqslant i \leqslant n\big\}.$$

One can easily check that $K_1 K_2 \ldots K_n$ forms an $R(M)$-module under the below operations:

$$(1_R, k_1, k_2, \ldots, k_n) + (1_R, k'_1, k'_2, \ldots, k'_n) =$$

$$= (1_R, (k_1 + k'_1), (k_2 + k'_2), \ldots, (k_n + k'_n)),$$

$$(r, m)(1_R, k_1, k_2, \ldots, k_n) = (1_R, rk_1, rk_2, \ldots, rk_n).$$

For convenience, 1_R and $M \ldots M$ (n-times) will be denoted by 1 and M^n, respectively.

In this section, we shall investigate the multiplication, quasimultiplication, projective, flat, faithfully flat, cancellation and weak cancellation properties under this new product submodules.

Let P be a maximal ideal of R and let $T_P(M) = \{m \in M \mid (1-p)m = 0$, for some $p \in P\}$. Then $T_P(M)$ is a submodule of M. An R-module M is called P-torsion if $T_P(M) = M$. On the other hand, M is called P-cyclic provided there exist $x \in M$ and $q \in P$ such that $(1-q)M \subseteq Rx$. El-Bast and Smith [8] (Theorem 1.2), showed that M is multiplication if and only if M is P-torsion or P-cyclic for each maximal ideal P of R.

Next we prove that if $K_1 \ldots K_n$ is a multiplication module, then each K_i, $1 \leq i \leq n$, is a multiplication module. But first we need the following lemma.

Lemma 1. *Let $K_1, \ldots, K_n$ be submodules of an R-module M and P a maximal ideal of R. Then*:

(i) $T_{P(+)M}(K_1 \ldots K_n) = \{(1, m_1, m_2, \ldots, m_n) \mid m_i \in T_P(K_i)\}$. *In particular, $K_1 \ldots K_n$ is $P(+)M$-torsion if and only if each K_i, $1 \leq i \leq n$, is P-torsion.*

(ii) *$K_1 \ldots K_n$ is $P(+)M$-cyclic if and only if each K_i ($1 \leq i \leq n$) is P-cyclic.*

Proof. (i) Let $(1, m_1, m_2, \ldots, m_n) \in T_{P(+)M}(K_1 \ldots K_n)$. Then there exists $(p, m) \in P(+)M$ such that $((1,0) - (p,m))(1, m_1, m_2, \ldots, m_n) = (1, 0, 0, \ldots, 0)$. So $(1, (1-p)m_1, (1-p)m_2, \ldots, (1-p)m_n) = (1, 0, 0, \ldots, 0)$. It follows that $m_i \in T_P(K_i)$, for all $1 \leq i \leq n$. Now, suppose that $m_i \in T_P(K_i)$, $1 \leq i \leq n$. Then there exists $p_i \in P$

such that $(1-p_i)m_i = 0$. If we put $q = 1 - \prod_{i=1}^{n}(1-p_i) \in P$, then $(1-q)m_i = 0$, for each $1 \leq i \leq n$ and hence $((1,0)-(q,0))(1,m_1,m_2,\ldots,m_n) = (1,0,0,\ldots,0)$. Thus $(1,m_1,m_2,\ldots,m_n) \in T_{P(+)M}(K_1\ldots K_n)$.

(ii) To see why (ii) is true, let K_i, $1 \leq i \leq n$, be P-cyclic. Then there exist $m_i \in K_i$ and $p_i \in P$ such that $(1-p_i)K_i \subseteq Rm_i$. If we put $q = 1 - \prod_{i=1}^{n}(1-p_i) \in P$, then $((1,0)-(q,0))K_1\ldots K_n \subseteq R(M)(1,m_1,m_2,\ldots,m_n)$. Hence $K_1\ldots K_n$ is $P(+)M$-cyclic.

Conversely, let $K_1\ldots K_n$ be $P(+)M$-cyclic. Then there exist $(1,k_1,\ldots,k_n) \in \in K_1\ldots K_n$ and $(p,m) \in P(+)M$ such that $((1,0)-(p,m))K_1\ldots K_n \subseteq \subseteq R(M)(1,k_1,\ldots,k_n)$. Thus $(1-p)K_i \subseteq Rk_i$, for all $1 \leq i \leq n$ and so K_i is P-cyclic, as required.

Lemma 1 is proved.

