UDC 517.9

E. Tunç (Gaziosmanpaşa Univ., Turkey),
H. Avcı (Ondokuz Mayıs Univ., Turkey)

NEW OSCILLATION THEOREMS FOR A CLASS OF SECOND-ORDER DAMPED NONLINEAR DIFFERENTIAL EQUATIONS

НОВІ ОСЦИЛЯЦІЙНІ ТЕОРЕМИ ДЛЯ ОДНОГО КЛАСУ ЗГАСАЮЧИХ НЕЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ

Some new oscillation criteria are established for the nonlinear damped differential equation

$$\left(r(t)k_1\left(x,x'\right)\right)'+p\left(t\right)k_2\left(x,x'\right)x'+q\left(t\right)f\left(x\left(t\right)\right)=0,\quad t\geq t_0.$$

The results obtained extend and improve some existing results in the literature.

Встановлено деякі нові осциляційні критерії для згасаючого нелінійного рівняння

$$\left(r(t)k_1\left(x,x'\right)\right)'+p\left(t\right)k_2\left(x,x'\right)x'+q\left(t\right)f\left(x\left(t\right)\right)=0,\quad t\geq t_0.$$

Отримані результати узагальнюють і посилюють деякі існуючі результати.

1. Introduction. We are concerned with oscillation behavior of solutions of second-order nonlinear differential equations with nonlinear damping of the form

$$\left(r(t)k_1\left(x,x'\right)\right)'+p\left(t\right)k_2\left(x,x'\right)x'+q\left(t\right)f\left(x\left(t\right)\right)=0, \tag{1.1}$$

where $t\geq t_0>0,\ r\in C^1([t_0,\infty);(0,\infty)),\ p,q\in C\left([t_0,\infty);R\right),\ f\in C\left(R,R\right),$ $k_1\in C^1\left(R^2,R\right)$ and $k_2\in C\left(R^2,R\right)$. Throughout the paper, it is assumed that

(a) $p(t)\geq 0$ for all $t\geq t_0$;

(b) $f(x)/x\geq K$ for some constant $K>0$ and all $x\in R\setminus\{0\}$;

(c) $q(t)\geq 0$ for all $t\geq t_0$ and $q(t)\not\equiv 0$ on $[t_*,0)$ for any $t_*\geq t_0$;

(d) $k_1^2(u,v)\leq\alpha_1 vk_1(u,v)$ for some constant $\alpha_1>0$ and all $(u,v)\in R^2$;

(e) $uvk_2(u,v)\geq\alpha_2 k_1^2(u,v)$ for some constant $\alpha_2>0$ and all $(u,v)\in R^2$;

(e_1) $uvk_2(u,v)\geq\alpha_2 uk_1(u,v)$ for some constant $\alpha_2>0$ and all $(u,v)\in R^2$.

We recall that a function $x\colon[t_0,t_1)\to(-\infty,\infty), t_1>t_0,$ is called a solution of equation (1.1) if $x(t)$ satisfies equation (1.1) for all $t\in[t_0,t_1)$. In the sequel, it will be always assumed that solutions of equation (1.1) exist on $[t_0,\infty)$. A solution $x(t)$ of equation (1.1) is called oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory. Eq. (1.1) is called oscillatory if all solutions are oscillatory.

In the relevant literature, till now, oscillation behaviors of solutions of linear and non-linear second order differential equations have been the subject of intensive investigations for many authors. For instance, one can refer to [1 – 39], as some related papers or books on the subject.

The oscillation of Eq. (1.1) was first studied by Rogovchenko and Rogovchenko in [1]. Afterward, under assumptions (a) – (e), Tiryaki and Zafer [21] established some oscillation criteria for (1.1), which extend and improve the results in [1]. We also note that the similar results for the differential equations those are near to (1) were established before (see, for example [36]).

The motivation for the present work has been inspired basically by the paper of [1, 21] and the works mentioned above. Our aim here is to improve the some results verified by [21] for the oscillation of solutions of Eq. (1.1) under assumptions (a) – (e). On the other hand, under assumptions (a) – (d) and (e_1) our results are new.

Let $D_0 = \{(t,s)\colon t > s \geq t_0\}$ and $D = \{(t,s)\colon t \geq s \geq t_0\}$. We say that the function $H = H(t,s) \in C\,(D, (-\infty,\infty))$ belongs to the class P if

(i) $H(t,t) = 0$ for $t \geq t_0, H(t,s) > 0$ on D_0;

(ii) $H(t,s)$ has a continuous and nonpositive partial derivative on D_0 with respect to the second variable, and there is a function $h \in C\,(D, [0,+\infty))$ such that

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \quad \text{for all} \quad (t,s) \in D_0.$$

2. Main results. The main results of this paper are the following theorems.

Theorem 1. *Let assumptions* (a) – (d) *and* (e_1) *hold. Further, suppose that there exists a function* $g \in C^1\left([t_0,\infty);\mathbb{R}\right)$ *such that, for some* $\beta \geq 1$ *and for some* $H \in P$,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1 \beta r(s) v(s)}{4} h^2(t,s) \right) ds = \infty, \tag{2.1}$$

where

$$v(t) = \exp\left(-2\int^{t} \left(\frac{g(s)}{\alpha_1} - \frac{\alpha_2 p(s)}{2r(s)} \right) ds \right) \tag{2.2}$$

and

$$\gamma(t) = v(t)\left(Kq(t) + \frac{r(t)g^2(t)}{\alpha_1} - \alpha_2 p(t) g(t) - \left(r(t)g(t)\right)' \right). \tag{2.3}$$

