UDC 517.5

A. Razani, V. Parvaneh (Islamic Azad Univ., Karaj, Iran),
M. Abbas (Lahore Univ. Management Sci., Pakistan)

A COMMON FIXED POINT FOR GENERALIZED $(\psi, \varphi)_{f,g}$-WEAK CONTRACTIONS

СПІЛЬНА НЕРУХОМА ТОЧКА ДЛЯ УЗАГАЛЬНЕНИХ $(\psi, \varphi)_{f,g}$-СЛАБКИХ СТИСКУЮЧИХ ВІДОБРАЖЕНЬ

We extend the common fixed point theorem established by Zhang and Song in 2009 to generalized $(\psi, \varphi)_{f,g}$-weak contractions. Moreover, we give an example that illustrates the main result. Finally, some common fixed point results are obtained for mappings satisfying a contraction condition of the integral type in complete metric spaces.

Теорему про спільну нерухому точку, що була встановлена Чжаном і Суном у 2009 році, поширено на узагальнені $(\psi, \varphi)_{f,g}$-слабкі стискуючі відображення. Наведено приклад, що ілюструє основний результат. Отримано деякі результати про спільну нерухому точку для відображень, що задовольняють умову стиску інтегрального типу у повних метричних просторах.

1. Introduction. Let (X, d) be a metric space. A mapping $T\colon X \to X$ is said to be φ-weak contraction, if there exists a map $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y))$$

for all $x, y \in X$.

The above notion has been defined by Alber et al. [2] in 1997. They obtained some fixed point results in Hilbert spaces. Then Rhoades [14] extended those results in Banach spaces. In 2006, Beg and Abbas [5] studied some generalizations of Rhoades's results [14] for a pair of mappings such that one is weakly contractive with respect to the other.

In 2009, Zhang et al. [15] introduced the concept of generalized φ-weak contraction as follows:

Definition 1.1. *Two mappings $T, S\colon X \to X$ are called generalized φ-weak contractions, if there exists a lower semicontinuous function $\varphi\colon [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that*

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)),$$

for all $x, y \in X$, where

$$N(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Sy),\ \frac{1}{2}\big[d(x, Sy) + d(y, Tx)\big]\right\}.$$

Zhang et al. proved the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space, and $T, S\colon X \to X$ are generalized φ-weak contractions mappings where $\varphi\colon [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then there exists a unique point $u \in X$ such that $u = Tu = Su$.*

So far, many authors extended Theorem 1.1 (see [1, 7, 12]). Moreover, Doric [7] generalized it, by the definition of generalized (ψ, φ)-weak contractions.

Definition 1.2. *Two mappings $T,S\colon X\to X$ are called generalized (ψ,φ)-weak contractive, if there exist two maps $\varphi,\psi\colon[0,\infty)\to[0,\infty)$ such that*

$$\psi(d(Tx,Sy))\le\psi(N(x,y))-\varphi(N(x,y)),$$

for all $x,y\in X$, where N and φ are as in Definition 1.1 and $\psi\colon[0,\infty)\to[0,\infty)$ is a continuous monotone nondecreasing function with $\psi(0)=0$ and $\psi(t)>0$ for all $t>0$.

Theorem 1.2 [7]. *Let (X,d) be a complete metric space, and $T,S\colon X\to X$ be generalized (ψ,φ)-weak contractive self-mappings. Then there exists a unique point $u\in X$ such that $u=Tu=Su$.*

Moradi et al. [12] extended the Zhang and Song's result by introducing the notion of φ_f-weak contractive mappings.