Theorem 1. *Let $K_1,\ldots,K_n$ be submodules of an R-module M. If $K_1\ldots K_n$ is a multiplication $R(M)$-module, then each K_i is a multiplication R-module. Moreover, if M is a multiplication R-module, then M^n is a multiplication $R(M)$-module.*

Proof. Let $K_1\ldots K_n$ be a multiplication $R(M)$-module and P a maximal ideal of R. If each K_i, $1 \leq i \leq n$, is P-torsion, there is nothing to prove. Now, assume that there exists i, $1 \leq i \leq n$, such that K_i is not P-torsion. By Lemma 1(i), $K_1\ldots K_n$ is not $P(+)M$-torsion and so there exist $(p,m) \in P(+)M$ and $(1,k_1,\ldots,k_n) \in K_1\ldots K_n$ such that

$$((1,0)-(p,m))K_1\ldots K_i\ldots K_n \subseteq R(+)M(1,k_1,\ldots,k_i,\ldots,k_n).$$

Hence $(1-p)K_i \subseteq Rk_i$, i.e., K_i is P-cyclic. It follows that each K_i is a multiplication submodule of M.

Now, let M be a multiplication R-module and $P(+)M$ be a maximal ideal of $R(M)$. Suppose that M^n is not $P(+)M$-torsion. By Lemma 1(i), M is not P-torsion and hence there exist $p \in P$ and $m \in M$ such that $(1-p)M \subseteq Rm$. It follows that $((1,0)-(p,0))M^n \subseteq R(+)M(1,m,\ldots,m)$. Thus M^n is $P(+)M$-cyclic and this completes the proof.

For an R-module M, following [7], we set

$$M(P) = \{x \in M \mid sx \in PM, \text{ for some } s \in R \setminus P\},$$

in which P is a prime ideal of R. In [7], it is shown that $M(P) = M$ or $M(P)$ is a submodule of M, for every $P \in \operatorname{Spec}(R)$. As usual, we will denote the Support of M by

$$\operatorname{Supp}_R M = \big\{P \in \operatorname{Spec}(R) \mid \text{there exists } 0 \neq x \in M \text{ such that } \operatorname{ann}(x) \subseteq P\big\}.$$

Recall that an R-module M is called quasimultiplication if $M(P) = PM$, for all $P \in \operatorname{Supp}_R M$. For a reference on quasimultiplication module see [7].

The next result will be used in the Theorem 2.

Lemma 2. *Let M be an R-module. Then*:

(i) $\operatorname{Supp}_{R(M)} M^n = \{P(+)M \mid P \in \operatorname{Supp}_R M\}$.

(ii) $M^n(P(+)M) = \{(1,m_1,\ldots,m_n) \in M^n \mid m_i \in M(P), \textit{for all } 1 \leq i \leq n\}$.

Proof. (i) If $P \in \operatorname{Supp}_R M$, then there exists $0 \neq x \in M$ such that $\operatorname{ann}(x) \subseteq P$. Clearly, $\operatorname{ann}(1, x, x, \ldots, x) = \{(r, m) \mid r \in \operatorname{ann}(x)\} \subseteq P(+)M$. Hence $P(+)M \in \in \operatorname{Supp}_{R(+)M} M^n$ and so $\{P(+)M \mid P \in \operatorname{Supp}_R M\} \subseteq \operatorname{Supp}_{R(M)} M^n$. Now, let $P(+)M \in \operatorname{Supp}_{R(+)M} M^n$. Then there exists $0 \neq (1, x_1, \ldots, x_n) \in M^n$ such that $\operatorname{ann}(1, x_1, \ldots, x_n) \subseteq P(+)M$, and hence $\operatorname{ann} x_1 \cap \operatorname{ann} x_2 \cap \ldots \cap \operatorname{ann} x_n \subseteq P$. Since P is a prime ideal, there exists $1 \leq i \leq n$ such that $\operatorname{ann} x_i \subseteq P$, i.e., $P \in \operatorname{Supp}_R M$ and so $\operatorname{Supp}_{R(M)} M^n \subseteq \{P(+)M \mid P \in \operatorname{Supp}_R M\}$. By the above argument the proof is finished.

(ii) Let $T = \{(1, m_1, \ldots, m_n) \in M^n \mid m_i \in M(P) \text{ for all } 1 \leq i \leq n\}$, and $(1, m_1, m_2, \ldots, m_n) \in M^n(P(+)M)$. Then there exists $(s', m') \in R(+)M \setminus P(+)M$ such that $(s', m')(1, m_1, m_2, \ldots, m_n) \in (P(+)M)M^n$. Hence $s'm_i \in PM$, for all $1 \leq i \leq n$ and so $m_i \in M(P)$ for all $1 \leq i \leq n$. It follows that $M^n(P(+)M) \subseteq T$.

Conversely, let $(1, m_1, \ldots, m_n) \in T$. For every $1 \leq i \leq n$, there exists $s_i \in \in R \setminus P$ such that $s_i m_i \in PM$. Put $s = s_1 \ldots s_n$. Clearly, $s \notin P$ and $sm_i \in PM$, for each $1 \leq i \leq n$. So $(s, 0)(1, m_1, m_2, \ldots, m_n) \in (P(+)M)M^n$ and therefore, $(1, m_1, \ldots, m_n) \in M^n(P(+)M)$. Thus $T \subseteq M^n(P(+)M)$. By the above argument it follows that

$$M^n(P(+)M) = \big\{(1, m_1, \ldots, m_n) \in M^n \mid m_i \in M(P), \text{ for all } 1 \leq i \leq n\big\}.$$

Lemma 2 is proved.

