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# ON THE ENERGY AND PSEUDO-ANGLE OF FRENET VECTOR FIELDS IN $R_v^n$

# ПРО ЕНЕРГІЮ ТА ПСЕВДОКУТ ВЕКТОРНОГО ПОЛЯ ФРЕНЕ В $R_v^n$

In this paper, we compute the energy of a Frenet vector field and the pseudo-angle between Frenet vectors for a given non-null curve $C$ in semi-Euclidean space of signature $(n, \nu)$. It is shown that the energy and pseudo-angle can be expressed in terms of the curvature functions of $C$.

Обчислено енергію векторного поля Френе та псевдокут між векторами Френе для заданої ненульової кривої $C$ у напівевклідовому просторі сигнатури $(n, \nu)$. Показано, що енергія та псевдокут можуть бути виражені через функції кривини $C$.

**1. Introduction.** It is well known that the studies on the energy of a unit vector field on a compact oriented $m$-dimensional Riemannian manifold $M$ basically consider the equality $M = S^{2n+1}$ (see [1 – 3]). Let $C$ be a curve with a pair $(I, \alpha)$ of parametric unit speed in $R^n$. Let us take an initial point $a \in I$ and the Frenet frames $\{V_1(\alpha(a)), \ldots, V_r(\alpha(a))\}$ and $\{V_1(\alpha(s)), \ldots, V_r(\alpha(s))\}$ at the points $\alpha(a)$ and $\alpha(s)$, respectively. In [4], we calculated the energy of a Frenet vector field and the angle between each vector $V_i(\alpha(a))$ and $V_i(\alpha(s))$ where $1 \leq i \leq r$. Further, we observed that the energy and angle may be expressed in terms of the curvature functions of the given curve $C$. In this paper, we consider the Frenet frame at the point $\alpha(a)$ for a given non-null curve $C$ in semi-Euclidean space

$$R_\nu^n = \left(R^n, -\sum_{i=1}^{\nu} dx_i + \sum_{i=\nu+1}^{n} dx_i\right).$$

Note that the Frenet vectors may not be the same type here, i.e., both of the spacelike and timelike vectors are contained in the collection of Frenet vectors. It is well known that the angle between timelike vectors is not defined. In [5], the integral $\int_a^s \left\| \frac{dV_i}{du} \right\| du$ is called *the pseudo-angle* between timelike vectors $V_i(\alpha(a))$ and $V_i(\alpha(s))$. Recall that the angle between spacelike vectors is $\int_a^s \left\| \frac{dV_i}{du} \right\| du$ [4]. In this work, the angle between arbitrary vectors $V_i(\alpha(a))$ and $V_i(\alpha(s))$ will be called *pseudo-angle*.

**Definition 1.1** [6]. *Let $\alpha\colon I \subset R \longrightarrow R_v^n$ be a regular curve in $R_v^n$. Then $\alpha$ is called spacelike (timelike, null) if for all $t \in I$ the velocity vector $\alpha'(t)$ is spacelike (timelike, null). If $\langle \alpha'(t),\ \alpha'(t)\rangle = 1$ or $-1$, then $\alpha$ is called a unit speed curve where $\langle\, ,\rangle$ denotes the scalar product of $R_v^n$.*

**Definition 1.2.** *Let $\alpha$ be a regular curve in $R_v^n$ and $\psi = \{\alpha'(t), \alpha''(t), \ldots, \alpha^r(t)\}$ be a linear independent and non-null system. Further, let $\alpha^m(t) \in Sp\psi$ for all $\alpha^m(t)$ where $m > r \geq 2$. Then the orthonormal system $\{V_1(t), V_2(t), \ldots, V_r(t)\}$ obtained from $\psi$ is called r-Frenet frame at the point $\alpha(t)$.*

In this paper, curve means that $\{\alpha', \alpha'', \ldots, \alpha^r\}$ is a set of derivatives of $\alpha$ such that $\alpha^l$ is non-null for $1 \leq l \leq r$.