Definition 1.3. *Two mappings $T,S\colon X\to X$ are called generalized φ_f-weak contractive, if there exist two maps $\varphi\colon[0,\infty)\to[0,\infty)$ and $f\colon X\to X$ where φ is a lower semicontinuous function with $\varphi(0)=0$ and $\varphi(t)>0$ for all $t>0$ such that*

$$d(Tx,Sy)\le P(x,y)-\varphi(P(x,y)),$$

for all $x,y\in X$, where

$$P(x,y)=\max\left\{d(fx,fy),d(fx,Tx),d(fy,Sy),\frac{1}{2}\big[d(fy,Tx)+d(fx,Sy)\big]\right\}.$$

Moradi et al. [12] proved the following theorem:

Theorem 1.3. *Let (X,d) be a complete metric space and E be a nonempty closed subset of X. Let $T,S\colon E\to E$ be two generalized φ_f-weak contractive.*

Assume that f is a continuous function on E and

(I) *$TE\subseteq fE$ and $SE\subseteq fE$.*

(II) *The pairs (T,f) and (S,f) are weakly compatible.*

If for all $x\in X$

$$d(fTx,Tfx)\le d(fx,Tx)\quad\textit{and}\quad d(fSx,Sfx)\le d(fx,Sx),$$

then f, T and S have a unique common fixed point.

2. Main results. In this paper, we establish common fixed point theorems for mappings satisfying $(\psi,\varphi)_{f,g}$-weakly contractive condition in a complete metric space. Our result is an extension of Theorem 1.1 and Theorem 1.2. In fact, our generalization is different from other generalization in [1, 7, 12].

From now as in [1], we assume:

$$\Phi=\Big\{\varphi\big|\varphi\colon[0,\infty)\to[0,\infty)\ \text{ is a lower semicontinuous function,}$$
$$\varphi(t)>0\ \text{ for all } t>0\ \text{ and }\ \varphi(0)=0\Big\},$$

and

$$\Psi=\Big\{\psi\big|\psi\colon[0,\infty)\to[0,\infty)\ \text{ is a continuous and nondecreasing function}$$
$$\text{and }\ \psi(t)=0\Longleftrightarrow t=0\Big\}.$$

We introduce the following definitions.

Definition 2.1. *Two mappings $T, S\colon X \to X$ are called generalized $(\psi,\varphi)_{f,g}$-weak contractive, if there exists maps $\varphi,\psi\colon [0,\infty) \to [0,\infty)$ and $f, g\colon X \to X$ such that*

$$\psi(d(Tx,Sy)) \le \psi(M(x,y)) - \varphi(M(x,y)), \tag{2.1}$$

for all $x, y \in X$, where

$$M(x,y) = \max\left\{d(fx,gy), d(fx,Tx), d(gy,Sy), \frac{1}{2}\big[d(gy,Tx) + d(fx,Sy)\big]\right\},$$

$\psi \in \Psi$ and $\varphi \in \Phi$.

Abbas et al. extended Zhang and Song's theorem by the above concept [1]. We call this class of mappings, generalized $(\psi,\varphi)_{f,g}$-weak contractive mappings.

Definition 2.2. *Let T and S be two self mappings of a metric space (X, d). T and S are said to be weakly compatible, if for all $x \in X$ the equality $Tx = Sx$ implies $TSx = STx$.*

With respect to the above definition, we prove a common fixed point theorem as follows:

Theorem 2.1. *Let (X, d) be a complete metric space and E be a nonempty closed subset of X. Suppose f and g are continuous functions of X. Let $T, S\colon E \to E$ be two generalized $(\psi,\varphi)_{f,g}$-weak contractive maps, such that*

(A) *$TE \subseteq gE$ and $SE \subseteq fE$,*

(B) *T and f as well as S and g are weakly compatible.*

In addition, for all $x \in X$

$$d(fTx, Tfx) \le d(fx, Tx) \quad \text{and} \quad d(gSx, Sgx) \le d(gx, Sx), \tag{2.2}$$

and for all $x, y \in X$

$$d(fgx, gfy) \le d(gx, fy). \tag{2.3}$$

Then T, f, S and g have a unique common fixed point.

