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ON GENERALIZED DERIVATIONS SATISFYING CERTAIN IDENTITIES

ПРО УЗАГАЛЬНЕНІ ДИФЕРЕНЦІЮВАННЯ, ЩО ЗАДОВОЛЬНЯЮТЬ ДЕЯКІ ТОТОЖНОСТІ

Let R be a prime ring with char $R \neq 2$ and d be a generalized derivation on R. The goal of this study is to investigate the generalized derivation d satisfying any one of the following identities:

(i) $d[(x, y)] = [d(x), d(y)]$ for all x, $y \in R$;

(ii) $d[(x, y)] = [d(y), d(x)]$ for all x, $y \in R$;

(iii) either $d([x, y]) = [d(x), d(y)]$ or $d([x, y]) = [d(y), d(x)]$ for all x, $y \in R$.

Припустимо, що R — просте кільце з $R \neq 2$, а d — узагальнене диференціювання на R. Мета цієї роботи полягає у дослідженні диференціювання d, що задовольняє будь-яку з наступних тотожностей:

(i) $d[(x, y)] = [d(x), d(y)]$ для всіх x, $y \in R$;

(ii) $d[(x, y)] = [d(y), d(x)]$ для всіх x, $y \in R$;

(iii) $d([x, y]) = [d(x), d(y)]$ або $d([x, y]) = [d(y), d(x)]$ для всіх x, $y \in R$.

1. Introduction. Let R always denote an associative ring with center Z, extended centroid C, Utumi quotient ring U. Recall that an additive mapping $\alpha: R \to R$ is called a derivation if $\alpha(xy) = \alpha(x)y + x\alpha(y)$ holds for all x, $y \in R$. The study of prime rings with derivations was initiated by Posner [16]. Many related generalizations have been done on this subject (see [16, 8], where further references can be found). Following Bresar [8], $d: R \to R$ is called a *generalized derivation* if there exists a derivation α of R such that $d(xy) = d(x)y + x\alpha(y)$ for all x, $y \in R$. It is clear that the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier (i.e., an additive mapping $f: R \to R$ satisfying $f(xy) = f(x)y$ for all x, $y \in R$). In [10], Hvala initiated the study of generalized derivations from the algebraic viewpoint. Many authors have studied generalized derivations in the context of prime and semiprime rings (see [1–4, 13, 14, 17]). In [13], T. K. Lee extended the definition of generalized derivations as follows: By a generalized derivation we mean an additive mapping $d: I \to U$ such that $d(xy) = d(x)y + x\alpha(y)$ for all x, $y \in I$, where U is the right Utumi quotient ring, I is a dense right ideal of R and α is a derivation from I into U. Moreover Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U and he obtained the following results:

Theorem ([13], Theorem 3)**.** *Every generalized derivation d on a dense right ideal of R can be uniquely extended to U and assumes the form $d(x) = ax + \alpha(x)$ for some $a \in U$ and a derivation α on U.*

Over the last three decades, several authors have proved the commutativity theorems for prime or semiprime rings admitting derivations or generalized derivations satisfying some relations (see [3, 4, 7, 17]). In [4], M. Ashraf et al. investigated the commutativity of a prime ring R admitting a generalized derivation F with associated derivation d satisfying any one of the following conditions: $d(x)\circ F(y)=0$, $[d(x), F(y)]=0$, $d(x)\circ F(y)=x\circ y$, $d(x)\circ F(y)+x\circ y=0$, $d(x)\circ F(y)-xy\in Z$, $d(x)\circ F(y)+xy\in Z$, $[d(x),F(y)]=[x,y]$, $[d(x),F(y)]+[x,y]=0$ for all x, $y\in I$, where I is a nonzero ideal of R, $[x,y]=xy-yx$ and $x\circ y=xy+yx$. In [3], the authors proved the commutativity of a prime ring R in which a generalized derivation F satisfies any one of the following properties: (i) $F(xy)-xy\in Z$, (ii) $F(xy)+xy\in Z$, (iii) $F(xy)-yx\in Z$, (iv) $F(xy)+yx\in Z$, (v) $F(x)F(y)-xy\in Z$ and (vi) $F(x)F(y)+xy\in Z$, for all x, $y\in R$. In [17], Shuliang proved that if L is a lie ideal of a prime ring R such that $u^2\in L$ for all $u\in L$ and if F is a generalized derivation on R associated with a derivation d on R satisfying any one of the following conditions: (1) $d(x)\circ F(y)=0$, (2) $[d(x),F(y)]=0$, (3) either $d(x)\circ F(y)=x\circ y$ or $d(x)\circ F(y)+x\circ y=0$, (4) either $d(x)\circ F(y)=[x,y]$ or $d(x)\circ F(y)+[x,y]=0$, (5) either $d(x)\circ F(y)-xy\in Z$ or $d(x)\circ F(y)+xy\in Z$, (6) $[d(x),F(y)]=[x,y]$ or $[d(x),F(y)]+[x,y]=0$, (7) either $[d(x),F(y)]=x\circ y$ or $[d(x),F(y)]+x\circ y=0$ for all x, $y\in L$, then either $d=0$ or $L\subseteq Z$.