Then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then there exists a $T_0 \geq$ $\geq t_0$ such that $x(t) \neq 0$ for all $t \geq T_0 \geq t_0$. For $t \geq T_0$, define a generalized Riccati transformation by

$$w(t) = v(t)\left[\frac{r(t)k_1\left(x\left(t\right), x'\left(t\right)\right)}{x\left(t\right)} + r(t)g(t) \right] \tag{2.4}$$

where $v(t)$ is given by (2.2). Differentiating (2.4) and using (1.1), we have

$$w'(t) = \frac{v'(t)}{v(t)} w(t) + v(t)\left[\frac{-p(t)k_2\left(x\left(t\right), x'\left(t\right)\right)x'\left(t\right)}{x(t)} - \right.$$

$$\left. -\frac{q(t)f(x(t))}{x(t)} - \frac{r(t)k_1\left(x\left(t\right), x'\left(t\right)\right)x'(t)}{x^2\left(t\right)} + \left(r(t)g(t)\right)' \right]. \tag{2.5}$$

Using (a) – (d) and (e_1) in (2.5), we obtain

$$w'(t) \leq \frac{v'(t)}{v(t)} w(t) - \frac{\alpha_2 p(t) v(t) k_1\left(x\left(t\right), x'\left(t\right)\right)}{x(t)} - Kq(t)v(t) -$$

$$-\frac{r(t)v(t)k_1^2\left(x\left(t\right), x'\left(t\right)\right)}{\alpha_1 x^2(t)} + v(t)\left(r(t)g(t)\right)' =$$

$$= \frac{v'(t)}{v(t)} w(t) - \alpha_2 p(t) v(t)\left[\frac{w(t)}{v(t)r(t)} - g(t) \right] - Kq(t)v(t) -$$

$$-\frac{r(t)v(t)}{\alpha_1}\left[\frac{w(t)}{v(t)r(t)}-g(t)\right]^2+v(t)\left(r(t)g(t)\right)'=$$

$$=\left[-\frac{2g(t)}{\alpha_1}+\frac{\alpha_2 p(t)}{r(t)}\right]w(t)-\frac{\alpha_2 p(t)w(t)}{r(t)}+\alpha_2 p(t)g(t)v(t)-Kq(t)v(t)-$$

$$-\frac{w^2(t)}{\alpha_1 v(t)r(t)}+\frac{2g(t)w(t)}{\alpha_1}-\frac{r(t)g^2(t)v(t)}{\alpha_1}+v(t)\left(r(t)g(t)\right)'. \tag{2.6}$$

So, (2.6) yields, for all $t\geq T_0$,

$$w'(t)\leq-\gamma(t)-\frac{w^2(t)}{\alpha_1 v(t)r(t)} \tag{2.7}$$

where $\gamma(t)$ is defined by (2.3). Multiplying both sides of (2.7) by $H(t,s)$ and integrating from T to t, we have for some $\beta\geq 1$ and for all $t\geq T\geq T_0$,

$$\int_T^t H(t,s)\gamma(s)ds\leq-\int_T^t H(t,s)w'(s)ds-\int_T^t H(t,s)\frac{w^2(s)}{\alpha_1 r(s)v(s)}ds=$$

$$=H(t,T)w(T)-\int_T^t\left[-\frac{\partial H(t,s)}{\partial s}\right]w(s)ds-\int_T^t H(t,s)\frac{w^2(s)}{\alpha_1 r(s)v(s)}ds=$$

$$=H(t,T)w(T)-\int_T^t\left[h(t,s)\sqrt{H(t,s)}w(s)+H(t,s)\frac{w^2(s)}{\alpha_1 r(s)v(s)}\right]ds=$$

$$=H(t,T)w(T)-\int_T^t\left(\sqrt{\frac{H(t,s)}{\beta\alpha_1 r(s)v(s)}}w(s)+\frac{1}{2}\sqrt{\beta\alpha_1 r(s)v(s)}h(t,s)\right)^2ds+$$

$$+\frac{\beta\alpha_1}{4}\int_T^t r(s)v(s)h^2(t,s)ds-\int_T^t\frac{(\beta-1)H(t,s)}{\beta\alpha_1 r(s)v(s)}w^2(s)ds.$$

Hence, for all $t\geq T\geq T_0$,

$$\int_T^t\left(H(t,s)\gamma(s)-\frac{\alpha_1\beta}{4}r(s)v(s)h^2(t,s)\right)ds\leq$$

$$\leq H(t,T)w(T)-\int_T^t\frac{(\beta-1)H(t,s)}{\beta\alpha_1 r(s)v(s)}w^2(s)ds-$$

$$-\int_T^t\left(\sqrt{\frac{H(t,s)}{\beta\alpha_1 r(s)v(s)}}w(s)+\frac{1}{2}\sqrt{\beta\alpha_1 r(s)v(s)}h(t,s)\right)^2ds. \tag{2.8}$$

So, for every $t \geq T_0$,

$$\int\limits_{T_0}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1 \beta}{4} r(s)v(s)h^2(t,s) \right) ds \leq$$

$$\leq H(t,T_0)w(T_0) \leq H(t,T_0)\left|w(T_0)\right| \leq H(t,t_0)\left|w(T_0)\right|.$$

Therefore,

$$\int\limits_{t_0}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1 \beta}{4} r(s)v(s)h^2(t,s) \right) ds =$$

$$= \int\limits_{t_0}^{T_0} \left(H(t,s)\gamma(s) - \frac{\alpha_1 \beta}{4} r(s)v(s)h^2(t,s) \right) ds +$$

$$+ \int\limits_{T_0}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1 \beta}{4} r(s)v(s)h^2(t,s) \right) ds \leq$$

$$\leq \int\limits_{t_0}^{T_0} H(t,s)\left|\gamma(s)\right| ds + H(t,t_0)\left|w(T_0)\right| \leq$$

$$\leq H(t,t_0) \left[\int\limits_{t_0}^{T_0} \left|\gamma(s)\right| ds + \left|w(T_0)\right| \right]$$

for all $t \geq T_0$. This gives

$$\int\limits_{t_0}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1 \beta}{4} r(s)v(s)h^2(t,s) \right) ds \leq H(t,t_0) \left[\int\limits_{t_0}^{T_0} \left|\gamma(s)\right| ds + \left|w(T_0)\right| \right]. \tag{2.9}$$

It follows from (2.9) that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1 \beta}{4} r(s)v(s)h^2(t,s) \right) ds \leq$$

$$\leq \left[\int\limits_{t_0}^{T_0} \left|\gamma(s)\right| ds + \left|w(T_0)\right| \right] < +\infty$$

which contradicts (2.1). Therefore, Eq. (1.1) is oscillatory.