Proof. Let $x_0 \in E$ be arbitrary. From (A), we can find two sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_1 = Tx_0 = gx_1$, $y_2 = Sx_1 = fx_2$, $y_3 = Tx_2 = gx_3, \ldots$ $\ldots, y_{2n+1} = Tx_{2n} = gx_{2n+1}$, $y_{2n+2} = Sx_{2n+1} = fx_{2n+2}, \ldots$, respectively.

The rest of the proof is done in three steps as follows:

Step I. For all $n = 0, 1, \ldots$

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$$

Define $d_n = d(y_n, y_{n+1})$. Suppose $d_{n_0} = 0$ for some n_0. Then $y_{n_0} = y_{n_0+1}$. Consequently, the sequence y_n is constant for $n \ge n_0$. Indeed, let $n_0 = 2k$. Then $y_{2k} = y_{2k+1}$ and we obtain from (2.1)

$$\psi(d(y_{2k+1}, y_{2k+2})) = \psi(d(Tx_{2k}, Sx_{2k+1})) \le$$

$$\le \psi(M(x_{2k}, x_{2k+1})) - \varphi(M(x_{2k}, x_{2k+1})), \tag{2.4}$$

where

$$M(x_{2k}, x_{2k+1}) = \max\Big\{d(y_{2k}, y_{2k+1}), d(y_{2k}, y_{2k+1}), d(y_{2k+2}, y_{2k+1}),$$

$$\frac{1}{2}\big[d(y_{2k},y_{2k+2})+d(y_{2k+1},y_{2k+1})\big]\Big\}=$$

$$=\max\Big\{0,0,d(y_{2k+1},y_{2k+2}),\frac{1}{2}[d(y_{2k},y_{2k+2})]\Big\}=d(y_{2k+1},y_{2k+2}).$$

Now from (2.1)

$$\psi(d(y_{2k+1},y_{2k+2}))=\psi(d(Sx_{2k+1},Tx_{2k}))\le$$

$$\le\psi(d(y_{2k+1},y_{2k+2}))-\varphi(d(y_{2k+1},y_{2k+2})),$$

and so $\varphi(d(y_{2k+1},y_{2k+2}))=0$, that is, $y_{2k+1}=y_{2k+2}$.

Similarly, if $n_0=2k+1$, one can easily obtain $y_{2k+2}=y_{2k+3}$ and so the sequence y_n is constant (for $n\ge n_0$) and y_{n_0} is a common fixed point of T, S, f and g. If we set $z=y_{n_0}$, then z is a unique common fixed point for T, S, f and g.

Suppose $d_n=d(y_n,y_{n+1})>0$ for all n. We prove for each $n=1,2,3,\ldots$

$$d(y_{n+1},y_{n+2})\le M(x_{n+1},x_{n+2})=d(y_n,y_{n+1}). \tag{2.5}$$

Let $n=2k$. Using condition (2.1), we obtain

$$\psi(d(y_{2k+1},y_{2k+2}))=\psi(d(Tx_{2k},Sx_{2k+1}))\le$$

$$\le\psi(M(x_{2k},x_{2k+1}))-\varphi(M(x_{2k},x_{2k+1}))\le$$

$$\le\psi(M(x_{2k},x_{2k+1}))$$

and since the function ψ is nondecreasing, it follows

$$d(y_{2k+1},y_{2k+2})\le M(x_{2k},x_{2k+1}). \tag{2.6}$$

Here,

$$M(x_{2k},x_{2k+1})=\max\Big\{d(fx_{2k},gx_{2k+1}),d(fx_{2k},Tx_{2k}),d(gx_{2k+1},Sx_{2k+1}),$$