The next result shows how quasimultiplication property of an R-module M can be transferred to an $R(M)$-module M^n and conversely.

Theorem 2. *An R-module M is quasimultiplication if and only if M^n is a quasimultiplication $R(M)$-module.*

Proof. First, note that $M(P) = M$ if and only if $M^n(P(+)M) = M^n$. Suppose that M is a quasimultiplication R-module and $P(+)M \in \operatorname{Supp}_{R(+)M} M^n$. Then $M^n(P(+)M) = \{(1, m_1, \ldots, m_n) \in M^n \mid m_i \in M(P) = PM, \text{ for all } 1 \leq i \leq n\}$. It follows that for all $1 \leq i \leq n$ there exist $p_{ij} \in P$ and $m'_{ij} \in M$ such that $m_i = \sum_{j=1}^{t} p_{ij} m'_{ij}$. Hence

$$(1, m_1, \ldots, m_n) = \left(1, \sum_{j=1}^{t} p_{1j} m'_{1j}, \ldots, \sum_{j=1}^{t} p_{nj} m'_{nj}\right) = \sum_{j=1}^{t} (1, p_{1j} m'_{1j}, \ldots, p_{nj} m'_{nj}).$$

Therefore, $(1, m_1, \ldots, m_n) \in (P(+)M)M^n$. This yields that

$$M^n(P(+)M) \subseteq (P(+)M)M^n.$$

Now, let $\sum_{i=1}^{m} (1, p_i m_{i1}, \ldots, p_i m_{in}) \in (P(+)M)M^n$. Since $M(P) = PM$, $p_i m_{ij} \in \in M(P)$, for each $1 \leq i \leq m$ and $1 \leq j \leq n$. It follows that

$$\left(1, \sum_{i=1}^{m} p_i m_{i1}, \ldots, \sum_{i=1}^{m} p_i m_{in}\right) \subseteq (1, M(P), \ldots, M(P)),$$

and by Lemma 2, $\sum_{i=1}^{m} (1, p_i m_{i1}, \ldots, p_i m_{in}) \in M^n(P(+)M)$. Therefore,

$$(P(+)M)M^n \subseteq M^n(P(+)M),$$

i.e., M^n is a quasimultiplication $R(M)$-module.

Conversely, let M^n be a quasimultiplication $R(M)$-module and $P \in \operatorname{Supp}_R M$. Then $P(+)M \in \operatorname{Supp}_{R(M)} M^n$, by Lemma 2. Since M^n is quasimultiplication, $M^n(P(+)M) = (P(+)M)M^n$, and hence $M(P) = PM$. Thus M is a quasimultiplication R-module.

Theorem 2 is proved.

The following question is araised: Does M^n as an $R(M)$-module have all properties of R-module M? It is easily checked that $\mathbb{Q}$ is a faithful $\mathbb{Z}$-module, but $\operatorname{ann}(\mathbb{Q}^n) = \operatorname{ann}(\mathbb{Q})(+)\mathbb{Q} = 0(+)\mathbb{Q}$. In fact, it shows that every property of M can not be transferred to M^n.

Before we state and prove our next corollary, we need the following proposition.

Proposition 1. *For an R-module M*

$$\operatorname{Tr}(M^n) = \operatorname{Tr}(M)(+) \sum_{g \in \operatorname{Hom}(M,M)} g(M) = \operatorname{Tr}(M)(+)M.$$

Proof. Let $f \in \operatorname{Hom}(M^n, R(+)M)$. It is clear that there exist $g_1 \in \operatorname{Hom}(M, R)$ and $g_2 \in \operatorname{Hom}(M, M)$ such that $f = g_1(+)g_2$. Hence

$$\operatorname{Tr}(M^n) = \sum_{f \in \operatorname{Hom}(M^n, R(+)M)} f(M^n) =$$

$$= \sum_{g_1 \in \operatorname{Hom}(M,R), g_2 \in \operatorname{Hom}(M,M)} g_1(M)(+)g_2(M) =$$

$$= \sum_{g_1 \in \operatorname{Hom}(M,R)} g_1(M) + \sum_{g_2 \in \operatorname{Hom}(M,M)} g_2(M) \subseteq \operatorname{Tr}(M)(+)M.$$

Conversely, let $g \in \operatorname{Hom}(M, R)$. Define $f\colon M^n \longrightarrow R(+)M$ as follows: for each $(1, m_1, m_2, \ldots, m_n) \in M^n$, $f(1, m_1, m_2, \ldots, m_n) = g(m_1+m_2+\ldots+m_n)(+)id(m_1+m_2+\ldots+m_n)$. It is clear that f is well-defined and $R(M)$-homomorphism. Therefore,

$$\operatorname{Tr}(M)(+)M = \sum_{g \in \operatorname{Hom}(M,R)} g(M)(+)M \subseteq \sum f(M^n) \subseteq \operatorname{Tr}(M^n).$$

It follows that $\operatorname{Tr}(M^n) = \operatorname{Tr}(M)(+)M$.

