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ON STRONGLY $\bigoplus$-SUPPLEMENTED MODULES

ПРО СИЛЬНО $\bigoplus$-ДОПОВНЕНІ МОДУЛІ

In this work, strongly $\oplus$-supplemented and strongly cofinitely $\oplus$-supplemented modules are defined and some properties of strongly $\oplus$-supplemented and strongly cofinitely $\oplus$-supplemented modules are investigated. Let R be a ring. Then every R-module is strongly $\oplus$-supplemented if and only if R is perfect. Finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented. But this is not true for strongly $\oplus$-supplemented modules. Any direct sum of cofinitely $\oplus$-supplemented modules is cofinitely $\oplus$-supplemented but this is not true for strongly cofinitely $\oplus$-supplemented modules. We also prove that a supplemented module is strongly $\oplus$-supplemented if and only if every supplement submodule lies above a direct summand.

Визначено сильно $\oplus$-доповнені та сильно кофінітно $\oplus$-доповнені модулі і досліджено деякі властивості сильно $\oplus$-доповнених та сильно кофінітно $\oplus$-доповнених модулів. Припустимо, що R — кільце. У цьому випадку кожен R-модуль є сильно $\oplus$-доповненим тоді і тільки тоді, коли R є досконалим. Скінченна пряма сума $\oplus$-доповнених модулів є $\oplus$-доповненою. Але це не справджується для сильно $\oplus$-доповнених модулів. Будь-яка пряма сума кофінітно $\oplus$-доповнених модулів є кофінітно $\oplus$-доповненою, але це не справджується для сильно кофінітно $\oplus$-доповнених модулів. Доведено також, що доповнений модуль є сильно $\oplus$-доповненим модулем тоді і тільки тоді, коли кожен підмодуль-доповнення розташований над прямим доданком.

1. Introduction. In this work R will denote an arbitrary ring with unity and M will state for an unitary left R-module. Let M be an R-module. $N \leq M$ will mean N is a submodule of M. Let $K \leq M$. If $L = M$ for every submodule L of M such that $K + L = M$ then K is called a **small submodule** of M and written by $K \ll M$. Let $U \leq M$ and $V \leq M$. If V is minimal with respect to $M = U + V$ then V is called a **supplement** of U in M. This equivalent to $M = U + V$ and $U \cap V \ll V$. M is called **supplemented** if every submodule of M has a supplement in M. M is called **finitely supplemented** if every finitely generated submodule of M has a supplement in M. M is called $\oplus$**-supplemented** if every submodule of M has a supplement that is a direct summand of M. M is called **completely** $\oplus$**-supplemented** if every direct summand of M is $\oplus$-supplemented. A submodule U of M is called **cofinite** if M/U is finitely generated. M is called **cofinitely supplemented** if every cofinite submodule of M has a supplement in M. We say a submodule U of the R-module M has **ample supplements** in M if for every $V \leq M$ with $U + V = M$, there exists a supplement V' of U with $V' \leq V$. If every submodule of M has ample supplements in M, then we call M is **amply supplemented**.

M is called a **projective cover** of N, if M is a projective module and there exists an epimorphism $f\colon M \to N$ such that $\operatorname{Ke} f \ll M$. A module M is called **semiperfect** if every factor module of M has a projective cover. M is **called** $\boldsymbol{\pi}$**-projective** module if there exists an endomorphism f of M such that $\operatorname{lm} f \leq U$, $\operatorname{Im}(1 - f) \leq V$ for every submodules U, V of M such that $M = U + V$.

Let $V \leq M$. V is called **lies above a direct summand** of M if there exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2, M_1 \leq V, V \cap M_2 \ll M_2$.

In this work **Jac**R will denote intersection of all maximal left ideals of R.

Let M be an R-module. We consider the following conditions.

(D_1) Every submodule of M lies above a direct summand of M.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

Lemma 1.1 (Modular law). *Let M be an R-module, K, N and H are submodules of M and $H \leq N$. Then $N \cap (H + K) = H + N \cap K$ (see* [1]).