Under a modification of the hypotheses of Theorem 1, we can obtain the following result.

***Corollary* 1.** *In Theorem* 1, *if condition* (2.1) *is replaced by the conditions*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} H(t,s)\gamma(s)ds = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} r(s)v(s)h^2(t,s)ds < \infty,$$

then (1.1) *is oscillatory.*

Theorem 2. *Let assumptions* (a)–(d) *and* (e_1) *be fulfilled. Suppose that there exists function* $H \in P$ *and*

$$0 < \inf_{s\geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)}\right] \leq \infty. \tag{2.10}$$

Assume that there exist functions $g \in C^1([t_0,\infty);\mathbb{R})$ *and* $b \in C([t_0,\infty);\mathbb{R})$ *such that, for some* $\beta > 1$, *all* $t > t_0$ *and* $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int\limits_{T}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1\beta}{4} r(s)v(s)h^2(t,s)\right) ds \geq b(T), \tag{2.11}$$

where $\gamma(s), v(t)$ *are as in Theorem* 1. *If*

$$\limsup_{t\to\infty} \int\limits_{t_0}^{t} \frac{b_+^2(s)}{r(s)v(s)} ds = \infty, \tag{2.12}$$

where $b_+(t) = \max\{b(t), 0\}$. *Eq.* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We may assume that $x(t) \neq 0$ for some $t \geq T_0 \geq t_0$. Define $w(t)$ as in (2.4). Then, following the proof of Theorem 1, we obtain (2.8). Further, it follows that

$$\frac{1}{H(t,T)} \int\limits_{T}^{t} \left(H(t,s)\gamma(s) - \frac{\alpha_1\beta}{4} r(s)v(s)h^2(t,s)\right) ds \leq$$

$$\leq w(T) - \frac{1}{H(t,T)} \int\limits_{T}^{t} \frac{(\beta-1)H(t,s)}{\beta\alpha_1 r(s)v(s)} w^2(s)ds -$$

$$-\frac{1}{H(t,T)} \int\limits_{T}^{t} \left(\sqrt{\frac{H(t,s)}{\beta\alpha_1 r(s)v(s)}} w(s) + \frac{1}{2}\sqrt{\beta\alpha_1 r(s)v(s)} h(t,s)\right)^2 ds \leq$$

$$\leq w(T) - \frac{1}{H(t,T)} \int\limits_{T}^{t} \frac{(\beta-1)H(t,s)}{\beta\alpha_1 r(s)v(s)} w^2(s)ds.$$

Hence, for $t > T \geq T_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\gamma(s) - \frac{\alpha_1\beta}{4} r(s)v(s)h^2(t,s) \right) ds \leq$$

$$\leq w(T) - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)H(t,s)}{\beta\alpha_1 r(s)v(s)} w^2(s)ds.$$

Thus, by (2.11) we have

$$w(T) \geq b(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)H(t,s)}{\beta\alpha_1 r(s)v(s)} w^2(s)ds,$$

for all $T \geq T_0$ and for any $\beta > 1$. This implies that

$$w(T) \geq b(T) \quad \text{for all} \quad T \geq T_0 \tag{2.13}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \frac{H(t,s)}{r(s)v(s)} w^2(s)ds \leq \frac{\beta\alpha_1}{(\beta-1)} \left(w(T_0) - b(T_0)\right) < \infty. \tag{2.14}$$

Now, we calim that

$$\int_{T_0}^\infty \frac{w^2(s)}{r(s)v(s)} ds < \infty. \tag{2.15}$$

Suppose to the contrary that

$$\int_{T_0}^\infty \frac{w^2(s)}{r(s)v(s)} ds = \infty. \tag{2.16}$$

By (2.10), there exists a positive constant δ such that

$$\inf_{s\geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] > \delta > 0. \tag{2.17}$$

From (2.17),

$$\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} > \delta > 0,$$

and there exists a $T_2 \geq T_1$ such that $H(t,T_1)/H(t,t_0) \geq \delta$, for all $t \geq T_2$. On the other hand, by (2.16), for any positive number κ, there exist a $T_1 > T_0$, such that, for all $t \geq T_1$,

$$\int_{T_0}^t \frac{w^2(s)}{r(s)v(s)} ds \geq \frac{\kappa}{\delta}.$$

Using integration by parts, we conclude that, for all $t \geq T_1$,

$$\frac{1}{H(t,T_0)}\int\limits_{T_0}^{t}\frac{H(t,s)}{r(s)v(s)}w^2(s)ds=\frac{1}{H(t,T_0)}\int\limits_{T_0}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int\limits_{T_0}^{s}\frac{w^2(\tau)}{r(\tau)v(\tau)}d\tau\right]ds\geq$$

$$\geq\frac{\kappa}{\delta}\frac{1}{H(t,T_0)}\int\limits_{T_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]ds=\frac{\kappa}{\delta}\frac{H(t,T_1)}{H(t,T_0)}. \tag{2.18}$$

Hence, we have from (2.18)

$$\frac{1}{H(t,T_0)}\int\limits_{T_0}^{t}\frac{H(t,s)}{r(s)v(s)}w^2(s)ds\geq\kappa\quad\text{for all}\quad t\geq T_2.$$

Since κ is an arbitrary positive constant,

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int\limits_{T_0}^{t}\frac{H(t,s)}{r(s)v(s)}w^2(s)ds=+\infty,$$

which contradicts assumption (2.14). Therefore we proved that (2.16) fails, so (2.15) holds true. Then, it follows from (2.13) that $b_+^2(T)\leq w^2(T)$ for all $T\geq T_0$, and

$$\int\limits_{T_0}^{\infty}\frac{b_+^2(s)}{r(s)v(s)}ds\leq\int\limits_{T_0}^{\infty}\frac{w^2(s)}{r(s)v(s)}ds<+\infty,$$

which contradicts (2.12). Hence, Eq. (1.1) is oscillatory.