$$\frac{1}{2}\big[d(gx_{2k+1},Tx_{2k})+d(fx_{2k},Sx_{2k+1})\big]\Big\}=$$

$$=\max\Big\{d(y_{2k},y_{2k+1}),d(y_{2k},y_{2k+1}),d(y_{2k+1},y_{2k+2}),$$

$$\frac{1}{2}\big[d(y_{2k+1},y_{2k+1})+d(y_{2k},y_{2k+2})\big]\Big\}\le$$

$$\le\max\Big\{d(y_{2k},y_{2k+1}),d(y_{2k+2},y_{2k+1}),$$

$$\frac{1}{2}\big[d(y_{2k},y_{2k+1})+d(y_{2k+1},y_{2k+2})\big]\Big\}=$$

$$=\max\Big\{d(y_{2k},y_{2k+1}),d(y_{2k+1},y_{2k+2})\Big\}.$$

If $d(y_{2k+1}, y_{2k+2}) \geq d(y_{2k}, y_{2k+1}) > 0$, then

$$M(x_{2k+2}, x_{2k+1}) = d(y_{2k+2}, y_{2k+1}),$$

and this implies

$$\psi(d(y_{2k+2}, y_{2k+1})) \leq \psi(d(y_{2k+2}, y_{2k+1})) - \varphi(d(y_{2k+2}, y_{2k+1}))$$

which is only possible when $d(y_{2k+2}, y_{2k+1}) = 0$. This is a contradiction.

Hence, $d(y_{2k+1}, y_{2k+2}) \leq d(y_{2k+1}, y_{2k})$ and

$$M(x_{2k+2}, x_{2k+1}) \leq d(y_{2k+1}, y_{2k}).$$

Since, by definition of $M(x, y)$,

$$M(x_{2k+2}, x_{2k+1}) \geq d(y_{2k+1}, y_{2k}),$$

(2.5) is proved for $d(y_{2k+1}, y_{2k+2})$. Similarly, one can obtain

$$d(y_{2k+2}, y_{2k+3}) \leq M(x_{2k+1}, x_{2k+2}) = d(y_{2k+1}, y_{2k+2}).$$

So, (2.5) holds for all n.

Thus (2.5) shows that the sequence $d(y_n, y_{n+1})$ is a nonincreasing sequence of real numbers and so there exists $\lim_{n\to\infty} d(y_n, y_{n+1}) = \lim_{n\to\infty} M(x_n, x_{n+1}) = r \geq 0$. Suppose $r > 0$. Then from

$$\psi(d(y_{n+1}, y_{n+2})) \leq \psi(M(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})),$$

if $n \to \infty$, it follows that

$$\psi(r) \leq \psi(r) - \liminf_{n\to\infty} \varphi(M(x_n, x_{n+1})) \leq \psi(r) - \varphi(r),$$

i.e., $\varphi(r) \leq 0$. But, $\varphi \in \Phi$, so $r = 0$, which is a contradiction. We conclude that

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = \lim_{n\to\infty} M(x_n, x_{n+1}) = 0. \tag{2.7}$$

Step II. $\{y_n\}$ is a Cauchy sequence.

It is sufficient to show the subsequence $\{y_{2n}\}$ is a Cauchy sequence. If not, there exists $\varepsilon > 0$ for which one can find subsequences $\{y_{2m(k)}\}$ and $\{y_{2n(k)}\}$ of $\{y_{2n}\}$ such that

$$n(k) > m(k) > k \quad \text{and} \quad d(y_{2m(k)}, y_{2n(k)}) \geq \varepsilon$$

and $n(k)$ is the least index with this property, that is,

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon. \tag{2.8}$$

From (2.8) and triangle inequality

$$\varepsilon \leq d(y_{2m(k)}, y_{2n(k)}) \leq$$

$$\leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \leq$$

$$\leq \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}).$$

If $k \to \infty$ and using (2.7) we have

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon. \tag{2.9}$$

In addition, from the known relation $|d(x,z)-d(x,y)|\le d(y,z)$, we obtain

$$\big|d(y_{2m(k)},y_{2n(k)+1})-d(y_{2m(k)},y_{2n(k)})\big|\le d(y_{2n(k)},y_{2n(k)+1}), \tag{2.10}$$