Proposition 1 is proved.

Lemma 3. *Let M be a projective R-module. Then* $\operatorname{Tr}(M)$ *is a finitely generated ideal of R if and only if* $\operatorname{Tr}(M^n)$ *is a finitely generated ideal of* $R(M)$.

Proof. By Proposition 1 and [9] (Proposition 3.3), $\operatorname{Tr}(M^n) = \operatorname{Tr}(M)(+)M = \operatorname{Tr}(M)(+)\operatorname{Tr}(M)M$. Hence, $\operatorname{Tr}(M^n)$ is a finitely generated ideal of $R(M)$ if and only if $\operatorname{Tr}(M)$ is a finitely generated ideal of R [1] (Theorem 7(1)).

Lemma 3 is proved.

It is shown in [9] (Lemma 3.23) that an R-module M is projective if and only if there exist a family of elements $\{m_i\}_{i \in I}$ in M and family $\{f_i\}_{i \in I}$ of elements in $M^* = \operatorname{Hom}_R(M, R)$ such that every $m \in M$ is a finite sum $m = \Sigma m_i f_i(m)$, where $f_i(m) = 0$ almost for every $i \in I$. In the next theorem we prove that M is a projective R-module if and only if M^n is a projective $R(M)$-module.

Theorem 3. *An R-module M is projective if and only if M^n is a projective $R(M)$-module.*

Proof. Let M be a projective R-module and $(1, m_1, m_2, \ldots, m_n) \in M^n$. There exist a family $\{m_{i,j}\}_{(i,j)\in I}$ in M and family $\{f_{(i,j)}\}_{(i,j)\in I}$ of elements in $M^* = \mathrm{Hom}_R(M, R)$ such that $m_j = \sum_i m_{(i,j)} f_{(i,j)}(m_j)$. Hence,

$$(1, m_1, m_2, \ldots, m_n) = (1, \Sigma m_{(i,1)} f_{(i,1)}(m_1), \ldots, \Sigma m_{(i,n)} f_{(i,n)}(m_n)) =$$

$$= \Sigma(1, m_{(i,1)} f_{(i,1)}(m_1), \ldots, m_{(i,n)} f_{(i,n)}(m_n)) =$$

$$= \Sigma(f_{(i,1)}(m_1), 0)(1, m_{(i,1)}, 0, \ldots, 0) + \ldots + (f_{(i,n)}(m_n), 0)(1, 0, \ldots, m_{(i,n)}).$$

For each $(i, j) \in I$, define $g_{(i,j)} \colon M^n \to R(M)$ as follows:

for each $(1, m_1, m_2, \ldots, m_n) \in M^n$, $g_{(i,j)}(1, m_1, m_2, \ldots, m_n) = (f_{(i,j)}(m_j), 0)$. It is clear that $g_{(i,j)}$ is well-defined and $R(M)$-homomorphism. Let $m_{i,j} = \Big(1, 0, \ldots, \ldots, \overbrace{m_{(i,j)}}^{j\text{ th}}, \ldots, 0\Big)$. So $(1, m_1, m_2, \ldots, m_n) = \Sigma g_{(i,j)}(1, m_1, m_2, \ldots, m_n) m_{i,j}$ and hence M^n is a projective $R(M)$-module.

Conversely, let M^n be a projective $R(M)$-module and $m \in M$. Since M^n is projective, there exist a family of elements $f_i \in \mathrm{Hom}(M^n, R(+)M)$ and family $(1, m_{1,i}, \ldots, m_{n,i})$ of elements M^n such that

$$(1, m, \ldots, 0) = \Sigma f_i(1, m, \ldots, 0)(1, m_{1,i}, \ldots, m_{n,i}).$$

Because $f_i \in \mathrm{Hom}(M^n, R(+)M)$, there exist $g_i \in \mathrm{Hom}(M, R)$ and $g'_i \in \mathrm{Hom}(M, M)$ such that $f_i = g_i(+)g'_i$. Thus $m = \Sigma g_i(m) m_{1,i}$, i.e., M is a projective R-module.

Theorem 3 is proved.

It is well-known that a projective module is weak cancellation if and only if its trace is a finitely generated ideal [14] (Theorem 4.1). The following question raises: If M is a weak cancellation module, can we deduce that M^n is a weak cancellation module? The following corollary gives an affirmative answer in case the projective modules.