In this paper we aim to investigate the generalized derivation d on a prime ring R associated with a derivation α on satisfying any one of the following identities: (i) $d([x,y])=[d(x),d(y)]$ for all x, $y\in R$, (ii) $d([x,y])=[d(y),d(x)]$ for all x, $y\in R$, (iii) either $d([x,y])=[d(x),d(y)]$ or $d([x,y])=[d(y),d(x)]$ for all x, $y\in R$.

In all that follows, unless stated otherwise, R will be a prime ring. The related object we need to mention is the two-sided Quotient ring Q of R, the right Utumi quotient ring U of R (sometimes, as in [6], U is called the maximal ring of quotients). The definitions, the axiomatic formulations and the properties of this quotient ring U can be found in [6] and [5].

We make a frequent use of the theory of generalized polynomial identities and differential identities (see [6, 9, 11, 12, 15]). In particular we need to recall that when R is a prime ring and I a nonzero two-sided ideal of R, then I, R, Q and U satisfy the same generalized polynomial identities [9] and also the same differential identities [12].

We will also make frequent use of the following result due to Kharchenko [11] (see also [12]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R. Let $f\left(x_1,\ldots,x_n, d(x_1),\ldots,d(x_n)\right)$ be a differential identity in I, that is the relation

$$f\left(r_1,\ldots,r_n, d(r_1),\ldots,d(r_n)\right) = 0$$

holds for all $r_1,\ldots,r_n\in I$. Then one of the following holds:

1) either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d(x) = [q, x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f\left(r_1, \ldots, r_n, [q, r_1], \ldots, [q, r_n]\right);$$

2) or I satisfies the generalized polynomial identity

$$f\left(x_1, \ldots, x_n, y_1, \ldots, y_n\right).$$

In [14], T. K. Lee and W. K. Shiue proved a version of Kharchenko's theorem for generalized derivations and presented some results concerning certain identities with generalized derivations. More details about generalized derivations can be found in [10, 11, 13, 14].

2. The results. In the following, we assume that R is a prime ring with char $R \neq 2$ and that Z is the center of R unless stated otherwise. We denote the identity map of a ring R by I_{id} (i.e., the map $I_{id}: R \to R$ defined by $I_{id}(x) = x$ for all $x \in R$). By a map $-I_{id}: R \to R$ we mean the map defined by $(-I_{id})(x) = -x$ for all $x \in R$.

We begin with the following.

Lemma 1. *Let R be a prime ring with char $R \neq 2$ and d be a generalized derivation on R associated with a derivation α on R. If $d\left([x, y]\right) = \left[d(x), d(y)\right]$ holds for all x, $y \in R$ then either R is commutative, or $d = 0$, or $d = I_{id}$.*

Proof. As we stated as theorem we can take the generalized derivation d as the form $d(x) = ax + \alpha(x)$ where $a \in U$ and α is a derivation on U.