$$\big|d(y_{2m(k)},y_{2n(k)+2})-d(y_{2m(k)},y_{2n(k)+1})\big|\le d(y_{2n(k)+2},y_{2n(k)+1}), \tag{2.11}$$

$$\big|d(y_{2n(k)+1},y_{2m(k)+1})-d(y_{2n(k)+1},y_{2m(k)})\big|\le d(y_{2m(k)},y_{2m(k)+1}), \tag{2.12}$$

$$\big|d(y_{2n(k)+2},y_{2m(k)+1})-d(y_{2n(k)+1},y_{2m(k)+1})\big|\le d(y_{2n(k)+1},y_{2n(k)+2}), \tag{2.13}$$

and using (2.7), (2.9), (2.10), (2.11), (2.12) and (2.13) we get

$$\lim_{k\to\infty} d(y_{2m(k)},y_{2n(k)+1})=\lim_{k\to\infty} d(y_{2m(k)},y_{2n(k)+2})=$$

$$=\lim_{k\to\infty} d(y_{2n(k)+1},y_{2m(k)+1})=\lim_{k\to\infty} d(y_{2n(k)+2},y_{2m(k)+1})=\varepsilon. \tag{2.14}$$

From the definition of $M(x,y)$ and the above limits,

$$\lim_{k\to\infty} M(x_{2m(k)},x_{2n(k+1)})=\varepsilon.$$

Because,

$$M(x_{2m(k)},x_{2n(k)+1})=\max\Big\{d(fx_{2m(k)},gx_{2n(k)+1}),d(fx_{2m(k)},Tx_{2m(k)}),$$

$$d(gx_{2n(k)+1},Sx_{2n(k)+1}),$$

$$\frac{1}{2}\big[d(gx_{2n(k)+1},Tx_{2m(k)})+d(fx_{2m(k)},Sx_{2n(k)+1})\big]\Big\}=$$

$$=\max\Big\{d(y_{2m(k)},y_{2n(k)+1}),d(y_{2m(k)},y_{2m(k)+1}),$$

$$d(y_{2n(k)+1},y_{2n(k)+2}),$$

$$\frac{1}{2}\big[d(y_{2n(k)+1},y_{2m(k)+1})+d(y_{2m(k)},y_{2n(k)+2})\big]\Big\},$$

and if $k\to\infty$, we have

$$M(x_{2m(k)},x_{2n(k)+1})\to\max\Big\{\varepsilon,0,0,\frac{1}{2}[\varepsilon+\varepsilon]\Big\}=\varepsilon.$$

Now, we apply condition (2.1), to obtain

$$\psi(d(y_{2m(k)+1},y_{2n(k)+2}))\le\psi(M(x_{2m(k)},x_{2n(k)+1}))-\varphi(M(x_{2m(k)},x_{2n(k)+1})).$$

Again, if $k\to\infty$, we obtain $\psi(\varepsilon)\le\psi(\varepsilon)-\varphi(\varepsilon)$ which is a contradiction with $\varepsilon>0$. Thus, $\{y_{2n}\}$ is a Cauchy sequence and hence $\{y_n\}$ is a Cauchy sequence.

Step III. There exists t such that $gt = ft = St = Tt = t$.

Since (X, d) is complete and $\{y_n\}$ is Cauchy, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$. Since E is closed and $\{y_n\} \subseteq E$, we have $z \in E$. We know that

$$z = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} Sx_{2n-1} =$$

$$= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Tx_{2n}.$$

Since f and g are continuous, we have $fy_n \to fz$ and $gy_n \to gz$.