***Corollary* 1.** *Let M be a projective R-module. Then M is a weak cancellation R-module if and only if M^n is a weak cancellation $R(M)$-module.*

Proof. Let M be a projective weak cancellation R-module. Then $\mathrm{Tr}(M)$ is finitely generated by [13] (Theorem 4.1). By the above theorem, Lemma 3 and [13] (Theorem 4.1), M^n is a weak cancellation module. The proof of the converse is similar.

Our the following result is taken from [13] (Theorem 4.2).

***Corollary* 2.** *Let M be a projective R-module. Then M is a cancellation R-module if and only if M^n is a cancellation $R(M)$-module.*

Proof. Let M be a projective cancellation R-module. By [13] (Theorem 4.2), $\mathrm{Tr}(M) = R$. By Proposition 1, $\mathrm{Tr}(M^n) = \mathrm{Tr}(M)(+)M = R(M)$. Hence, by [13] (Theorem 4.2), M^n is a cancellation module. The proof of the converse is similar.

It is shown in [11] (Theorem 7.6), that M is flat if and only if for every pair of finite subsets $\{x_1, \ldots, x_n\}$ and $\{a_1, \ldots, a_n\}$ of M and R, respectively, such that $\sum_{i=1}^n a_i x_i = 0$ there exist elements $z_1, \ldots, z_k \in M$ and $b_{ij} \in R$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ such that $\sum_{i=1}^n b_{ij} a_i = 0$ $(j = 1, \ldots, k)$ and $x_i = \sum_{j=1}^k b_{ij} z_j$. Our main concern in this part is to show that M is flat if and only if M^n is flat.

Theorem 4. *An R-module M is flat if and only if M^n is a flat $R(M)$-module.*

Proof. Let M be a flat R-module and $\sum_{i=1}^{m}(a_i,0)(1,x_{1,i},x_{2,i},\ldots,x_{n,i}) = 0$. Then $\sum_{i=1}^{m} a_i x_{j,i} = 0$, for every $1 \le j \le n$. Since M is flat, there exist elements $z_{j,1},\ldots,z_{j,s} \in M$ and $b_{j,ik} \in R$, $i = 1,\ldots,m$, $k = 1,\ldots,s$ and $j = 1,\ldots,n$ such that $\sum_{i=1}^{m} b_{j,ik} a_i = 0$, $k = 1,\ldots,s$ and $x_{j,i} = \sum_{k=1}^{s} b_{j,ik} z_{j,k}$, for $j = 1,\ldots,n$. Hence $\sum_{i=1}^{m}(b_{j,ik},0)(a_i,0) = 0$ and

$$(1,x_{1,i},\ldots,x_{n,i}) =$$

$$= \sum_{k=1}^{s}(b_{1,ik},0)(1,z_{1,k},0,\ldots,0) + \ldots + (b_{n,ik},0)(1,0,\ldots,z_{n,k}), \quad 1 \le i \le m.$$

Therefore, M^n is a flat $R(M)$-module.

Conversely, let M^n be flat. Suppose that $\{x_1,\ldots,x_m\}$ and $\{a_1,\ldots,a_m\}$ are two finite subsets of M and R, respectively, such that $\sum_{i=1}^{m} a_i x_i = 0$. This implies that

$$\sum_{i=1}^{m}(a_i,0)(1,x_i,0,\ldots,0) = 0.$$

Since M^n is flat, there exist elements $(1,z_1,\ldots,0),\ldots,(1,z_s,\ldots,0) \in M^n$ and $(b_{i,k},0) \in R(M)$ $(i = 1,\ldots,m,\ k = 1,\ldots,s)$, such that

$$\sum_{i=1}^{m}(b_{i,k},0)(a_i,0) = 0, \quad k = 1,\ldots,s,$$

and

$$(1,x_i,0,\ldots,0) = \sum_{k=1}^{s}(b_{i,k},0)(1,z_k,0,\ldots,0).$$

Hence

$$\sum_{i=1}^{m} b_{i,k} a_i = 0, \qquad x_i = \sum_{k=1}^{s} b_{i,k} z_k.$$

By the above argument, M is a flat R-module.

Theorem 4 is proved.

It is known that an R-module M is called faithfully flat if M is flat and $N \otimes M \neq 0$ for non-zero R-module N. Equivalently, M is flat and $PM \neq M$ for every maximal ideal P of R [11] (Theorem 7.2). We next show that if M is faithfully flat, then M^n is faithfully flat.

***Corollary* 3.** *An R-module M is faithfully flat if and only if M^n, $n \ge 2$, is a faithfully flat $R(M)$-module.*

Proof. It is easy to check that for every maximal ideal P of R, $PM \neq M$ if and only if $(P(+)M)M^n \neq M^n$. The result now follows by Theorem 4.