Lemma 1.2. *Let V be a supplement of U in M, K and T be submodules of V. Then T is a supplement of K in V if and only if T is a supplement of $U + K$ in M.*

Proof. $(\Rightarrow)$ Let T be a supplement of K in V. Let $U + K + L = M$ for a submodule $L \leq T$. In this case $K + L \leq V$ and because V is a supplement of U, $K + L = V$. Since $L \leq T$ and T is a supplement of K in V, $L = T$ and then T is a supplement of $U + K$ in M.

$(\Leftarrow)$ Let T be a supplement of $U + K$ in M. This can be found that because of $U + K + T = M$ and $K + T \leq V$, then we can have $K + T = V$. Since $K \cap T \leq$ $\leq (U + K) \cap T \ll T$, $K \cap T \ll T$ and then T is a supplement of K in V.

2. Strongly ⊕-supplemented modules.

Definition 2.1. *Let M be a supplemented module. If every supplement submodule of M is a direct summand of M then M is called a* **strongly ⊕-supplemented module.**

***Corollary* 2.1.** *Strongly ⊕-supplemented modules are ⊕-supplemented.*

Lemma 2.1. *Let M be supplemented and π-projective module. Then M is a strongly ⊕-supplemented module.*

Proof. See [21].

Lemma 2.2. *Let M be a strongly ⊕-supplemented module. Then every direct summand of M is strongly ⊕-supplemented.*

Proof. Let L be a direct summand of M and $M = L \oplus T$. Let K be a supplement of U in L. By Lemma 1.2 K is a supplement of $U \oplus T$ in M. Because M is strongly ⊕-supplemented, K is a direct summand of M. Let $M = K \oplus P$. By Modular law $L = L \cap M = L \cap (K \oplus P) = K \oplus (L \cap P)$. Thus K is a direct summand of L and L is strongly ⊕-supplemented.

***Corollary* 2.2.** *Strongly ⊕-supplemented modules are completely ⊕-supplemented.*

Theorem 2.1. *Every (D_1) module is strongly ⊕-supplemented.*

Proof. See [21].

Theorem 2.2. *Let R be a Prüfer ring. Then every finitely generated torsion free supplemented R-module is strongly ⊕-supplemented.*

Proof. Because R is a Prüfer ring, then every finitely generated torsion free R-module is projective (see [21]). Because every projective module is π-projective, by Lemma 2.1 every finitely generated torsion free supplemented R-module is strongly ⊕-supplemented.

Theorem 2.3. *Let M_i, $1 \leq i \leq n$, are projective modules. Then $\oplus_{i=1}^{n} M_i$ is strongly ⊕-supplemented if and only if every M_i is strongly ⊕-supplemented.*

Proof. $(\Rightarrow)$ Because every M_i is direct summand of $\oplus_{i=1}^{n} M_i$, by Lemma 2.2 every M_i is strongly ⊕-supplemented.

$\Leftarrow$ Because every M_i is supplemented by [21], $\oplus_{i=1}^{n} M_i$ is supplemented. Because every M_i is projective modules by [21], $\oplus_{i=1}^{n} M_i$ is projective module. Thus by Lemma 2.1 $\oplus_{i=1}^{n} M_i$ is strongly ⊕-supplemented.

Lemma 2.3. *Let M be a projective module. Then the followings are equivalent.*

(i) *M is semiperfect.*

(ii) *M is supplemented.*

(iii) *M is ⊕-supplemented.*

(iv) *M is strongly ⊕-supplemented.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are proved in [10]. (ii) $\Rightarrow$ (iv) Because M is a projective module, M is a π-projective module. Thus by Lemma 2.1 M is strongly $\oplus$-supplemented.

(iv) $\Leftrightarrow$ (ii) Clear.