On the other hand, from (2.2) and (2.3)

$$d(Ty_{2n}, gz) \le d(Ty_{2n}, fy_{2n+1}) + d(fy_{2n+1}, gy_{2n}) + d(gy_{2n}, gz) =$$

$$= d(Tfx_{2n}, fTx_{2n}) + d(fgx_{2n+1}, gfx_{2n}) + d(gy_{2n}, gz) \le$$

$$\le d(Tx_{2n}, fx_{2n}) + d(gx_{2n+1}, fx_{2n}) + d(gy_{2n}, gz) =$$

$$= d(y_{2n+1}, y_{2n}) + d(y_{2n+1}, y_{2n}) + d(gy_{2n}, gz).$$

Therefore, from (2.7) and continuity of g,

$$\lim_{n\to\infty} d(Ty_{2n}, gz) = 0.$$

Also, from (2.3) we have

$$d(Ty_{2n}, fz) \le d(Ty_{2n}, fy_{2n+1}) + d(fy_{2n+1}, fz) =$$

$$= d(Tfx_{2n}, fTx_{2n}) + d(fy_{2n+1}, fz) \le$$

$$\le d(Tx_{2n}, fx_{2n}) + d(fy_{2n+1}, fz) =$$

$$= d(y_{2n+1}, y_{2n}) + d(fy_{2n+1}, fz).$$

Therefore, from (2.7)

$$\lim_{n\to\infty} d(Ty_{2n}, fz) = 0.$$

From (2.1)

$$\psi(d(Ty_{2n}, Sz)) \le \psi(M(y_{2n}, z)) - \varphi(M(y_{2n}, z)),$$

where

$$M(y_{2n}, z) = \max\Big\{d(fy_{2n}, gz), d(fy_{2n}, Ty_{2n}), d(gz, Sz),$$

$$\frac{1}{2}\big[d(gz, Ty_{2n}) + d(fy_{2n}, Sz)\big]\Big\}.$$

Also,

$$\lim_{n\to\infty} d(Ty_{2n}, gz) = \lim_{n\to\infty} d(Ty_{2n}, fz) = 0.$$

Consequently, $fz = gz$.

If $n \to \infty$, we have

$$\lim_{n\to\infty} M(y_{2n}, z) = \max\Big\{d(fz, gz), d(fz, fz), d(gz, Sz),$$

$$\frac{1}{2}\big[d(gz, fz) + d(fz, Sz)\big]\Big\}.$$

So, we have

$$\lim_{n\to\infty} M(y_{2n}, z) = d(fz, Sz).$$

Therefore,

$$\psi(d(fz, Sz)) \le \psi(d(fz, Sz)) - \varphi(d(fz, Sz))).$$

This implies $\varphi(d(fz, Sz)) = 0$, and hence $Sz = fz$. We can analogously prove $Tz = = gz$. Therefore, $Tz = gz = fz = Sz = t$.

Using weak compatibility of the pairs (T, f) and (S, g), we have $Tt = ft$ and $gt = St$. So,

$$\psi(d(Tt, t)) = \psi(d(Tt, Sz)) \le \psi(M(t, z)) - \varphi(M(t, z)) =$$

$$= \psi\left(\max\left\{d(ft, gz), d(ft, Tt), d(gz, Sz), \frac{1}{2}[d(gz, Tt) + d(ft, Sz)]\right\}\right) -$$

$$-\varphi\left(\max\left\{d(ft, gz), d(ft, Tt), d(gz, Sz), \frac{1}{2}[d(gz, Tt) + d(ft, Sz)]\right\}\right) =$$

$$= \psi\left(\max\left\{d(Tt, t), d(Tt, Tt), d(t, t), \frac{1}{2}[d(t, Tt) + d(Tt, t)]\right\}\right) -$$

$$-\varphi\left(\max\left\{d(Tt, t), d(Tt, Tt), d(t, t), \frac{1}{2}[d(t, Tt) + d(Tt, t)]\right\}\right) =$$

$$= \psi(d(Tt, t)) - \varphi(d(Tt, t)).$$

That is, $\varphi(d(Tt, t)) = 0$ and this implies $Tt = t$. Therefore, $ft = Tt = t$. Analogously, $gt = St = t$. Hence $gt = St = t = ft = Tt$.

