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STABILITY OF SMOOTH SOLITARY WAVES FOR THE GENERALIZED KORTEWEG – DE VRIES EQUATION WITH COMBINED DISPERSION*

СТІЙКІСТЬ ГЛАДКИХ ВІДОКРЕМЛЕНИХ ХВИЛЬ ДЛЯ УЗАГАЛЬНЕНОГО РІВНЯННЯ КОРТЕВЕГА – ДЕ ФРІЗА З КОМБІНОВАНОЮ ДИСПЕРСІЄЮ

The orbital stability problem of the smooth solitary waves in the generalized Korteweg – de Vries equation with combined dispersion is considered. The results show that the smooth solitary waves are stable for an arbitrary speed of wave propagation.

Розглянуто задачу про орбітальну стійкість гладких відокремлених хвиль для узагальненого рівняння Кортевега – де Фріза з комбінованою дисперсією. Отримані результати показують, що гладкі відокремлені хвилі є стійкими при довільній швидкості поширення хвиль.

1. Introduction. In order to understand the effect of nonlinear dispersion on pattern formation as well as the formation of nonlinear structures like liquid drops etc, Rosenau and Hyman [1] gave and studied the nonlinear dispersive Korteweg – de Vries (KdV) equation

$$u_t + (u^2)_x + (u^2)_{xxx} = 0. \tag{1.1}$$

The nonlinear dispersion term leads to a singular solitary solution, called compacton (i.e., solitary waves with compact support). There are many researches on compactons (see [2 – 5]). However, the nonlinear dispersive KdV equation was not equivalent to a Hamiltonian dynamical system. Hence this equation does not exhibit the usual energy conservation law, Cooper, Shepard and Sodano considered instead a related generalized KdV equation [6]

$$u_t + uu_x + \alpha(2uu_{xxx} + 4u_x u_{xx}) = 0 , \tag{1.2}$$

which can be derived from a Lagrangian. Eq. (1.2) possesses the same terms as in Eq. (1.1), except for the relative weights of the terms. Eq. (1.2) also admits compacton solutions. The stability of the compacton solutions to Eq. (1.2) was considered in [7].

From the above fact, nonlinear dispersion plays a very important role in the formation of solutions. Many well-known equations present interesting singular solutions because of the nonlinear dispersion. For example, the Camassa – Holm equation [8]

$$u_t - u_{xxt} + 3uu_x = uu_{xxx} + 2u_x u_{xx} \tag{1.3}$$

has singular solitary waves called peakons. Peakons have attracted many attentions be-

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cause they have a discontinuous first derivative at the wave peak. The peakons for Eq. (1.3) are orbitally stable under small perturbations [9]. These novel solitary waves have been researched by many authors [10 – 14].

We also know that, having linear dispersion, the well-known KdV equation and the Boussinesq equation both have smooth solitary wave. This means linear dispersion might be responsible for smooth solitary wave.

What is more interesting, due to the missing of the linear dispersion, all the above three equations (1.1) – (1.3) do not have smooth solitary wave decaying to zero. If one introduces a linear dispersion u_{xxx} to the Camassa – Holm equation, solitary waves appear [9].

Our first interesting is that whether the smooth solitary wave would exist if a linear dispersion is added to (1.2). After introducing a linear dispersion term, Eq. (1.2) is modified to the generalized KdV equation with combined dispersion

$$u_t + uu_x + \alpha(2uu_{xxx} + 4u_x u_{xx}) + \beta u_{xxx} = 0 , \tag{1.4}$$

which can model the role of nonlinear dispersion and linear dispersion on pattern formation as well as the formation of nonlinear structures like liquid drops. In fact, when $\alpha = 0$, Eq. (1.4) becomes the KdV equation. When $\beta = 0$, it becomes Eq. (1.3). Eq. (1.4) has the following two important conservative laws

$$E(u) = \frac{1}{2}\int_R u^2 dx , \qquad F(u) = -\frac{1}{2}\int_R \left(\frac{u^3}{3} - 2\alpha u u_x^2 - \beta u_x^2 \right) dx . \tag{1.5}$$

Another aspect absorbing our attention is that whether it is stable if the smooth solitary wave exists. The stability problem of the generalized KdV equation with combined dispersion is very attractive because of the following two points:

i) Similar to the method in [1], the generalized KdV equation with combined dispersion is not integrable. This suggests that the appearance of the smooth solitary waves is probably not due to the integrability. The mechanism responsible for the coherence and robustness of the solitary waves is still unknown. One can turn to the stability analysis of the smooth solitary waves for help.

ii) As we know, the nonlinear term in (1.4) might lead to wave collapse. The phenomena may be changed because of the existence of linear dispersion. What role does the combination of nonlinear and linear dispersion play in the stability of the solitary waves of this type equation? Will the solitary waves be stable?