Theorem 2.4. *For every ring R, the following statements are equivalent.*

(i) *R is semiperfect.*

(ii) *Every finitely generated free R-module is $\oplus$-supplemented.*

(iii) *Every finitely generated free R-module is strongly $\oplus$-supplemented.*

(iv) *${}_RR$ is $\oplus$-supplemented.*

(v) *${}_RR$ is strongly $\oplus$-supplemented.*

(vi) *For every left ideal A of R, there exists an idempotent $e \in R\backslash A$ such that $A \cap eR \subset \operatorname{Jac} R$.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) are proved in [11].

Because ${}_RR$ is a projective module, (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) are hold.

Theorem 2.5. *A commutative ring R is semiperfect if and only if every π-projective cyclic R-module is strongly $\oplus$-supplemented.*

Proof. ($\Rightarrow$) Let R be semiperfect. By [11] every cyclic R-module is $\oplus$-supplemented. Thus by Lemma 2.1 every π-projective cyclic R-module is strongly $\oplus$-supplemented.

($\Leftarrow$) Since ${}_RR$ is cyclic and π-projective, by hypothesis ${}_RR$ is strongly $\oplus$-supplemented. By Lemma 2.3 ${}_RR$ is semiperfect.

Theorem 2.6. *Let M be a finitely generated strongly $\oplus$-supplemented R-module. Then M is direct sum of cyclic submodules.*

Proof. Since M is a strongly $\oplus$-supplemented module, by Corollary 2.2 M is completely $\oplus$-supplemented. In case by [11] M is direct sum of cyclic submodules.

Theorem 2.7. *For any ring R, the following statements are equivalent.*

(i) *R is perfect.*

(ii) *$R^{(N)}$ is $\oplus$-supplemented.*

(iii) *$R^{(N)}$ is strongly $\oplus$-supplemented.*

(iv) *Every countable generated free R-module is $\oplus$-supplemented.*

(v) *Every countable generated free R-module is strongly $\oplus$-supplemented.*

(vi) *Every free R-module is $\oplus$-supplemented.*

(vii) *Every free R-module is strongly $\oplus$-supplemented.*

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) are proved in [11].

Since ${}_RR$ is a projective module, every free R-module is projective. Thus every free R-module is π-projective. By Lemma 2.1 (iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) are hold.

Theorem 2.8. *For a supplemented module M, the following statements are equivalent.*

(i) *M is strongly $\oplus$-supplemented.*

(ii) *Every supplement submodule of M lies above a direct summand.*

(iii) (a) *Every non zero supplement submodule of M contains a non zero direct summand of M.*

(b) *Every supplement submodule of M contains a maximal direct summand of M.*

Proof. (i) $\Rightarrow$ (ii) Clear from definitions.

(ii) $\Rightarrow$ (i) Let V be any supplement submodule of M. Let V is a supplement of U in M. By hypothesis there exist $M_1 \leq M$ and $M_2 \leq M$ such that $M = M_1 \oplus M_2$, $M_1 \leq V$ and $V \cap M_2 \ll M_2$. In this case $V = V \cap M = M_1 \oplus V \cap M_2$ and by

$V \cap M_2 \ll M$, $M = U + V = U + V \cap M_2 + M_1 = U + M_1$. Since V is supplement of U, $V = M_1$. Thus $M = V \oplus M_2$ and V is a direct summand of M. That is M is strongly $\oplus$-supplemented.

(i) $\Rightarrow$ (iii) Clear from definitions.

(iii) $\Rightarrow$ (i) Let V be a supplement of U in M and assume X to be a maximal direct summand of M with $X \leq V$ and $M = X \oplus Y$. Then $V = X \oplus V \cap Y$ and by Lemma 1.2 $V \cap Y$ is a supplement of $U + X$ in M. If $V \cap Y$ is not zero then by (iii, a) there exists a non zero direct summand N of M such that $N \leq V \cap Y$. In this case $X \oplus N$ is a direct summand of M and $X \oplus N \leq V$. This contradicts the choice of X. Thus $V \cap Y = 0$ and $V = X$. In this case V is direct summand of M and M is a strongly $\oplus$-supplemented module.