**Lemma 1.1** [8]. *For a curve $\alpha$ in $R_v^n$, r-Frenet frame exists if and only if the space $Sp\,\{\alpha'(t), \ldots, \alpha^r(t)\}$ is non-degenerated for $k = 1, \ldots, r$.*

**Definition 1.3.** *Let $\alpha$ be a regular curve in $R_v^n$ and $\{V_1(t), V_2(t), \ldots, V_r(t)\}$ be the Frenet frame at the point $\alpha(t)$. If $\varepsilon_i(t) = \langle V_i(t), V_i(t)\rangle = 1$ or $-1$, then the function $k_i\colon I \longrightarrow R$ defined by*

$$k_i(t) = \varepsilon_i(t)\varepsilon_{i+1}(t)\langle V_i'(t), V_{i+1}(t)\rangle, \quad 1 \le i \le r-1, \tag{1}$$

*is called curvature function on $\alpha$. Moreover, the real number $k_i(t)$ is called the $i$ th curvature on $\alpha$ at the point $\alpha(t)$.*

**Theorem 1.1** [8]**.** *Let $\alpha$ be a unit speed curve in $R_v^n$ and $k_i(t)$ be the $i$ th curvature. If $\{V_1(t), V_2(t), \ldots, V_r(t)\}$ is the Frenet frame at the point $\alpha(t)$, then the following equalities hold*:

$$V_1'(t) = \varepsilon_1(t)k_1(t)V_2(t), \tag{2}$$

$$V_i'(t) = -\varepsilon_i(t)k_{i-1}(t)V_{i-1}(t) + \varepsilon_i(t)k_i(t)V_{i+1}(t), \quad 1 < i < r, \tag{3}$$

$$V_r'(t) = -\varepsilon_r(t)k_{r-1}(t)V_{r-1}(t). \tag{4}$$

**Definition 1.4.** *The energy of a differentiable map $f\colon (M, \langle,\rangle) \to (N, h)$ between Riemannian manifolds is given by*

$$\mathcal{E}(f) = \frac{1}{2}\int\limits_M \sum_{a=1}^{n} h\big(df(e_a), df(e_a)\big)\upsilon, \tag{5}$$

*where $\upsilon$ is the canonical volume form in $M$ and $\{e_a\}$ is a local basis of the tangent space* (*see for example* [1, 3]).

The energy of a unit vector field $X$ is defined to be the energy of the section $X\colon M \to T^1M$, where $T^1M$ is the unit tangent bundle equipped with the restriction of the Sasaki metric on $TM$. Now let $\pi\colon T^1M \to M$ be the bundle projection, and let $T(T^1M) = \mathcal{V} \oplus \mathcal{H}$ denote the vertical/horizontal splitting induced by the Levi-Civita connection. Further, let us write $TM = \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F}$ denotes the line bundle generated by $X$, and $\mathcal{G}$ is the orthogonal complement [3].

**Proposition 1.1** [9]**.** *The connection map $K\colon T(T^1M) \to T^1M$ verifies the following conditions*:

1) $\pi \circ K = \pi \circ d\pi$ *and* $\pi \circ K = \pi \circ \widetilde{\pi}$, *where* $\widetilde{\pi}\colon T(T^1M) \to T^1M$ *is the tangent bundle projection*;

2) *for* $\omega \in T_xM$ *and a section* $\xi\colon M \to T^1M$, *we have*

$$K(d\xi(\omega)) = \nabla_\omega \xi,$$

*where $\nabla$ is the Levi-Civita covariant derivative.*

**Definition 1.5** [9]**.** *For $\eta_1, \eta_2 \in T_\xi(T^1M)$ define*

$$g_{\mathcal{S}}(\eta_1, \eta_2) = \langle d\pi(\eta_1), d\pi(\eta_2)\rangle + \langle K(\eta_1), K(\eta_2)\rangle. \tag{6}$$

*This gives a Riemannian metric on $TM$. Recall that $g_{\mathcal{S}}$ is called the Sasaki metric. The metric $g_s$ makes the projection $\pi\colon T^1M \to M$ a Riemannian submersion.*

**2. The energy and pseudo-angle of a Frenet vector field in $R_v^n$.**

**Definition 2.1.** *Let $C$ be a non-null curve where with a pair $(I,\alpha)$ of parametric unit speeds in a space $R_v^n$ at a initial point $a \in I$. Further, let*

$$\{V_1(\alpha(a)),\ldots,V_r(\alpha(a))\} \quad \text{and} \quad \{V_1(\alpha(s))\ldots,V_r(\alpha(s))\}$$

*be the Frenet frames at the points $\alpha(a)$ and $\alpha(s)$, respectively. The integral* $\displaystyle\int_a^s \left\|\frac{dV_i}{du}\right\| du$ *is called the pseudo-angle between the vectors $V_i(\alpha(a))$ and $V_i(\alpha(s))$.*