Theorem 2.1 is proved.

Note that the proof of Steps I and II is approximately analogous to what which has been done in the other papers such as [1, 7, 12, 13], specially.

***Example* 2.1.** Let $X = R$ be endowed with the Euclidean metric and $E = [0, 1]$. Suppose $T, S\colon E \to E$ is defined by $Tx = \dfrac{1}{2} = Sx$, for all $x \in E$. We define functions $f, g\colon E \to X$ by

$$f(x) = \begin{cases} x, & 0 \le x \le \dfrac{1}{2}, \\ \dfrac{1}{2}, & \dfrac{1}{2} \le x \le 1, \end{cases}$$

$$g(x) = \begin{cases} \dfrac{1}{2}, & 0 \le x \le \dfrac{1}{2}, \\ x, & \dfrac{1}{2} \le x \le 1, \end{cases}$$

and function $\psi, \varphi\colon [0,\infty) \to [0,\infty)$ by $\varphi(t) = t^3$ and $\psi(t) = t^2$.

Thus for all $x \in X$

$$d(fTx, Tfx) \le d(fx, Tx), \quad d(gSx, Sgx) \le d(gx, Sx),$$

and

$$d(fgx, gfy) \le d(gx, fy) \quad \text{for all} \quad x, y \in X.$$

Since $0 \le M(x,y) \le 1$ and $d(Tx, Sy) = 0$, we have

$$\psi(d(Tx, Sy)) \le \psi(M(x,y)) - \varphi(M(x,y)).$$

So mappings T and S satisfy relation (2.1). This example cannot be studied by the Theorem 1.3 (Theorem 2.1 of [12]). But, all conditions of Theorem 2.1 are hold, and T, S, f and g have a unique common fixed point $\left(x = \dfrac{1}{2}\right)$.

3. Applications. In this section, we obtain some common fixed point theorems for mappings satisfying a contraction condition of integral type in a complete metric space.

In [6], Branciari obtained a fixed point result for a single mapping satisfying an integral type inequality. Then Altun et al. [3] established a fixed point theorem for weakly compatible maps satisfying a general contractive inequality of integral type.

As in [13], we denote by Υ the set of all functions $\phi\colon [0,+\infty) \to [0,+\infty)$ verifying the following conditions:

(I) ϕ is a positive Lebesgue integrable mapping on each compact subset of $[0,+\infty)$.

(II) For all $\varepsilon > 0$, $\displaystyle\int_0^\varepsilon \phi(t)dt > 0$.

***Corollary* 3.1.** *Replace the generalized* $(\psi,\varphi)_{f,g}$*-weak contractive condition of Theorem* 2.1 *by the following condition*:

There exists a $\phi \in \Upsilon$ *such that*

$$\int_0^{\psi(d(Tx,Sy))} \phi(t)dt \le \int_0^{\psi(M(x,y))} \phi(t)dt - \int_0^{\varphi(M(x,y))} \phi(t)dt. \tag{3.1}$$

If other conditions of Theorem 2.1 *satisfy, then* T, S, f *and* g *have a unique common fixed point.*

Proof. Consider the function $\Gamma(x) = \displaystyle\int_0^x \phi(t)dt$. Then (3.1) becomes

$$\Gamma(\psi(d(Tx,Sy))) \le \Gamma(\psi(M(x,y))) - \Gamma(\varphi(M(x,y)),$$

and taking $\psi_1 = \Gamma o \psi$ and $\varphi_1 = \Gamma o \varphi$ and applying Theorem 2.1, we obtain the proof (it is easy to verify that $\psi_1 \in \Psi$ and $\varphi_1 \in \Phi$).