Let M be an R-module. If $rM = M$ for every $r \in R$ which not zero divisor, then M is called **a divisible *R*-module.** Let R be a domain. If every submodule of left R-module ${}_RR$ is projective, then R is called a **Dedekind domain.** Let R be a principal ideal domain. If R has the unique prime element (up to unit), then R is called **a discrete valuation ring.**

***Remark* 2.1.** Let R be a discrete valuation ring and p be the unique prime element of R. Then every ideal of R is of the form Rp^k which $k \in \mathbf{Z}$. If we take these ideals to be neighborhoods of 0 in R, we define a topology in R, making R a topological ring. If R is complete in this topology, we call it **a complete discrete valuation ring.**

***Example* 2.1.** Let R be a discrete valuation ring which not complete and K be a quotient field of R. Then $M = K^2$ is strongly $\oplus$-supplemented but not amply supplemented.

Proof. By [21] Theorem 2.2, M is supplemented but not amply supplemented. Let V be a supplement submodule in M. Assume that V is a supplement of U in M. Since M is divisible, then $M = rM = rU + rV = U + rV$ for every $r \in R$ which $r \neq 0$. Since V is a supplement of U in M, $V = rV$ and then V is divisible. Since R is a Dedekind domain, V is injective (see [19], 40.5) and a direct summand of M. Thus M is strongly $\oplus$-supplemented.

***Example* 2.2.** Let R be a discrete valuation ring with quotient field K, let p be the unique prime element and let $N = Rp$. Then $M = K/R \oplus R/N$ is completely $\oplus$-supplemented but is not strongly $\oplus$-supplemented.

Proof. By [10] Example 2.1, M is completely $\oplus$-supplemented but not (D_1). Moreover M satisfies (D_3). Let $L = R(p^{-2} + R, 1 + N) \leq M$. Then we can prove $K/R + L = M$. Let $x \in (K/R) \cap L$. Then $x = (rp^{-2} + R, r + N)$ for some $r \in R$. Since $(rp^{-2} + R, r + N) \in K/R$, $r + N = 0$ and then there exists $r' \in R$ with $r = r'p$. Then $x = (r'pp^{-2} + R, 0) = (r'p^{-1} + R, 0) \in R(p^{-1} + R, 0)$. Since $R(p^{-1} + R, 0) \leq (K/R) \cap L$, $K/R \cap L = R(p^{-1} + R, 0)$. Let $R(p^{-1} + R, 0) + T = L$ with $T \leq L$. Then there exists $s \in R$ such that $s(p^{-2} + R, 1 + N) \in T$ and $s + N \neq 0$. Since $s + N \neq 0$, $s \notin N$. Since p is the unique prime element of R, s is invertible in R, i.e., there exists $s' \in R$ with $s's = 1$. Then $(p^{-2} + R, 1 + N) = s's(p^{-2} + R, 1 + N) \in T$ and then $L = R(p^{-2} + R, 1 + N) \leq T$. Thus $T = L$, $R(p^{-1} + R, 0) \ll L$ and L is a supplement of K/R in M. If L is a direct summand of M, by $M = K/R + L$ and M satisfying (D_3), $(K/R) \cap L = R(p^{-1} + R, 0)$ is also direct summand of M. This contradicts $R(p^{-1} + R, 0) \ll M$. Hence L is not a direct summand of M and M is not strongly $\oplus$-supplemented.

***Remark* 2.2.** In Example 2.2 K/R is hollow and strongly $\oplus$-supplemented. Since R/N is simple, it is strongly $\oplus$-supplemented. But the direct sum of K/R and R/N is not strongly $\oplus$-supplemented. Zöschinger has proved that if R is a Dedekind domain then an R-module M is supplemented if and only if M is $\oplus$-supplemented. But this not true for strongly $\oplus$-supplemented by Example 2.2.