Theorem 3. *The conclusion of Theorem* 2 *remains valid, if assumptions* (2.11) *is replaced by*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int\limits_{T}^{t}\left(H(t,s)\gamma(s)-\frac{\alpha_1\beta}{4}r(s)v(s)h^2(t,s)\right)ds\geq b(T)$$

for all $T\geq t_0$ and for some $\beta>1$.

Proof. Due to the fact that

$$b(T)\leq\liminf_{t\to\infty}\frac{1}{H(t,T)}\int\limits_{T}^{t}\left(H(t,s)\gamma(s)-\frac{\alpha_1\beta}{4}r(s)v(s)h^2(t,s)\right)ds\leq$$

$$\leq\limsup_{t\to\infty}\frac{1}{H(t,T)}\int\limits_{T}^{t}\left(H(t,s)\gamma(s)-\frac{\alpha_1\beta}{4}r(s)v(s)h^2(t,s)\right)ds,$$

the conclusion follows immediately from Theorem 2.

From now on, we shall consider the oscillation for (1.1) under assumptions (a) – (e).

Theorem 4. *Let assumptions* (a) – (e) *hold. Suppose that there exists a function $g\in C^1([t_0,\infty);\mathbb{R})$ such that, for some $\beta\geq 1$ and for some $H\in P$,*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \left(H(t,s)\gamma_1(s) - \frac{\beta\alpha_1}{4(\alpha_1\alpha_2 p(s) + r(s))} r^2(s)v(s)h^2(t,s) \right) ds = \infty, \tag{2.19}$$

where

$$v(t) = \exp\left(-2 \int^{t} \left(\frac{g(s)}{\alpha_1} + \frac{\alpha_2 p(s) g(s)}{r(s)} \right) ds \right), \tag{2.20}$$

$$\gamma_1(t) = v(t)\left(Kq(t) + \frac{r(t)g^2(t)}{\alpha_1} + \alpha_2 p(t) g^2(t) - (r(t)g(t))' \right). \tag{2.21}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a non-oscillatory solution of the differential equation (1.1). Then there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T_0$. Define

$$w(t) = v(t)\left[\frac{r(t)k_1\left(x\left(t\right), x'\left(t\right)\right)}{x\left(t\right)} + r(t)g(t) \right] \quad \text{for all} \quad t \geq T_0,$$

where $v(t)$ is given by (2.20). This and (1.1) imply

$$w'(t) = \frac{v'(t)}{v(t)} w(t) + v(t)\left[\frac{-p(t)k_2\left(x\left(t\right), x'\left(t\right)\right)x'\left(t\right)}{x(t)} - \right.$$

$$\left. - \frac{q(t)f(x(t))}{x(t)} - \frac{r(t)k_1\left(x\left(t\right), x'\left(t\right)\right)x'(t)}{x^2\left(t\right)} + (r(t)g(t))' \right].$$

Taking into account (a) – (e), we conclude that for all $t \geq T_0$,

$$w'(t) \leq \frac{v'(t)}{v(t)} w(t) - \frac{\alpha_2 p(t) v(t) k_1^2\left(x\left(t\right), x'(t)\right)}{x^2(t)} - Kq(t)v(t) -$$

$$- \frac{r(t)v(t)k_1^2\left(x\left(t\right), x'\left(t\right)\right)}{\alpha_1 x^2(t)} + v(t)\left(r(t)g(t)\right)' =$$

$$= \frac{v'(t)}{v(t)} w(t) - \alpha_2 p(t) v(t) \left[\frac{w(t)}{v(t)r(t)} - g(t) \right]^2 - Kq(t)v(t) -$$

$$- \frac{r(t)v(t)}{\alpha_1} \left[\frac{w(t)}{v(t)r(t)} - g(t) \right]^2 + v(t)\left(r(t)g(t)\right)' =$$

$$= \left[-\frac{2g(t)}{\alpha_1} - \frac{2\alpha_2 p(t) g(t)}{r(t)} \right] w(t) - \frac{\alpha_2 p(t) w^2(t)}{v(t) r^2(t)} + \frac{2\alpha_2 p(t) g(t) w(t)}{r(t)} -$$

$$- \alpha_2 p(t) g^2(t) v(t) - Kq(t)v(t) - \frac{w^2(t)}{\alpha_1 v(t) r(t)} +$$

$$+ \frac{2g(t)w(t)}{\alpha_1} - \frac{r(t)g^2(t)v(t)}{\alpha_1} + v(t)\left(r(t)g(t)\right)'$$

which yields

$$w'(t) \le -\gamma_1(t) - \frac{(\alpha_1\alpha_2 p(t) + r(t))\, w^2(t)}{\alpha_1 v(t) r^2(t)}, \tag{2.22}$$

where $\gamma_1(t)$ is defined by (2.21). Multiplying both sides of (2.22) by $H(t,s)$ and integrating from T to t, we have for some $\beta \ge 1$ and for all $t \ge T \ge T_0$,

$$\int_T^t H(t,s)\gamma_1(s)ds \le -\int_T^t H(t,s)w'(s)ds - \int_T^t H(t,s)\frac{(\alpha_1\alpha_2 p(s) + r(s))\, w^2(s)}{\alpha_1 r^2(s) v(s)}ds =$$

$$= H(t,T)w(T) - \int_T^t \left[-\frac{\partial H(t,s)}{\partial s}\right] w(s)ds - \int_T^t H(t,s)\frac{(\alpha_1\alpha_2 p(s) + r(s))\, w^2(s)}{\alpha_1 r^2(s) v(s)}ds =$$