If $\alpha = 0$, then by the hypothesis we have $a[x, y] = [ax, ay]$ for all x, $y \in R$. Replacing yz by y we have $ay[x, z] = ay[ax, z]$, hence $ay[x - ax, z] = 0$ for all x, y, $z \in R$. By the primeness of R we get either $a = 0$ or $[x - ax, z] = 0$ for all $x, z \in R$. The first case gives us that $d = 0$, as desired. For the second case, let $[x - ax, z] = 0$ for all $x, z \in R$. Substituting xyr by x we have $0 = \left[(xy - axy)r, z\right] = (xy - axy)[r, z] = (x - ax)y[r, z]$ for all x, y, $r \in R$. By the primeness of R we obtain that either R is commutative, or $x - ax = 0$ for all $x \in R$ implying that $d(x) = ax = x$, i.e., $d = I_{id}$, as desired.

Now we may consider the case that R is not commutative. Suppose $\alpha \neq 0$. Since R and U satisfy the same differential identities [12], we get

$$a[x, y] + \left[\alpha(x), y\right] + \left[x, \alpha(y)\right] = \left[ax + \alpha(x), ay + \alpha(y)\right] \quad \text{for all} \quad x, \ y \in U. \quad (1)$$

In light of Kharchenko's theory [11] we can divide the proof into two cases.

Assume first that α is an outer derivation of U. By Kharchenko's theorem in [11, 12], we get

$$a[x, y] + [z, y] + [x, w] = [ax, ay] + [ax, w] + [z, ay] + [z, w]$$

for all x, y, z, $w \in U$. In particular, taking $w = z = 0$ we obtain $a[x, y] = [ax, ay]$. By the same argument as above we have either R is commutative or

$a = 0$. Let $a = 0$. Using this fact and taking $w = y$ in the above relation we have $[x, w] = 0$ for all $x, w \in U$ implying R is commutative. It is seen that the two cases give us a contradiction.

Assume now that a is an inner derivation of U induced by an element $q \in U$, that is $\alpha(x) = [q, x]$, for all $x \in U$. In this case $d(x) = ax + \alpha(x) = ax + [q, x]$. Replacing 1 for y in (1) we have

$$[ax + \alpha(x), a] = 0 \quad \text{for all} \quad x \in U. \tag{2}$$

Replacing q by x in (2) we get $[aq, a] = a[q, a] = 0$, i.e., $a\alpha(a) = 0$. Using (2), we have

$$a[x, a] + [\alpha(x), a] = 0 \quad \text{for all} \quad x \in U. \tag{3}$$

Taking rx in place of x in (3)

$$ar[x, a] + \alpha(r)[x, a] + [r, a]\alpha(x) + r[\alpha(x), a] = 0 \quad \text{for all} \quad x, r \in U. \tag{4}$$

Say $\beta(x) = [a, x]$, $x \in U$. By (3) we have $0 = a[x, a] + \alpha(x)a - a\alpha(x) = -a(\beta(x) + \alpha(x)) + \alpha(x)a$ for all $x \in U$. Hence we get

$$a(\beta(x) + \alpha(x)) = \alpha(x)a \quad \text{for all} \quad x \in U. \tag{5}$$

By (3) we have $r[\alpha(x), a] = -ra[x, a]$ for all $r, x \in U$. Using this fact in (4) we arrive at $0 = ar[x, a] + \alpha(r)[x, a] + [r, a]\alpha(x) - ra[x, a] = [a, r][x, a] + \alpha(r)[x, a] + [r, a]\alpha(x) = -\beta(r)\beta(x) - \alpha(r)\beta(x) - \beta(r)\alpha(x)$. The last relation implies that

$$(\beta(r) + \alpha(r))\beta(x) + \beta(r)\alpha(x) = 0 \quad \text{for all} \quad r, x \in U. \tag{6}$$

Multiplying (6) by a from the left-hand side and using (5) we find that $0 = a(\beta(r) + \alpha(r))\beta(x) + a\beta(r)\alpha(x) = \alpha(r)a\beta(x) + a\beta(r)\alpha(x)$, i.e.,

$$\alpha(r)a\beta(x) + a\beta(r)\alpha(x) = 0 \quad \text{for all} \quad r, x \in U. \tag{7}$$