Definition 2.2. *Let M be an R-module. If M is cofinitely supplemented and every supplement of cofinite submodules of M is a direct summand of M then M is called a strongly cofinitely $\oplus$-supplemented module.*

***Corollary* 2.3.** *Every strongly cofinitely $\oplus$-supplemented module is cofinitely supplemented.*

Theorem 2.9. *Let M be a strongly cofinitely $\oplus$-supplemented module. Then every direct summand of M is strongly cofinitely $\oplus$-supplemented.*

Proof. Let N be a direct summand of M and let $M = N \oplus T$. Since M is cofinitely supplemented, $N \cong M/T$ is also cofinitely supplemented. Let U be a cofinite submodule of N and V be a supplement of U in N. Then by Lemma 1.2 V is a supplement of $U \oplus T$ in M. Since $U \oplus T$ is a cofinite submodule of M and M is strongly cofinitely $\oplus$-supplemented, V is a direct summand of M. Let $M = V \oplus X$. Then by Modular law $N = V \oplus (N \cap X)$ and then V is a direct summand of N. Hence N is strongly cofinitely $\oplus$-supplemented.

Theorem 2.10. *Let M be a π-projective and finitely supplemented R-module. If M is cofinitely supplemented then M is strongly cofinitely $\oplus$-supplemented.*

Proof. Let U be a cofinite submodule of M and V be a supplement of U in M. Then V is finitely generated. Since M is finitely supplemented, V has a supplement X in M. Since M is π-projective, there exists $f \in \mathrm{End}(M)$ such the $\mathrm{Im}\, f \leq U$, $\mathrm{Im}(1-f) \leq V$. Then we can prove $(1-f)(U) \leq U$ and $f(V) \leq V$. Then $M = f(M) + (1-f)(M) = f(V) + f(X) + V = V + f(X)$. Let $\nu \in V \cap f(X)$. Then there exists $x \in X$ with $\nu = f(x)$. Since $x - \nu = x - f(x) = (1-f)(x) \in V$, $x \in V$. Hence $\nu = f(x) \in f(V \cap X)$. Since $V \cap X \ll X$, $f(V \cap X) \ll f(X)$ and $V \cap f(X) \leq f(V \cap X) \ll f(X)$. Hence $f(X)$ is a supplement of V in M. Since $f(X) \leq U$, then V is a supplement of $f(X)$ in M. Hence V and $f(X)$ are mutual supplements in M. Since M is π-projective, then by [19] $M = V \oplus f(X)$ and V is a direct summand of M. Thus M is strongly cofinitely $\oplus$-supplemented.

Theorem 2.11. *If M is cofinitely supplemented, then $M/\mathrm{Rad}\,(M)$ is strongly cofinitely $\oplus$-supplemented.*

Proof. Since M is cofinitely supplemented, $M/\operatorname{Rad}(M)$ is also cofinitely supplemented. Let $U/\operatorname{Rad}(M)$ be a cofinite submodule of $M/\operatorname{Rad}(M)$ and $V/\operatorname{Rad}(M)$ be a supplement of $U/\operatorname{Rad}(M)$ in $M/\operatorname{Rad}(M)$. Since

$$U/\operatorname{Rad}(M) \cap V/\operatorname{Rad}(M) \ll M/\operatorname{Rad}(M),$$

$$U/\operatorname{Rad}(M) \cap V/\operatorname{Rad}(M) \leq \operatorname{Rad}(M/\operatorname{Rad}(M)) = 0$$

and then $M/\operatorname{Rad}(M) = U/\operatorname{Rad}(M) \oplus V/\operatorname{Rad}(M)$. Hence $V/\operatorname{Rad}(M)$ is a direct summand of $M/\operatorname{Rad}(M)$ and $M/\operatorname{Rad}(M)$ is strongly cofinitely $\oplus$-supplemented.

***Example* 2.3.** Let M be a direct sum of an infinite number of copies of the Prüferp-group Z_{p^∞}. Then M is strongly cofinitely $\oplus$-supplemented but not strongly $\oplus$-supplemented.