The remainder of the paper is organized as follows. In Section 2 the existence of smooth solitary wave solutions to Eq. (1.4) is considered. In Section 3 the orbital stability problem of the smooth solitary waves is studied by extending the method in [15]. The result shows that the smooth solitary waves are stable for arbitrary wave speed of propagation. The last section is the conclusions.

2. Existence of smooth solitary waves. We assume a solitary wave with speed c is a solution to (1.4) with

$$u(x,t) = \phi_c(x - ct) , \tag{2.1}$$

where ϕ_c is a one variable function vanishing at infinity and $c > 0$. Substituting

(2.1) into (1.4), we obtain

$$-c\phi_{cx} + \phi_c\phi_{cx} + \alpha(2\phi_c\phi_{cxxx} + 4\phi_{cx}\phi_{cxx}) + \beta\phi_{cxxx} = 0 . \quad (2.2)$$

In view of the decay of ϕ_c at infinity, by integration, we obtain from (2.2)

$$-c\phi_c + \frac{1}{2}\phi_c^{\,2} + \beta\phi_{cxx} + \alpha(2\phi_c\phi_{cxx} + \phi_{cx}^{\,2}) = 0 . \quad (2.3)$$

Multiplying by ϕ_{cx} both sides of (2.3) and integrating again, we have

$$\phi_{cx}^{\,2} = \frac{\phi_c^2\left(c - \frac{\phi_c}{3}\right)}{\beta + 2\alpha\phi_c} . \quad (2.4)$$

In order to study the existence of smooth solitary waves in Eq. (2.4), we will explore the qualitative behavior of solutions of $\phi_x^2 = F(\phi) = \dfrac{\phi^2\left(c - \frac{\phi}{3}\right)}{\beta + 2\alpha\phi}$ near points where F has a zero or a pole as follows. Firstly, if $\phi = 3c$ is a simple zero of $F(\phi)$, that is $F(3c) = 0$, $F'(3c) \neq 0$. It is easy to obtain that $\phi_x^2 = (\phi - 3c)F'(3c) + O\left((\phi - 3c)^2\right)$ as $\phi \to 3c$. Hence $\phi(x) = 3c + \frac{1}{4}(x - x_0)^2 F'(3c) + O\left((x - x_0)^4\right)$ as $x \to x_0$, where $\phi(x_0) = 3c$. Secondly, if $\phi = 0$ is a double zero of $F(\phi)$, that is $F(0) = 0$, $F'(0) = 0$, $F''(0) \neq 0$. We have $\phi_x^2 = \phi^2 F''(0) + O(\phi^3)$ as $\phi \to 0$. Hence $\phi(x) \sim \eta \exp\left(-|x|\sqrt{|F''(0)|}\right)$ as $x \to \infty$ for some constant η.

Remark. If $\phi(x) = -\dfrac{\beta}{2\alpha}$ for certain x, the solution to (2.4) is unsmooth. In order to get smooth solutions, the domain of ϕ can not include $-\dfrac{\beta}{2\alpha}$. Another condition to the existence of the solution to (2.4) is $F(\phi) = \dfrac{\phi^2\left(c - \frac{\phi}{3}\right)}{\beta + 2\alpha\phi} \geq 0$.

Now we will show the existence of the smooth solitary wave solutions to Eq. (2.4).

According to the above analysis, there are only two cases leading to smooth solitary wave solutions.

Case 1: $\alpha > 0$ and $-\dfrac{\beta}{2\alpha} < 0$.