3. Product submodules and decomposition. Recall that a proper submodule N of a module M is said to be primary submodule if the condition $ra \in N$, $r \in R$ and $a \in M$, implies that $a \in N$ or $r^n M \subseteq N$, for some positive integer n. Let $T = K_1 \ldots K_n$ be a primary submodule of M^n and $r_i k_i \in K_i$ where $r_i \in R$, $k_i \in M$. Then

$$(r_i,0)\Big(1,0,\ldots,\overbrace{k_i}^{i\,\text{th}},\ldots,0\Big)\in K_1\ldots K_i\ldots K_n.$$

Since $K_1\ldots K_i\ldots K_n$ is primary, either $(1,0,\ldots,k_i,0,\ldots,0)\in K_1\ldots K_i\ldots K_n$ or $(r_i,0)^mM^n\subseteq K_1\ldots K_i\ldots K_n$, for some positive integer m and this means that $k_i\in$ $\in K_i$ or $r_i^mM\subseteq K_i$. By the above argument it follows that K_i is a primary submodule for all $1\le i\le n$.

Conversely, let N be a primary submodule and $(r,m)(1,m_1,\ldots,m_n)\in N^n$, for some $(r,m)\in R(M)$, and $(1,m_1,\ldots,m_n)\in M^n$. Hence $rm_i\in N$ for each $1\le i\le n$. Since N is primary, $m_i\in N$ or $r^tM\subseteq N$, for some positive integer t. It follows that N^n is a primary submodule of M^n.

By the above argument we have the following theorem.

Theorem 5. *Let $T=K_1\ldots K_n$ be a primary submodule of M^n. Then K_i is a primary submodule of M for all $1\le i\le n$. Furthermore, let N be a primary submodule of M. Then N^n, $n\ge 2$, is a primary submodule of M^n.*

Before we state and prove our next theorem, the following lemma is needed.

Lemma 4. *Let $Q_1,\ldots,Q_m$ be submodules of an R-module M. Then*:

(i) $(Q_1+Q_2+\ldots+Q_m)^n=Q_1^n+Q_2^n+\ldots+Q_m^n$,

(ii) $(Q_1\cap Q_2\cap\ldots\cap Q_m)^n=Q_1^n\cap Q_2^n\cap\ldots\cap Q_m^n$.

Proof. (i) First suppose that $m=2$. We have only to prove that $(Q_1+Q_2)^n\subseteq Q_1^n+$ $+Q_2^n$. Let $(1,m_1,\ldots,m_n)\in(Q_1+Q_2)^n$. Then there exist $q_i\in Q_1$ and $q_i'\in Q_2$ such that $(1,m_1,\ldots,m_n)=(1,q_1+q_1',\ldots,q_n+q_n')=(1,q_1,\ldots,q_n)+(1,q_1',\ldots,q_n')\in$ $\in Q_1^n+Q_2^n$ and hence $(Q_1+Q_2)^n\subseteq Q_1^n+Q_2^n$. If $m>2$, then the assertion follows by the case $m=2$ and induction on m. Hence $(Q_1+Q_2+\ldots+Q_m)^n=Q_1^n+Q_2^n+\ldots+Q_m^n$.

(ii) It is easy to check that $(Q_1\cap Q_2)^n=Q_1^n\cap Q_2^n$. Induction on m shows that $(Q_1\cap Q_2\cap\ldots\cap Q_m)^n=Q_1^n\cap Q_2^n\cap\ldots\cap Q_m^n$, as desired.

Lemma 4 is proved.

We record the following theorem.

Theorem 6. *Let N be a submodule of an R-module M. If N has a primary decomposition, then N^n, $n\ge 2$, has a primary decomposition.*

Proof. Suppose that N has a primary decomposition. Then $N=Q_1\cap Q_2\cap\ldots\cap Q_m$, where each Q_i is a P_i-primary submodule of M. Hence $N^n=Q_1^n\cap Q_2^n\cap\ldots\cap Q_m^n$, by Lemma 5. To see why this is a primary decomposition of N^n, note first that $\dfrac{M^n}{Q_i^n}\ne 0$, because $\dfrac{M}{Q_i}\ne 0$. Next, if $(r,m)\in Zdv_{R(M)}\left(\dfrac{M^n}{Q_i^n}\right)$, then there exists a positive integer n_1 such that $r^{n_1}\left(\dfrac{M}{Q_i}\right)=0$. Since $r^{n_1}M\subseteq Q_i$, $(r,m)^{n_1}M^n=(r^{n_1},n_1r^{n_1-1}m)M^n\subseteq$ $\subseteq Q_i^n$ and hence $(r,m)^{n_1}\dfrac{M^n}{Q_i^n}=0$. It remains to be shown that Q_i^n is $P_i(+)M$-primary $\left(\text{where } P_i=\operatorname{rad}\left(\operatorname{ann}_R\dfrac{M}{Q_i}\right)\right)$. Let $(t,m)\in\operatorname{rad}\left(\operatorname{ann}_{R(M)}\dfrac{M^n}{Q_i^n}\right)=P_i(+)M$. Then there exists a positive integer n_1 such that $(t^{n_1},n_1t^{n_1-1}m)M^n\subseteq Q_i^n$, so $t^{n_1}M\subseteq Q_i$. It follows that $t\in\operatorname{rad}\left(\operatorname{ann}_R\dfrac{M}{Q_i}\right)=P_i$. Therefore, $\operatorname{rad}\left(\operatorname{ann}_{R(M)}\dfrac{M^n}{Q_i^n}\right)\subseteq P_i(+)M$.