Substituting zx by x in (7) and using (7) we have

$$\alpha(r)az\beta(x) + a\beta(r)z\alpha(x) = 0.$$

Taking $\alpha(z)$ instead of z in the last relation and using (7) again we get

$$\alpha(r)a(\alpha(z)\beta(x) - \beta(z)\alpha(x)) = 0.$$

Replacing rs by r we arrive at

$$\alpha(r)sa(\alpha(z)\beta(x) - \beta(z)\alpha(x)) = 0.$$

Since U is prime and $\alpha \neq 0$ we obtain $a(\alpha(z)\beta(x) - \beta(z)\alpha(x)) = a\alpha(z)\beta(x) - a\beta(z)\alpha(x) = 0$. Using (7) in the last relation we have $(a\alpha(z) + \alpha(z)a)\beta(x) = 0$ for all $x, z \in U$. Substituting rx by x in the last relation we get

$$\big(a\alpha(z)+\alpha(z)a\big)r\beta(x) \;=\; 0 \quad \text{for all} \quad x,\, z\in U\,.$$

By the primeness of U we obtain that either $\beta=0$, or $a\alpha(z)+\alpha(z)a=0$ for all $z\in U$.

The first case implies that $a\in C$. Using this fact in (1) we have

$$(a-a^2)[x,y]+(1-a)\big([\alpha(x),y]+[x,\alpha(y)]\big) \;=\; \big[\alpha(x),\alpha(y)\big] \quad \text{for all} \quad x\,,\ y\in U\,. \tag{8}$$

Replacing q by y in (8) and using the facts that $\alpha(x)=[q,x]$ and $\alpha^2(x) = [q,\alpha(x)]$ we get

$$(a-a^2)\alpha(x)+(1-a)\alpha^2(x)=0\,.$$

Taking xy for x and using $a\in C$ we have $2(1-a)\alpha(x)\alpha(y)=0$. Since char $R\neq 2$ and $a\in C$ we have either $\alpha(x)\alpha(y)=0$ for all x, $y\in U$ or $a=1$. If $\alpha(x)\alpha(y)=0$, then taking ry for y we get $\alpha(x)r\alpha(y)=0$ implying that $\alpha=0$ by the primeness of U, a contradiction. If $a=1$, then we find $\big[\alpha(x),\alpha(y)\big]=0$. Substituting yq by y in the last relation we have $\alpha(y)\alpha^2(x)=0$ for all x, $y\in U$. Since $\alpha\neq 0$ and U is prime we get $\alpha^2(x)=0$, implying that $\alpha=0$, a contradiction.

So we are forced to conclude that

$$a\alpha(z)+\alpha(z)a \;=\; 0 \quad \text{for all} \quad z\in U\,. \tag{9}$$

Using (9) in (3) we have $0=a[x,a]+\alpha(x)a-a\alpha(x) \;=\; -a\beta(x)-a\alpha(x)-a\alpha(x) = -a\big(\beta(x)+2\alpha(x)\big)$. Hence we get $a\big(\beta(x)+2\alpha(x)\big)=0$. Replacing rx by x in the last relation and using the primeness of U we obtain that either $a=0$ or $\beta(x)=-2\alpha(x)$ for all $x\in U$.

If $a=0$, (1) is reduced to

$$\big[\alpha(x),y\big]+\big[x,\alpha(y)\big] \;=\; \big[\alpha(x),\alpha(y)\big].$$

Substituting q by y we have $\alpha^2(x)=0$, implying that $\alpha=0$, a contradiction. So we arrive at the case $\beta(x)=-2\alpha(x)$ for all $x\in U$. Replacing yx by y in the hypothesis we get

$$[x,y]\alpha(x) \;=\; d(y)\big[d(x),x\big]+\big[d(x),y\big]\alpha(x)+y\big[d(x),\alpha(x)\big] \quad \text{for all} \quad x\,,\ y\in U\,. \tag{10}$$

Taking yz instead of y in (10) and using (10) we have

$$\big[x-d(x),y\big]z\alpha(x) \;=\; [a,y]z\big[d(x),x\big]+\alpha(y)z\big[d(x),x\big].$$