In this case, we can observe that $F(\phi)$ has a double zero $\phi = 0$, a simple zero $\phi = 3c$, and $F(\phi) > 0$ for $-\dfrac{\beta}{2\alpha} < 0 < \phi < 3c$. Let ϕ be a solution in this interval. We have $\phi_x \to 0$ as $\phi \to 0$ and as $\phi \to 3c$, hence ϕ is strictly monotonic in any interval where $F(\phi) > 0$. In view of (2.4), It is easy to obtain that ϕ is symmetric with respect to x_0, where $\phi(x_0) = 3c$. Note that the domain of the solution never touch the pole point $-\frac{\beta}{2\alpha}$. Hence the interval of the solution to (2.4) must be the whole real line. Also from the above analysis, $\phi \to 0$ as $|x| \to \infty$ at the double zeros of $F(\phi)$. Therefore, when $\alpha > 0$, and $-\frac{\beta}{2\alpha} < 0$, that is, $\alpha > 0$, and $\beta > 0$, Eq. (2.4) has smooth solitary wave solutions with $0 = \min_{x\in R} \phi(x)$, $3c = \max_{x \in R} \phi(x)$ (see Fig. 2.1).

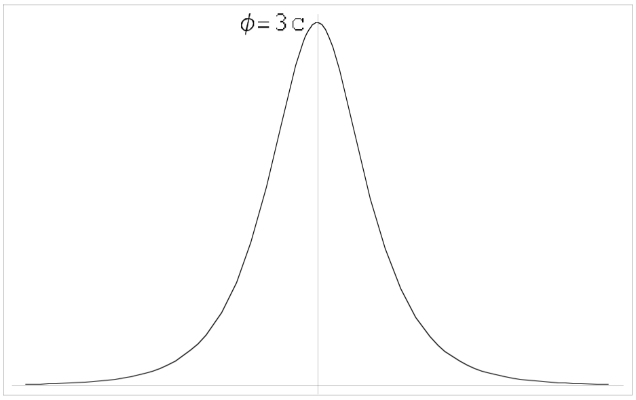

Fig. 2.1. The solitary wave of Eq. (1.4).

Case 2: $\alpha < 0$ and $-\frac{\beta}{2\alpha} > 3c$.

In this case, we can observe that $F(\phi)$ has a double zero $\phi = 0$, a simple zero $\phi = 3c$, and $F(\phi) > 0$ for $0 < \phi < 3c < -\frac{\beta}{2\alpha}$. Similar to the discussion in Case 1, Eq. (1.4) has the smooth solitary wave solutions with $0 = \min_{x\in R} \phi(x)$, $3c = \max_{x\in R} \phi(x)$ when $\alpha < 0$, $-\frac{\beta}{2\alpha} > 3c$, namely when $\alpha < 0$, $c < -\frac{\beta}{6\alpha}$.

3. Stability of smooth solitary waves. To study the stability problem of the smooth solitary wave solution to Eq. (1.4), we need the following results.

In terms of the functions E and F in (1.5), one can easily have

$$F'(\phi_c) + cE'(\phi_c) = 0,$$

where E' and F' are the Frechet derivatives of E and F, respectively in $H^1(R)$.

The linearized operator H_c of $F'(\phi_c) + cE'(\phi_c)$ around ϕ_c is defined by

$$H_c = F''(\phi_c) + cE''(\phi_c) = -\partial_x\left((\beta + 2\alpha\phi_c)\partial_x\right) - \phi_c - 2\alpha\phi_{cxx} + c.$$

We know that, since ϕ_c, ϕ_{cx}, $\phi_{cxx} \to 0$ exponentially fast as $|x| \to \infty$, smooth solitary waves exist when the following condition (a) and (b) hold:

(a) $0 < \phi_c < 3c$,

(b) $\alpha < 0$ and $2\alpha\phi_c + \beta > 0$ (or $\alpha > 0$ and $\beta > 0$).

It follows that the spectral equation $H_C v = \lambda v$ can be transformed by the Liouville transformation

$$\psi(y) = (\beta + 2\alpha\phi_c)^{1/4} v(x), \qquad y = \int_0^x \frac{1}{\sqrt{\beta + 2\alpha\phi_c(z)}} dz$$

into

$$H_c\psi(y) = \left(-\partial_y^2 + p_c(y) + c\right)\psi(y) = \lambda\psi(y),$$

where $p_c(y) = -\phi_c(x) - \frac{3\alpha}{2}\phi_c'' - \frac{\alpha^2(\phi_c'(x))^2}{4\left(\beta + 2\alpha\phi_c(x)\right)}$.