Now, let $(p_i, m) \in P_i(+)M = \operatorname{rad}\left(\operatorname{ann}_R \frac{M}{Q_i}\right)(+)M$. There exists a positive integer n_1 such that $p_i^{n_1} M \subseteq Q_i$ and so $(p_i, m)^{n_1} M^n \subseteq Q_i^n$. Since $(p_i, m)^{n_1} \dfrac{M^n}{Q_i^n} = 0$, $(p_i, m) \in \operatorname{rad}\left(\operatorname{ann}_{R(M)} \dfrac{M^n}{Q_i^n}\right)$. So $P_i(+)M \subseteq \operatorname{rad}\left(\operatorname{ann}_{R(M)} \dfrac{M^n}{Q_i^n}\right)$, as required.

Theorem 6 is proved.

It is naturally to ask when the converse of Theorem 6 is true. See the next theorem.

Theorem 7. *Let N^n be a submodule of M^n, $n \geq 2$. If N^n has a primary decomposition of the form $N^n = K_1^n \cap K_2^n \cap \ldots \cap K_m^n$, where K_i^n, $1 \leq i \leq m$, is $P_i(+)M$-primary, then N has a primary decomposition of the form $N = K_1 \cap K_2 \cap \ldots \cap K_m$.*

Proof. We show that $N = K_1 \cap K_2 \cap \ldots \cap K_m$ is a primary decomposition for N. First, it is clear that $N = K_1 \cap K_2 \cap \ldots \cap K_m$. Next, we show that each K_i, for $1 \leq i \leq m$, is P_i-primary. Suppose that $a \in Zdv_R \left(\dfrac{M}{K_i}\right)$. Then there exists $m \in M \setminus K_i$ such that $am \in K_i$. Since $m \notin K_i$, $(1, m, \ldots, m, \ldots, m) \notin K_i^n$. But $(a, 0)(1, m, \ldots, m, \ldots, m) = (1, am, \ldots, am, \ldots, am) \in K_i^n$. So $(a, 0) \in Zdv\left(\dfrac{M^n}{K_i^n}\right)$. Because K_i^n is $P_i(+)M$-primary, there exists a positive integer n_1 such that $(a, 0)^{n_1}\left(\dfrac{M^n}{K_i^n}\right) = 0$, and hence $a^{n_1}\left(\dfrac{M}{K_i}\right) = 0$. It remains to be shown $\operatorname{rad}\left(\operatorname{ann}_R \dfrac{M}{K_i}\right) = P_i$. Let $r \in \operatorname{rad}\left(\operatorname{ann}_R \dfrac{M}{K_i}\right)$. Then $r^{n_1} M \subseteq K_i$, for some positive integer n_1. Thus $(r, 0)^{n_1} M^n \subseteq K_i^n$ and so $(r, 0) \in \operatorname{rad}\left(\operatorname{ann}_{R(M)} \dfrac{M^n}{K_i^n}\right) = P_i(+)M$, i.e., $r \in P_i$. Therefore, $\operatorname{rad}\left(\operatorname{ann}_R \dfrac{M}{K_i}\right) \subseteq P_i$.

Conversely, let $r \in P_i$. Then $(r, 0) \in P_i(+)M = \operatorname{rad}\left(\operatorname{ann}_{R(M)} \dfrac{M^n}{K_i^n}\right)$ and hence $r^{n_1} M \subseteq K_i$, for some integer n_1. It follows that $P_i \subseteq \operatorname{rad}\left(\operatorname{ann}_R \dfrac{M}{K_i}\right)$.

Theorem 7 is proved.

I. G. Macdonald has developed the theory of attached prime ideals and secondary representations of a module, which is, in a certain sense, dual to the theory of associated prime ideals and primary decompositions. Let us recall from [11], the definition of secondary module. An R-module M is said to be secondary if $M \neq 0$ and, for each $a \in R$ the endomorphism $\varphi_a \colon M \to M$ defined by $\varphi_a(m) = am$ (for $m \in M$) is either surjective or nilpotent.

If M is secondary, then $P = \sqrt{\operatorname{ann} M}$ is a prime ideal, and M is said to be P-secondary. A secondary representation of an R-module M is an expression of M as a finite sum of secondary submodules:

$$M = N_1 + N_2 + \ldots + N_n.$$

Let N be a submodule of an R-module M, in the following theorem we investigate secondary representation of N^n.