$$= H(t,T)w(T) - \int_T^t \left[h(t,s)\sqrt{H(t,s)}w(s) + H(t,s)\frac{(\alpha_1\alpha_2 p(s) + r(s))\, w^2(s)}{\alpha_1 r^2(s) v(s)}\right] ds =$$

$$= H(t,T)w(T) - \int_T^t \left(\sqrt{\frac{H(t,s)\,(\alpha_1\alpha_2 p(s) + r(s))}{\beta\alpha_1 r^2(s) v(s)}}w(s) + \right.$$

$$\left. + \frac{1}{2}\sqrt{\frac{\beta\alpha_1 r^2(s) v(s)}{(\alpha_1\alpha_2 p(s) + r(s))}}h(t,s)\right)^2 ds + \frac{\beta\alpha_1}{4}\int_T^t \frac{r^2(s)v(s)}{(\alpha_1\alpha_2 p(s) + r(s))}h^2(t,s)ds -$$

$$-\int_T^t \frac{(\beta-1)H(t,s)\,(\alpha_1\alpha_2 p(s) + r(s))}{\beta\alpha_1 r^2(s) v(s)}w^2(s)ds.$$

Thus, for all $t \ge T \ge T_0$, we obtain that

$$\int_T^t \left(H(t,s)\gamma_1(s) - \frac{\beta\alpha_1 r^2(s) v(s)}{4\,(\alpha_1\alpha_2 p(s) + r(s))}h^2(t,s)\right) ds \le H(t,T)w(T) -$$

$$-\int_T^t \frac{(\beta-1)H(t,s)\,(\alpha_1\alpha_2 p(s) + r(s))}{\beta\alpha_1 r^2(s) v(s)}w^2(s)ds -$$

$$-\int_T^t \left(\sqrt{\frac{H(t,s)\,(\alpha_1\alpha_2 p(s) + r(s))}{\beta\alpha_1 r^2(s) v(s)}}w(s) + \frac{1}{2}\sqrt{\frac{\beta\alpha_1 r^2(s) v(s)}{(\alpha_1\alpha_2 p(s) + r(s))}}h(t,s)\right)^2 ds. \tag{2.23}$$

Following the same lines as in the proof of Theorem 1, we conclude that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left(H(t,s)\gamma_1(s) - \frac{\beta\alpha_1 r^2(s) v(s)}{4\,(\alpha_1\alpha_2 p(s) + r(s))}h^2(t,s)\right) ds \le$$

$$\leq \int\limits_{t_0}^{T_0} |\gamma_1(s)|\, ds + |w(T_0)| < +\infty,$$

which contradicts assumption (2.19). Therefore, we have proved that all solutions of Eq. (1.1) are oscillatory.

***Corollary* 2.** *The conclusion of Theorem* 4 *remains intact if assumption* (2.19) *is replaced with the following conditions*:

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} H(t,s)\gamma_1(s)ds = \infty$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^{t} \frac{r^2(s)v(s)}{\alpha_1\alpha_2 p(s) + r(s)} h^2(t,s)ds < \infty.$$

Theorem 5. *Let assumptions* (a) – (e) *hold. Suppose that there exists function* $H \in \in P$ *and*

$$0 < \inf_{s\geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] \leq \infty. \tag{2.24}$$

Assume that there exist functions $g \in C^1([t_0,\infty);\mathbb{R})$ *and* $b \in C([t_0,\infty);\mathbb{R})$ *such that, for all* $t > t_0$, *all* $T \geq t_0$, *and for some* $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int\limits_{T}^{t} \left(H(t,s)\gamma_1(s) - \frac{\beta\alpha_1 r^2(s)v(s)}{4(\alpha_1\alpha_2 p(s) + r(s))} h^2(t,s) \right) ds \geq b(T), \tag{2.25}$$

where $\gamma_1(s)$, $v(t)$ *are as in Theorem* 4, *and furthermore suppose that*

$$\limsup_{t\to\infty} \int\limits_{t_0}^{t} \frac{(\alpha_1\alpha_2 p(s) + r(s))\, b_+^2(s)}{r^2(s)v(s)} ds = \infty, \tag{2.26}$$

where $b_+(t) = \max\{b(t), 0\}$. *Then Eq.* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). We may assume that $x(t) \neq 0$ for some $t \geq T_0 \geq t_0$. Define $w(t)$ as in (2.4). As in the proof of Theorem 4, we can obtain (2.23). Then,

$$\frac{1}{H(t,T)} \int\limits_{T}^{t} \left(H(t,s)\gamma_1(s) - \frac{\beta\alpha_1 r^2(s)v(s)}{4(\alpha_1\alpha_2 p(s) + r(s))} h^2(t,s) \right) ds \leq w(T) -$$

$$- \frac{1}{H(t,T)} \int\limits_{T}^{t} \frac{(\beta - 1)H(t,s)(\alpha_1\alpha_2 p(s) + r(s))}{\beta\alpha_1 r^2(s)v(s)} w^2(s)ds -$$

$$- \frac{1}{H(t,T)} \int\limits_{T}^{t} \left(\sqrt{\frac{H(t,s)(\alpha_1\alpha_2 p(s) + r(s))}{\beta\alpha_1 r^2(s)v(s)}} w(s) + \right.$$

$$+\frac{1}{2}\sqrt{\frac{\beta\alpha_1 r^2(s)v(s)}{(\alpha_1\alpha_2 p(s)+r(s))}}h(t,s)\Bigg)^2 ds \leq$$

$$\leq w(T) - \frac{1}{H(t,T)}\int\limits_T^t \frac{(\beta-1)H(t,s)\left(\alpha_1\alpha_2 p(s)+r(s)\right)}{\beta\alpha_1 r^2(s)v(s)} w^2(s)ds.$$