For the smooth solitary wave $\phi_c(x)$, we can obtain that $p_c(y)$ decays exponentially at infinity, which gives the result that the operator $L_c : H^1 \to H^{-1}$ is self-adjoint with essential spectrum $[c, \infty)$ and there are infinitely many eigenvalues which are less than c. The function to the nth eigenvalue (in increasing order) have up to a constant multiple, a unique eigenfunction with exactly $(n-1)$ zeros. Referring to [15], the Liouville transformation ensures that the same spectral information holds for the operator H_c. Noting that (2.2) shows $H_c(\phi_{cx}) = 0$. The property of the function ϕ_c denotes that ϕ_{cx} has exactly one zero. The zero eigenvalue of H_c is simple, and there is exactly one negative eigenvalue, while the rest of the spectrum is positive and bounded away from zero.

Next we discuss orbital stability of the smooth solitary waves in Eq. (1.4). As we known, a solitary wave is called orbital stable if a wave with an initial profile close to the solitary wave remains close to some translate of it at all later times.

Definition 3.1. *The solitary solution* ϕ_c *is orbital stable if for any* $\varepsilon > 0$, *there is* $\delta > 0$ *such that if* $0 < T \le \infty$ *and* $u \in C\left([0,T); H^1(R)\right)$ *is a solution to* (1.4) *with*

$$|u_0 - \phi_c|_{H^1} \le \delta,$$

then

$$\inf_{\varepsilon \in R} |u(\cdot, t) - \phi_c(\cdot - \varepsilon)|_{H^1} \le \varepsilon \quad \textit{for every} \quad t \in [0, T).$$

As for the above results, the stability depends on the convexity properties of the function $d(c) = F(\phi_c) + cE(\phi_c)$ [15]. Then We give the following theorem.

Theorem 3.1. *The solitary wave* ϕ_c *is stable if the function* $d(c)$ *is strictly convex, i.e.,* $d''(c) > 0$ *and unstable if the function* $d(c)$ *is strictly concave, i.e.,* $d''(c) < 0$.

Next , we will prove the following theorem.

Theorem 3.2. *The solitary wave solutions with* $c \in R^+$ *for Eq.* (1.4) *are stable.*

Proof. Differentiating $d(c)$ with respect to c, we get

$$d'(c) = \left\langle F'(\phi_c) + cE'(\phi_c), \frac{\partial \phi_c}{\partial c} \right\rangle + E(\phi_c) = E(\phi_c).$$

Using (2.4) and the fact that ϕ_c is an even function, we have

$$d''(c) = \frac{d}{dc}\int_R \phi_c{}^2 dx = \frac{d}{dc}\int_0^{+\infty} \phi_c{}^2 dx =$$

$$= \frac{d}{dc}\int_0^{+\infty} -\phi_c \phi_{cx} \frac{\sqrt{2\alpha\phi_c + \beta}}{\sqrt{c - \frac{\phi_c}{3}}}\, dx =$$

$$= \frac{d}{dc}\int_{3c}^{0} -y \frac{\sqrt{2\alpha y + \beta}}{\sqrt{c - \frac{y}{3}}}\, dy =$$

$$= \frac{d}{dc}\int_{0}^{3c} y \frac{\sqrt{2\alpha y + \beta}}{\sqrt{c - \frac{y}{3}}}\, dy.$$

Let $y = 3cs$. Then we have

$$d''(c) = \frac{d}{dc}\int_0^1 9c^2 s \frac{\sqrt{6\alpha cs + \beta}}{\sqrt{c - cs}}\, ds =$$

$$= \frac{d}{dc}\int_0^1 9c^{3/2} s \frac{\sqrt{6\alpha cs + \beta}}{\sqrt{1 - s}}\, ds.$$

Let $P(c) = \int_0^1 9c^{3/2} s \frac{\sqrt{6\alpha cs + \beta}}{\sqrt{1 - s}}\, ds$. Clearly, $P(c)$ is increasing at s in the interval $[0,1]$, then $d''(c) > 0$. According to Theorem 3.1, we can finish the proof of Theorem 3.2 and the solitary waves are stable for any wave speed.

4. Conclusions. The generalized KdV equation with combined dispersion has smooth solitary wave solutions under the influence of the combined dispersion, which can not be seen in Eq. (1.2), (1.3) owing only nonlinear dispersion. By using qualitative analysis method, the existence scope of smooth solitary wave was obtained. In the existence scope, the solitary wave is orbit stable. Noticing the important role of the combined dispersion, we will study other proposition of the combined dispersion in our coming researches.

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