Theorem 8. *Let N be a submodule of an R-module M. If N has a secondary representation, then N^n, $n \geq 2$, has a secondary representation. Conversely, let N^n*

has a secondary representation of the form $N^n = Q_1^n + Q_2^n + \ldots + Q_m^n$ in which every Q_i^n is $P_i(+)M$-secondary. Then N has a secondary representation of the form $N = Q_1 + Q_2 + \ldots + Q_m$.

Proof. Let $N = Q_1 + Q_2 + \ldots + Q_m$ be a secondary representation of N with $\operatorname{rad}\left(\operatorname{ann} Q_i\right) = P_i$. Then $N^n = Q_1^n + Q_2^n + \ldots + Q_m^n$, by Lemma 1. To see why this is a secondary representation of N^n note first that for each $(r, m) \in R(M)$, the endomorphism $\phi_{(r,m)} \colon Q_i^n \to Q_i^n$ defined by $\phi_{(r,m)}((1, q_{1,i}, q_{2,i}, \ldots, q_{n,i})) = (r, m)(1, q_{1,i}, q_{2,i}, \ldots, q_{n,i}) = (1, rq_{1,i}, rq_{2,i}, \ldots, rq_{n,i})$ induces endomorphism $\varphi_r \colon Q_i \to Q_i$ defined by $\varphi_r(q) = rq$. Since Q_i is secondary, φ_r is either surjective or nilpotent. If φ_r is surjective, then it is clear that $\phi_{(r,m)}$ is surjective. If φ_r is nilpotent, then there exists a positive integer m_1 such that $(\varphi_r)^{m_1} = 0$ and therefore, $r^{m_1} q = 0$, for all $q \in Q$. It follows that $(r, m)^{m_1}(1, q_{1,i}, q_{2,i}, \ldots, q_{n,i}) = (1, r^{m_1} q_{1,i}, r^{m_1} q_{2,i}, \ldots, r^{m_1} q_{n,i}) = (1, 0, 0, \ldots, 0)$ for all $(1, q_{1,i}, q_{2,i}, \ldots, q_{n,i}) \in Q_i^n$ and hence $\phi_{(r,m)}^{m_1} = 0$, i.e., $\phi_{(r,m)}$ is nilpotent. This yields that Q_i^n is a secondary submodule of M^n. It remains to be shown that $\operatorname{rad}\left(\operatorname{ann} Q_i^n\right) = P_i(+)M$, for all $1 \le i \le n$. Suppose that $(t, m) \in \operatorname{rad}\left(\operatorname{ann} Q_i^n\right)$. Then there exists a positive integer m_1 such that $t^{m_1} Q_i = 0$. So $t \in \operatorname{rad}\left(\operatorname{ann} Q_i\right) = P_i$. It turns out that $\operatorname{rad}\left(\operatorname{ann} Q_i^n\right) \subseteq P_i(+)M$. One can check that $P_i(+)M \subseteq \operatorname{rad}\left(\operatorname{ann} Q_i^n\right)$. Thus Q_i^n is $P_i(+)M$-secondary.

Conversely, let N^n has a secondary representation of the form $N^n = Q_1^n + Q_2^n + \ldots + Q_m^n$, in which Q_i^n is $P_i(+)M$-secondary, for all $1 \le i \le n$. Clearly, $N = Q_1 + Q_2 + \ldots + Q_m$. We show that each Q_i is a P_i-secondary submodule of M. Let $\varphi_r \colon Q_i \to Q_i$ be an endomorphism defined by $\varphi_r(q) = rq$. We show that φ_r is either surjective or nilpotent. The endomorphism φ_r induces endomorphism $\phi_{(r,0)} \colon Q_i^n \to Q_i^n$ defined by $\phi_{(r,0)}(1, q_1, q_2, \ldots, q_n) = (1, rq_1, rq_2, \ldots, rq_n)$. Since Q_i^n is a secondary submodule, $\phi_{(r,0)}$ is either surjective or nilpotent, and so φ_r is either surjective or nilpotent. It is easy to check that $\operatorname{rad}\left(\operatorname{ann} Q_i\right) = P_i$. Hence Q_i is P_i-primary.

***Example* 1.** Let R be an integral domain and K be quotient field R. Then K^n is a $0(+)M$ secondary $R(+)K$-module. In particular, $\mathbb{Q}^2$ is a $0(+)\mathbb{Q}$ secondary $\mathbb{Z}(+)\mathbb{Q}$-module.

***Example* 2.** If P is a maximal ideal of R, then $\dfrac{R^n}{(P^m)^n}$ is a $P(R)$-secondary $R(R)$-module, for every positive integer m.

***Example* 3.** Let R be a local ring with maximal ideal P. If every element of P is nilpotent, then R^n is a $P(R)$-secondary $R(R)$-module.

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