Consequently,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int\limits_T^t\left(H(t,s)\gamma_1(s) - \frac{\beta\alpha_1 r^2(s)v(s)}{4\left(\alpha_1\alpha_2 p(s)+r(s)\right)}h^2(t,s)\right)ds \leq$$

$$\leq w(T) - \liminf_{t\to\infty}\frac{1}{H(t,T)}\int\limits_T^t \frac{(\beta-1)H(t,s)\left(\alpha_1\alpha_2 p(s)+r(s)\right)}{\beta\alpha_1 r^2(s)v(s)} w^2(s)ds$$

for $t > T \geq T_0$. It follows from (2.25)

$$w(T) \geq b(T) + \liminf_{t\to\infty}\frac{1}{H(t,T)}\int\limits_T^t \frac{(\beta-1)H(t,s)\left(\alpha_1\alpha_2 p(s)+r(s)\right)}{\beta\alpha_1 r^2(s)v(s)} w^2(s)ds$$

for all $T \geq T_0$ and for any $\beta > 1$. This implies that,

$$w(T) \geq b(T) \quad \text{for all} \quad T \geq T_0 \tag{2.27}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int\limits_{T_0}^t \frac{H(t,s)\left(\alpha_1\alpha_2 p(s)+r(s)\right)}{r^2(s)v(s)} w^2(s)ds \leq$$

$$\leq \frac{\beta\alpha_1}{(\beta-1)}\left(w(T_0)-b(T_0)\right) < \infty. \tag{2.28}$$

Now, we claim that

$$\int\limits_{T_0}^\infty \frac{(\alpha_1\alpha_2 p(s)+r(s))}{r^2(s)v(s)} w^2(s)ds < \infty. \tag{2.29}$$

Suppose the contrary, that is,

$$\int\limits_{T_0}^\infty \frac{(\alpha_1\alpha_2 p(s)+r(s))}{r^2(s)v(s)} w^2(s)ds = \infty. \tag{2.30}$$

It follows from (2.24) that there exists a positive constant δ such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] > \delta. \tag{2.31}$$

From (2.31),

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} > \delta > 0,$$

and there exists a $T_2 \geq T_1$ such that $H(t,T_1)/H(t,t_0) \geq \delta$, for all $t \geq T_2$. On the other hand, from (2.30), for any positive number κ, there exist a $T_1 > T_0$, such that, for all $t \geq T_1$,

$$\int_{T_0}^{t} \frac{(\alpha_1\alpha_2 p(s) + r(s))}{r^2(s)v(s)} w^2(s)ds \geq \frac{\kappa}{\delta}.$$

Using integration by parts, we obtain that, for all $t \geq T_1$,

$$\frac{1}{H(t,T_0)}\int_{T_0}^{t} \frac{H(t,s)\,(\alpha_1\alpha_2 p(s) + r(s))}{r^2(s)v(s)} w^2(s)ds =$$

$$= \frac{1}{H(t,T_0)}\int_{T_0}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_{T_0}^{s}\frac{(\alpha_1\alpha_2 p(\tau) + r(\tau))}{r^2(\tau)v(\tau)} w^2(\tau)d\tau\right] ds \geq$$

$$\geq \frac{\kappa}{\delta}\frac{1}{H(t,T_0)}\int_{T_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right] ds = \frac{\kappa}{\delta}\frac{H(t,T_1)}{H(t,T_0)}. \tag{2.32}$$

Hence, we have from (2.32)

$$\frac{1}{H(t,T_0)}\int_{T_0}^{t} \frac{H(t,s)\,(\alpha_1\alpha_2 p(s) + r(s))}{r^2(s)v(s)} w^2(s)ds \geq \kappa \quad \text{for all } t \geq T_2.$$

Since κ is an arbitrary positive constant,

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t} \frac{H(t,s)\,(\alpha_1\alpha_2 p(s) + r(s))}{r^2(s)v(s)} w^2(s)ds = +\infty,$$

which conradicts (2.28). Therefore, (2.29) holds, and from (2.27)

$$\int_{T_0}^{\infty} \frac{(\alpha_1\alpha_2 p(s) + r(s))}{r^2(s)v(s)} b_+^2(s)ds \leq \int_{T_0}^{\infty} \frac{(\alpha_1\alpha_2 p(s) + r(s))}{r^2(s)v(s)} w^2(s)ds < +\infty,$$

which contradicts (2.26). Therefore, Eq. (1.1) is oscillatory.

***Remark* 1.** In Theorem 5, the condition

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} \frac{r^2(s)v(s)}{\alpha_1\alpha_2 p(s) + r(s)} h^2(t,s)ds < \infty$$

is not necessary, however, an analogue of this condition is required in [21] (Theorem 2.3).

Theorem 6. *Let all assumptions of Theorem* 5 *satisfied except that condition* (2.25) *be replaced with*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int\limits_T^t \left(H(t,s)\gamma_1(s) - \frac{\beta\alpha_1 r^2(s)v(s)}{4\left(\alpha_1\alpha_2 p(s) + r(s)\right)} h^2(t,s) \right) ds \geq b(T).$$

Then Eq. (1.1) *is oscillatory.*

Proof. By an similar argument to that in the proof of Theorem 3, one can complete the proof of this theorem. Therefore, we omit the detailed proof for the theorem.

***Remark* 2.** In Theorem 6, the condition

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int\limits_{t_0}^t \frac{r^2(s)v(s)}{\alpha_1\alpha_2 p(s) + r(s)} h^2(t,s) ds < \infty$$

is not necessary, however, an analogue of this condition is required in [21] (Theorem 2.2).

3. Applications. Following the classical ideas of Kamenev [12], we define $H(t,s)$ as

$$H(t,s) = (t-s)^{n-1}, \quad (t,s) \in D,$$

where $n > 2$ is an integer. Evidently, $H \in P$ and

$$h(t,s) = (n-1)(t-s)^{(n-3)/2}, \quad (t,s) \in D.$$

Then, by Theorems 1 and 2 we have following two corollaries.