**Theorem 2.1.** *Let $C$ be a spacelike curve. Then we have the following conditions.*

(i) *If the energy of $V_i$ is $\mathcal{E}(V_i(s))$, i.e., let the function $\mathcal{E}(V_i)$ be defined as*

$$\mathcal{E}(V_i)\colon\ I \to R, \quad 1 \le i \le r.$$

*Then the following relations are valid*:

$$\mathcal{E}(V_1)(s) = \frac{1}{2}\int\limits_a^s \varepsilon_2 k_1^2(\alpha(u))\,du + \frac{1}{2}(s-a),$$

$$\mathcal{E}(V_i)(s) = \frac{1}{2}\int\limits_a^s (\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) \, + \varepsilon_{i+1}k_i^2(\alpha(u)))\,du + \frac{1}{2}(s-a), \quad 2 \le i \le r-1,$$

$$\mathcal{E}(V_r)(s) = \frac{1}{2}\int\limits_a^s \varepsilon_{r-1}k_{r-1}^2(\alpha(u))\,du + \frac{1}{2}(s-a).$$

(ii) *If the pseudo-angle between vectors $V_i(\alpha(a))$ and $V_i(\alpha(s))$ is $\theta_i(s)$, i.e., let the function $\theta_i$ be defined as*

$$\theta_i\colon\ I \to R, \quad 1 \le i \le r.$$

*Then the following relations are valid*:

$$\theta_1(s) = \int\limits_a^s k_1(\alpha(u))\,du,$$

$$\theta_i(s) = \int\limits_a^s \sqrt{\left|\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) \, + \varepsilon_{i+1}k_i^2(\alpha(u))\right|}\,du, \quad 2 \le i \le r-1,$$

$$\theta_r(s) = \int\limits_a^s \left|k_{r-1}(\alpha(u))\right|du.$$

***Proof.*** Note that by Lemma 1.1, the definitions of the energy and Sasaki metric in $R_v^n$ can be defined in the same manner in Riemannian manifolds.

(i) Let $TC$ be the tangent bundle and let $\{V_1,V_2,\ldots,V_r\}$ be the Frenet vector field of the curve $C$. Then we have $V_1\colon C \to TC = \bigcup_{t\in I} T_{\alpha(t)}C$. Let $\pi\colon TC \to C$ be the bundle projection and $T(TC) = \mathcal{V} \oplus \mathcal{H}$ be the vertical/horizontal splitting induced by the Levi-Civita connection. Let us write $TC = \mathcal{F} \oplus \mathcal{G}$ where $\mathcal{F}$ denotes the line bundle generated by $V_1$. Consider the Levi-Civita connection map $K\colon T(TC) \to TC$. By using equation (5), we obtain that the energy of $V_1$ is

$$\mathcal{E}(V_1)(s) = \frac{1}{2}\int_a^s g_{\mathcal{S}}(dV_1(V_1(\alpha(u)), dV_1(V_1\alpha(u)))du, \tag{7}$$

where $du$ is an arc length element. From (6) we have

$$g_{\mathcal{S}}(dV_1(V_1), dV_1(V_1)) =$$
$$= \langle d\pi(V_1(V_1)), d\pi(V_1(V_1))\rangle + \langle K(V_1(V_1)), K(V_1(V_1))\rangle.$$

Since $V_1$ is a section, then we obtain

$$d(\pi)\circ d(V_1) = d(\pi\circ V_1) = d(id_C) = id_{TC}.$$

On the other hand, by Proposition 1.1, we may write that

$$K(V_1(V_1)) = \nabla_{V_1}V_1 = V_1'.$$

Then we obtain

$$g_{\mathcal{S}}(dV_1(V_1), dV_1(V_1)) = \langle V_1, V_1\rangle + \langle V_1', V_1'\rangle.$$

Since $C$ is a spacelike curve, from (2) we get

$$g_{\mathcal{S}}(dV_1(V_1), dV_1(V_1)) = 1 + \varepsilon_2 k_1^2 \tag{8}$$

putting (8) in (7), and this yields that

$$\mathcal{E}(V_1(s)) = \frac{1}{2}\int_a^s \varepsilon_2 k_1^2(\alpha(u))\,du + \frac{1}{2}(s-a).$$