Since $\beta(x)=[a,x]=-2\alpha(x)$ and char $R\neq 2$ we get $\alpha(a)=0$. Using this fact and taking a in place of y in the above relation we obtain that

$$[x-d(x),a]z\alpha(x) = 0 \quad \text{for all} \quad x,\ z\in U.$$

By the primeness of U we have that for each $x\in U$, either $[x-d(x),a]=0$ or $\alpha(x)=0$. Let $H = \{x\in U:[x-d(x),a]=0\}$ and $K=\{x\in U:\alpha(x)=0\}$. It is clear that $(H,+)$ and $(K,+)$ are two additive subgroup of $(U,+)$ such that $(U,+) = (H,+)\cup(K,+)$. But a group can not be the union two proper subgroups. Therefore we get either $U=H$ or $U=K$. Since $\alpha\neq 0$ we arrive at $[x-d(x),a]=0$ for all $x\in U$. By (3) the last relation implies that $0=[x,a]-[d(x),a] = [x,a] - (a[x,a]+[\alpha(x),a]) = [x,a]=-\beta(x)$. Hence this last relation yields $\beta(x)=0$ whence $\alpha(x)=0$, a contradiction.

***Remark* 1.** If α is a derivation on a ring R then the map $-\alpha:R\to R$ defined by $(-\alpha)(x)=-\alpha(x)$ is also a derivation on R. Similarly, if d is a generalized derivation on a ring R associated with a derivation α on R then a map $-d:R\to R$ defined by $(-d)(x)=-d(x)$ is also a generalized derivation on R associated with a derivation $-\alpha$ on R.

Lemma 2. *Let R be a prime ring with char $R\neq 2$ and d be a generalized derivation on R associated with a derivation α on R. If $d([x,y])=[d(y),d(x)]$ holds for all x, $y\in R$ then either R is commutative, or $d=0$, or $d=-I_{id}$.*

Proof. Let $d([x,y])=[d(y),d(x)]$ for all x, $y\in R$. Replace $-x$ by x. Since

$$d([-x,y]) = d(-[x,y])=-d([x,y]) = (-d)([x,y])$$

and

$$[d(y),d(-x)] = [d(y),-d(x)] =$$

$$= -[d(y),d(x)]=[d(x),d(y)] = [-d(x),-d(y)]=[(-d)x,(-d)y]$$

we have $(-d)([x,y])=[(-d)(x),(-d)(y)]$ for all $x,\ y\in R$. In view of Remark 1 and Lemma 1 we obtain that either R is commutative, or $d=0$, or $d=-I_{id}$.

Theorem 1. *Let d be a generalized derivation on R be a prime ring with char $R\neq 2$ and R associated with a derivation α on R. If d satisfies either $d([x,y])=[d(x),d(y)]$ or $d([x,y])=[d(y),d(x)]$ for all x, $y\in R$ then either R is commutative, or $d=0$, or $d=I_{id}$, or $d=-I_{id}$.*

Proof. For each $x\in R$ we set $I_x=\{y\in R:\ d([x,y])=[d(x),d(y)]\}$ and $J_x = \{y\in R: d([x,y])=[d(y),d(x)]\}$. It is clear that for each $x\in R$, I_x and J_x are two additive subgroup of R and $(R,+)=(I_x,+)\cup(J_x,+)$. But a group can not be the union two proper subgroups. So we are forced to conclude that either $R=I_x$ or $R=J_x$. Now we set $I=\{x\in R: R=I_x\}$ and $J=\{y\in R: R=J_x\}$. The sets I and J are also two subgroups of R and $(R,+)=(I,+)\cup(J,+)$. By the similar manner as above we have $R = I$ or $R = J$. By Lemmas 1 and 2 we obtain desired results.

***Example* 1.** *Consider the matrix ring* $R = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x, y \in Z \right\}$, *where* $\mathbb{Z}$ *is the set of all integers. It is clear to see that a map* $\alpha : R \to R$ *defined by* $\alpha\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ *is a derivation on* R. *Then a map* $d : R \to R$ *defined by* $d\left(\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} x & x+y \\ 0 & 0 \end{bmatrix}$ *is a generalized derivation associated with* α *satisfying the condition* $d([X,Y]) = [d(X), d(Y)]$ *for all* X, $Y \in R$, *but neither* R *is commutative, nor* $d = 0$, *nor* $d = I_{id}$.

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