***Corollary* 3.** *Let* (a) – (d) *and* (e_1) *hold. Suppose that there exists a function* $g \in C^1([t_0,\infty);\mathbb{R})$ *such that, for some integer* $n > 2$ *and some* $\beta \geq 1$,

$$\limsup_{t\to\infty} t^{1-n} \int\limits_{t_0}^t (t-s)^{n-3} \left((t-s)^2 \gamma(s) - \frac{\alpha_1\beta r(s)v(s)}{4} (n-1)^2 \right) ds = \infty, \quad (2.33)$$

where $\nu(t)$ *and* $\gamma(t)$ *are as in Theorem* 1. *Then Eq.* (1.1) *is oscillatory.*

***Corollary* 4.** *Let assumptions* (a) – (d) *and* (e_1) *be fulfilled. Assume that there exist functions* $g \in C^1([t_0,\infty);\mathbb{R})$ *and* $b \in C([t_0,\infty);\mathbb{R})$ *such that, for all* $T \geq t_0$, *some* $\beta > 1$, *and some integer* $n > 2$,

$$\limsup_{t\to\infty} t^{1-n} \int\limits_T^t \left((t-s)^{n-1}\gamma(s) - \frac{\alpha_1\beta(n-1)^2}{4} r(s)v(s)(t-s)^{n-3} \right) ds \geq b(T), \quad (2.34)$$

where $\gamma(s)$, $v(t)$ *are as in Theorem* 1. *Suppose also that* (2.12) *is satisfied. Then Eq.* (1.1) *is oscillatory.*

***Example* 1.** For $t \geq 1$, consider the differential equation of the form

$$\left(\frac{x'}{1+x^2}\right) + \frac{2(1+\cos t)}{1+x^2} x' + \left(2 + 2\cos t - \sin t + \cos^2 t\right) x \left(1 + \frac{3}{7+x^2}\right) = 0. \quad (2.35)$$

It is easy to see that conditions (a) – (d) and (e_1) hold with $K = \alpha_1 = \alpha_2 = 1$. Let $g(t) = 1 + \cos t$; then $\nu(t) = 1$ and $\gamma(t) = 1$. Applying Corollary 3 with $n = 3$, we have

$$\limsup_{t\to\infty} \frac{1}{t^2} \int\limits_1^t \Big((t-s)^2 \cdot 1 - \beta \Big)\, ds = \infty.$$

Hence, Eq. (2.35) is oscillatory by Corollary 3. Since condition (e) is not satisfied, the results of [21] can not be used to Eq. (2.35).

***Example* 2.** Consider the differential equation

$$\left(t^2 \frac{x'}{1+x^2} \right) + \frac{2t^3}{1+x^2} x' + \left(6t^2 - 6t^2 \sin^2 t + t^4 + 2 \right) x(2+\cos x) = 0, \quad t \ge 1, \tag{2.36}$$

where

$$k_1(x,x') = \frac{x'}{1+x^2}, \qquad k_2(x,x') = \frac{1}{1+x^2}$$

$$(\alpha_1 = \alpha_2 = 1) \quad \text{and} \quad f(x) = x(2+\cos x) \quad (K=1).$$

We apply Corollary 4 with $n = 3$ and $g(t) = t$. Then, $\nu(t) = 1$ and $\gamma(t) = 3t^2 - 6t^2 \sin^2 t + 2$. Let $\beta = 2$. A direct computation yields

$$\limsup_{t\to\infty} \frac{1}{t^2} \int\limits_T^t \Big((t-s)^2 \left(3s^2 - 6s^2 \sin^2 s + 2 \right) - 2s^2 \Big)\, ds =$$

$$= \frac{3}{2} \left(1 + \cos T \sin T - 2T \cos^2 T - \frac{T}{3} - 2T^2 \cos T \sin T \right) = b(T).$$

Let $b_+(t) = \max\left(b(t), 0\right)$. The relation

$$\frac{b_+^2(t)}{r(t)\nu(t)} = O(t^2) \quad \text{as} \quad t \to \infty$$

implies that the condition (2.12) is satisfied. Therefore, (2.36) is oscillatory by Corollary 4. Note that in this example

$$\limsup_{t\to\infty} \frac{1}{t^2} \int\limits_1^t \frac{\alpha_1 \beta}{4} r(s) v(s) h^2(t,s) ds = \limsup_{t\to\infty} \frac{1}{t^2} \int\limits_1^t 2s^2 ds = \infty. \tag{2.37}$$

(2.37) show that we do not need to impose any condition similar to the condition

$$\limsup_{t\to\infty} \frac{1}{t^2} \int\limits_1^t \frac{\alpha_1 \beta}{4} r(s) v(s) h^2(t,s) ds < \infty$$

in Theorem 2, but the analogue of this condition is necessary for Theorem 3.4, 3.7 in [21].

1. *Rogovchenko S. P., Rogovchenko Yu. V.* Oscillation theorems for differential equations with a nonlinear damping // J. Math. Anal. and Appl. – 2003. – **279**. – P. 121 – 134.
2. *Rogovchenko Yu. V., Tuncay F.* Oscillation theorems for a class of second-order nonlinear differential equations with damping // Taiwan. J. Math. – 2009. – **13**, № 6B. – P. 1909 – 1928.