Let $N_iC$ be the $i$th normal bundle. Thus we have $V_i\colon C \to N_iC$ where $N_iC = \bigcup_{t\in I} N_{i\alpha(t)}C$. Here $N_{i\alpha(t)}C$ is generated by $V_i$. Now let $\pi_i\colon N_iC \to C$ be the $i$th bundle projection and $T(N_iC) = \mathcal{V}_i \oplus \mathcal{H}_i$ be the vertical/horizontal splitting induced by the Levi-Civita connection. Take the Levi-Civita connection map $K_i\colon T(N_iC) \to N_iC$. By using equation (5), we obtain that the energy of $V_i$ is

$$\mathcal{E}(V_i)(s) = \frac{1}{2}\int_a^s g_{\mathcal{S}}(dV_i(V_1(\alpha(u)), dV_i(V_1\alpha(u)))du, \tag{9}$$

where $2 \le i \le r$. From (6) we have

$$g_{\mathcal{S}}(dV_i(V_1), dV_i(V_1)) =$$
$$= \langle d\pi_i(V_i(V_1)), d\pi_i(V_i(V_1))\rangle + \langle K(V_i(V_1)), K(V_i(V_1))\rangle =$$
$$= \langle d(\pi_i\circ V_i)(V_1), d(\pi_i\circ V_i)(V_1)\rangle + \langle \nabla_{V_1}V_i, \nabla_{V_1}V_i\rangle =$$
$$= \langle V_1, V_1\rangle + \langle V_i', V_i'\rangle.$$

By (3) we obtain

$$g_{\mathcal{S}}(dV_i(V_1), dV_i(V_1)) = 1 + \varepsilon_{i-1}k_{i-1}^2 + \varepsilon_{i+1}k_i^2.$$

Using (9), we have

$$\mathcal{E}(V_i)(s) = \frac{1}{2}\int\limits_a^s \Big(\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) + \varepsilon_{i+1}k_i^2(\alpha(u))\Big)\,du + \frac{1}{2}(s-a),$$

$$2 \le i \le r-1.$$

So, (4) gives us that

$$g_{\mathcal{S}}(dV_r(V_1), dV_r(V_1)) = 1 + \varepsilon_{r-1}k_{r-1}^2.$$

Then (9) yields that

$$\mathcal{E}(V_r)(s) = \frac{1}{2}\int\limits_a^s \varepsilon_{r-1}k_{r-1}^2(\alpha(u))\,du + \frac{1}{2}(s-a).$$

This completes the proof of (i).

(ii) From Definition 2.1 we have

$$\theta_i(s) = \int\limits_a^s \left\|\frac{dV_i}{du}\right\| du.$$

Since $\alpha$ is spacelike, by using (2), we obtain

$$\theta_1(s) = \int\limits_a^s \sqrt{|\varepsilon_2 k_1^2(\alpha(u))|}du = \int\limits_a^s k_1(\alpha(u))\,du.$$

From (3) we have

$$\theta_i(s) = \int\limits_a^s \sqrt{\big|\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) + \varepsilon_{i+1}k_i^2(\alpha(u))\big|}\,du, \quad 2 \le i \le r-1.$$

By (4) we find

$$\theta_r(s) = \int\limits_a^s \sqrt{\big|\varepsilon_{r-1}k_{r-1}^2(\alpha(u))\big|}du = \int\limits_a^s \big|k_{r-1}(\alpha(u))\big|du.$$

**Theorem 2.2.** *Let C be a timelike curve.*

(i) *The energy of* $V_i$ *may be given by the following equalities*:

$$\mathcal{E}(V_1)(s) = \frac{1}{2}\int\limits_a^s \varepsilon_2 k_1^2(\alpha(u))\,du - \frac{1}{2}(s-a),$$

$$\mathcal{E}(V_i)(s) = \frac{1}{2}\int\limits_a^s \big(\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) + \varepsilon_{i+1}k_i^2(\alpha(u))\big)\,du - \frac{1}{2}(s-a), \quad 2 \le i \le r-1,$$

$$\mathcal{E}(V_r)(s) = \frac{1}{2}\int\limits_a^s \varepsilon_{r-1}k_{r-1}^2(\alpha(u))\,du - \frac{1}{2}(s-a).$$