Proof. By [10] M is not supplemented, i.e., not strongly $\oplus$-supplemented. By [2] M is cofinitely supplemented. Let L be a supplement submodule of M and L be a supplement of K in M. We can prove that M is a divisible **Z**-module. Let $n \in$ **Z**. Since $nM = M$, $M = nM = nK + nL = K + nL$. Since L is a supplement of K in M, $nL = L$ and L is divisible. Since **Z** is a Dedekind domain, L is injective ([19], 40.5) and a direct summand of M. Hence M is strongly cofinitely $\oplus$-supplemented.

***Remark* 2.3.** In Example 2.2 $M = K/R \oplus R/N$ is cofinitely supplemented but not strongly cofinitely $\oplus$-supplemented. Also in Example 2.2 K/R and R/N is strongly cofinitely $\oplus$-supplemented but the direct sum of K/R and R/N is not strongly cofinitely $\oplus$-supplemented.

1. *Alizade R., Pancar A.* Homoloji Cebire Giris. – Samsun: Ondokuz Mayis Üniv. Fen-Edebiyat Fakültesi, 1999.
2. *Alizade R., Bilhan G., Smith P. F.* Modules whose maximal submodules have supplements // Communs Algebra. – 2001. – **29** , № 6. – P. 2389 – 2405.
3. *Alizade R., Büyükaşik E.* Confinitely weak supplemented modules // Communs Algebra. – 2003. – **13**. – P. 5377 – 5390.
4. *Anderson F. V., Fuller K. R.* Rings and categories of modules. – Springer, 1992.
5. *Callialp F.* Cebir. – Sakarya: Sakarya Üniv. Yayin, 1995. – № 6.
6. *Hungerford T. W.* Algebra. – New York: Springer, 1973. – 504 p.
7. *Kasch F.* Modules and rings. – Acad. Press, 1982.
8. *Kaplansky I.* Infinite Abelian groups. – Ann Arbor, 1969.
9. *Kaplansky I.* Projective modules // Ann. Math. – 1958. – **68**. – P. 372 – 377.
10. *Keskin D., Harmanci A., Smith P. F.* On $\oplus$-supplemented modules // Acta math. hung. – 1999. – **83**, № 1-2. – P. 161 – 169.
11. *Keskin D., Smith P. F., Xue W.* Rings whose modules are $\oplus$-supplemented // J. Algebra. – 1999. – **218**. – P. 470 – 487.
12. *Lomp C.* On semilocal modules and rings // Communs Algebra. – 1999. – **27**, № 4. – P. 1921 – 1935.
13. *Lomp C.* On dual goldie dimension: Ph. D. Thesis. – Düsseldorf, 1996.
14. *Mohamed S. H., Müller B. J.* Continuous and discrete modules // London Math. Soc. – Cambridge: Cambridge Univ. Press, 1990. – **147**.
15. *Nebiyev C.* Amply weak supplemented modules // Int. J. Comput. Cognition. – 2005. – **3**, № 1. – P. 88 – 90.
16. *Sarath B., Varadarajan K.* Injectivity of direct sums // Communs Algebra. – 1974. – **1**. – P. 517 – 530.
17. *Sharpe W. D., Vamos P.* Injective modules. – Cambridge Univ. Press, 1972.
18. *Smith P. F.* Finitely generated supplemented modules are amply supplemented // Arab. J. Sci. and Eng. – 2000. – **25**, № 2. – P. 69 – 79.
19. *Wisbauer R.* Foundations of module and ring theory. – Philadelphia: Gordon and Breach, 1991.
20. *Zöschinger H.* Komplementierte Moduln über Dedekindringen // J. Algebra. – 1974. – **29**. – P. 42 – 56.
21. *Zöschinger H.* Komplemente als direkte Summanden // Arch. Math. – 1974. – **25**. – P. 241 – 243.
22. *Zöschinger H.* Projektive Moduln mit endlich erzeugten Radikalfaktormoduln // Math. Ann. – 1981. – **255**. – P. 199 – 206.
23. *Zöschinger H.* Komplemente als direkte Summanden II // Arch. Math. – 1982. – **38**. – P. 324 – 334.

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