3. *Rogovchenko Yu. V., Tuncay F.* Oscillation criteria for second-order nonlinear differential equations with damping // Nonlinear Anal. – 2008. – **62**. – P. 208 – 221.
4. *Rogovchenko Yu. V.* Oscillation theorems for second-order equations with damping // Nonlinear Anal. – 2000. – **41**. – P. 1005 – 1028.
5. *Rogovchenko S. P., Rogovchenko Yu. V.* Oscillation results for general second-order differential equations with damping term // J. Math. Anal. and Appl. – 2003. – **279**. – P. 139 – 152.
6. *Zhao A., Wang Y., Yan J.* Oscillation criteria for second-order nonlinear differential equations with nonlinear damping // Comp. Math. and Appl. – 2008. – **56**. – P. 542 – 555.
7. *Kirane M., Rogovchenko Yu. V.* On oscillation of nonlinear second order differential equation with damping term // Appl. Math. and Comput. – 2001. – **117**. – P. 177 – 192.
8. *Kirane M., Rogovchenko Yu. V.* Oscillation results for a second order damped differential equation with nonmonotonous nonlinearity // J. Math. Anal. and Appl. – 2000. – **250**. – P. 118 – 138.
9. *Mustafa O. G., Rogovchenko S. P., Rogovchenko Yu. V.* On oscillation of nonlinear second-order differential equations with damping term // J. Math. Anal. and Appl. – 2004. – **298**. – P. 604 – 620.
10. *Bobisud L. E.* Oscillation of solutions of damped nonlinear differential equations // SIAM J. Math. Anal. – 1970. – **18**. – P. 601 – 606.
11. *Zheng Z.* Oscillation criteria for nonlinear second order differential equations with damping // Acta math. hung. – 2006. – **110**, № 3. – P. 241 – 252.
12. *Kamenev I. V.* An integral criterion for oscillation of linear differential equations // Mat. Zametki. – 1978. – **23**. – P. 249 – 251.
13. *Philos Ch. G.* Oscillation theorems for linear differential equations of second order // Arch. Math. – 1989. – **53**. – P. 483 – 492.
14. *Wintner A.* A criterion of oscillatory stability // Quart. Appl. Math. – 1949. – **7**. – P. 115 – 117.
15. *Hartman P.* On nonoscillatory linear differential equations of second order // Amer. J. Math. – 1952. – **74**. – P. 389 – 400.
16. *Agarwal R. P., Grace S. R., O'Regan D.* Oscillation theory for difference and functional differential equations. – Dordrecht: Kluwer, 2000.
17. *Agarwal R. P., Grace S. R., O'Regan D.* Oscillation theory for second order linear, half-linear, superlinear and sublinear dynamic equations. – Dordrecht: Kluwer, 2002.
18. *Győri I., Ladas G.* Oscillation theory of delay differential equations. – New York: Clarendon Press, Oxford Univ. Press, 1991.
19. *Swanson C. A.* Comparison and oscillation theory of linear differential equations. – New York: Acad. Press, 1968.
20. *Baker J. W.* Oscillation theorems for a second order damped nonlinear differential equation // SIAM J. Math. Anal. – 1973. – **25**. – P. 37 – 40.
21. *Tiryaki A., Zafer A.* Oscillation of second-order nonlinear differential equations with nonlinear damping // Math. Comput. Modelling. – 2004. – **39**. – P. 197 – 208.
22. *Butler G. J.* The oscillatory behavior of a second order nonlinear differential equation with damping // J. Math. Anal. and Appl. – 1977. – **57**. – P. 273 – 289.
23. *Grace S. R.* Oscillation theorems for second order nonlinear differential equations with damping // Math. Nachr. – 1989. – **141**. – P. 117 – 127.
24. *Grace S. R.* Oscillation theorems for nonlinear differential equations of second order // J. Math. Anal. and Appl. – 1992. – **171**. – P. 220 – 241.
25. *Grace S. R., Lalli B. S.* Oscillation theorems for nonlinear second order differential equations with a damping term // Comment. math. Univ. carol. – 1989. – **30**. – P. 691 – 697.
26. *Grace S. R., Lalli B. S.* Integral averaging technique for the oscillation of second order nonlinear differential equations // J. Math. Anal. and Appl. – 1990. – **149**. – P. 277 – 311.
27. *Grace S. R., Lalli B. S., Yeh C. C.* Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term // SIAM J. Math. Anal. – 1984. – **15**. – P. 1082 – 1093.
28. *Grace S. R., Lalli B. S., Yeh C. C.* Addendum: Oscillation theorems for nonlinear second order differential equations with a nonlinear damping term // SIAM J. Math. Anal. – 1988. – **19**. – P. 1252 – 1253.
29. *Manojlović J. V.* Oscillation criteria for sublinear differential equations with damping // Acta math. hungar. – 2004. – **104**. – P. 153 – 169.
30. *Yan J.* Oscillation theorems for second order linear differential equations with damping // Proc. Amer. Math. Soc. – 1986. - **98**. – P. 276 – 282.
31. *Yang X.* Oscillation criteria for nonlinear differential equations with damping // Appl. Math. Comput. – 2003. – **136**. – P. 549 – 557.

32. *Yeh C. C.* Oscillation theorems for nonlinear second order differential equations with damping term // Proc. Amer. Math. Soc. – 1982. – **84**. – P. 397 – 402.
33. *Li H. J.* Oscillation criteria for second order linear differential equations // J. Math. Anal. and Appl. – 1995. – **194**. – P. 217 – 234.
34. *Elabbasy E. M., Elhaddad W. W.* Oscillation of second-order nonlinear differential equations with damping term // Electron. J. Qualitative Theory Different. Equat. – 2007. – № 5. – P. 1 – 19.
35. *Elabbasy E. M., Hassan T. S., Saker S. H.* Oscillation of second-order nonlinear differential equations with a damping term // Electron. J. Qualitative Theory Different. Equat. – 2005. – **76**. – P. 1 – 13.
36. *Elabbasy E. M.* Oscillation theorems for perturbed second order nonlinear differential equations with damping // Serdica Math. J. – 1997. – **13**. – P. 1 – 14.
37. *Tunç E.* A note on the oscillation of second order differential equations with damping // J. Comput. Anal. and Appl. – 2010. – **12**, № 2. – P. 444 – 453.
38. *Tunç C., Tunç E.* On the asymptotic behavior of solutions of certain second-order differential equations // J. Franklin Inst. – 2007. – **344**. – P. 391 – 398.
39. *Wang Q. R.* Oscillation criteria for nonlinear second order damped differential equation // Acta math. hung. – 2004. – **102**, № 1-2. – P. 117 – 139.

Received 10.01.11,
after revision — 18.07.11