(ii) *The pseudo-angle between vectors* $V_i(\alpha(a))$ *and* $V_i(\alpha(s))$ *are*

$$\theta_1(s) = \int\limits_a^s \big|k_1(\alpha(u))\big|\, du,$$

$$\theta_i(s) = \int\limits_a^s \sqrt{\big|\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) + \varepsilon_{i+1}k_i^2(\alpha(u))\big|}\, du, \quad 2 \le i \le r-1,$$

$$\theta_r(s) = \int\limits_a^s \big|k_{r-1}(\alpha(u))\big| du.$$

***Proof.*** (i) As in the proof of Theorem 2.1, we may write that

$$g_{\mathcal{S}}(dV_1(V_1), dV_1(V_1)) = \langle V_1, V_1\rangle + \langle V_1', V_1'\rangle\,.$$

Since $C$ is a timelike curve, from (2) we get

$$g_{\mathcal{S}}(dV_1(V_1), dV_1(V_1)) = -1 + \varepsilon_2 k_1^2 \tag{10}$$

and

$$\mathcal{E}(V_1(s)) = \frac{1}{2}\int\limits_a^s \varepsilon_2 k_1^2(\alpha(u))\, du - \frac{1}{2}(s-a).$$

By using (6) and (3) we have

$$g_{\mathcal{S}}(dV_i(V_1), dV_i(V_1)) = -1 + \varepsilon_{i-1}k_{i-1}^2 + \varepsilon_{i+1}k_i^2.$$

From (9), we obtain

$$\mathcal{E}(V_i)(s) = \frac{1}{2}\int\limits_a^s \Big(\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) \ + \varepsilon_{i+1}k_i^2(\alpha(u))\Big)\, du - \frac{1}{2}(s-a),$$

$$2 \le i \le r-1.$$

Therefore, (4) and (9) gives us that

$$\mathcal{E}(V_r)(s) = \frac{1}{2}\int\limits_a^s \varepsilon_{r-1}k_{r-1}^2(\alpha(u))\, du - \frac{1}{2}(s-a).$$

(ii) From Definition 2.1 we have

$$\theta_i(s) = \int\limits_a^s \left\|\frac{dV_i}{du}\right\| du.$$

Using (2), we obtain

$$\theta_1(s) = \int\limits_a^s \sqrt{\big|\varepsilon_2 k_1^2(\alpha(u))\big|} du = \int\limits_a^s \big|k_1(\alpha(u))\big|\, du.$$

From (3) we have

$$\theta_i(s) = \int\limits_a^s \sqrt{\left|\varepsilon_{i-1}k_{i-1}^2(\alpha(u)) \,+ \varepsilon_{i+1}k_i^2(\alpha(u))\right|}\, du, \quad 2 \le i \le r-1.$$

By (4) we obtain

$$\theta_r(s) = \int\limits_a^s \sqrt{\left|\varepsilon_{r-1}k_{r-1}^2(\alpha(u))\right|} du = \int\limits_a^s \left|k_{r-1}(\alpha(u))\right| du.$$

1. *Chacón P. M., Naveira A. M.* Corrected energy of distributions on riemannian manifold // Osaka J. Math. – 2004. – **41**. – P. 97 – 105.
2. *Higuchi A., Kay B. S., Wood C. M.* The energy of unit vector fields on the 3-sphere // J. Geom. and Phys. – 2001. – **37**. – P. 137 – 155.
3. *Wood C. M.* On the energy of a unit vector field // Geom. dedic. – 1997. – **64**. – P. 19 – 330.
4. *Altın A.* On the energy and the angle of a frenet vectors fields (in appear).
5. *Altın A.* The pitch and the pseudo-angle of pitch of a closed piece of $(k+1)$-dimensional ruled surface in $R_v^n$ // Turkish J. Math. – 2006. – **30**. – P. 1 – 7.
6. *O'Neill B.* Elementary differential geometry. – Acad. Press Inc., 1966.
7. *Gluk H.* Higher Curvatures of curves in Euclidean space // Amer. Math. Mon. – 1966. – **73**. – P. 699 – 704.
8. *Altın A.* Harmonic curvatures of non null curves and the helix in $R_v^n$ // Hacet. Bull. Nat. Sci. Eng. Ser. B. – 2001. – **30**. – P. 55 – 61.
9. *Chacón P. M., Naveira A. M., Weston J. M.* On the energy of distributions, with application to the quarternionic hopf fibration // Monatsh. Math. – 2001. – **133**. – P. 281 – 294.

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