***Corollary* 3.2.** *Replace the generalized* $(\psi,\varphi)_{f,g}$*-weak contractive condition of Theorem* 2.1 *by the following condition*:

There exists a $\phi \in \Upsilon$ *such that*

$$\psi\left(\int\limits_0^{d(Tx,Sy)}\phi(t)dt\right)\le\psi\left(\int\limits_0^{M(x,y)}\phi(t)dt\right)-\varphi\left(\int\limits_0^{M(x,y)}\phi(t)dt\right). \tag{3.2}$$

If other conditions of Theorem 2.1 *satisfy, then T, S, f and g have a unique common fixed point.*

Proof. Again, as in Corollary 3.1, define the function $\Gamma(x)=\int_0^x\phi(t)dt$. Then (3.2) changes to

$$\psi(\Gamma(d(Tx,Sy)))\le\psi(\Gamma(M(x,y)))-\varphi(\Gamma(M(x,y))).$$

Now, if we define $\psi_1=\psi o\Gamma$ and $\varphi_1=\varphi o\Gamma$ and applying Theorem 2.1, then the proof is complete (it is easy to verify $\psi_1\in\Psi$ and $\varphi_1\in\Phi$).

Now, we recall the definition of altering distance function as follows [10]:

Definition 3.1. *The function $\varphi\colon[0,+\infty)\to[0,+\infty)$ is called an altering distance function if the following properties are satisfied*:

(a) *φ is continuous and nondecreasing,*

(b) $\varphi(t)=0\Longleftrightarrow t=0$.

Remark **3.1.** In Theorem 2.1, assume ψ and φ are altering distance functions, then theorem is hold.

Corollary **3.3.** *Replace the generalized $(\psi,\varphi)_{f,g}$-weak contractive condition of Theorem* 2.1 *by the following condition*:

There exists a $\phi\in\Upsilon$ such that

$$\psi_1\left(\int\limits_0^{\psi_2(d(Tx,Sy))}\phi(t)dt\right)\le\psi_1\left(\int\limits_0^{\psi_2(M(x,y))}\phi(t)dt\right)-\varphi_1\left(\int\limits_0^{\varphi_2(M(x,y))}\phi(t)dt\right), \tag{3.3}$$

for altering distance functions ψ_1, ψ_2, φ_1 and φ_2. If other conditions of Theorem 2.1 *satisfy, then T, S, f and g have a unique common fixed point.*

Proof. Consider the function $\Gamma(x)=\int_0^x\phi(t)dt$. Then (3.3) will be

$$\psi_1(\Gamma(\psi_2(d(Tx,Sy))))\le\psi_1(\Gamma(\psi_2(M(x,y))))-\varphi_1(\Gamma(\varphi_2(M(x,y)))),$$

and taking $\hat{\Psi}=\psi_1 o\Gamma o\psi_2$ and $\hat{\Phi}=\varphi_1 o\Gamma o\varphi_2$ and applying Theorem 2.1, we obtain the proof (it is easy to verify that $\hat{\Psi}$ and $\hat{\Phi}$ are altering distance functions).

As in [13], let $N\in N^*$ be fixed. Let $\{\phi_i\}_{1\le i\le N}$ be a family of N functions which belong to Υ. For all $t\ge 0$, we define

$$I_1(t)=\int\limits_0^t\phi_1(s)ds,$$

$$I_2(t)=\int\limits_0^{I_1 t}\phi_2(s)ds=\int_0^{\int_0^t\phi_1(s)ds}\phi_2(s)ds,$$

$$I_3(t)=\int\limits_0^{I_2 t}\phi_3(s)ds=\int_0^{\int_0^{\int_0^t\phi_1(s)ds}\phi_2(s)ds}\phi_3(s)ds,$$

. . . .

$$I_N(t) = \int_0^{I_{(N-1)}t} \phi_N(s)ds.$$

We have the following result.

***Corollary* 3.4.** *Replace the generalized $(\psi, \varphi)_{f,g}$-weak contractive condition of Theorem* 2.1 *by the following condition*:

$$\psi\left(\int_0^{I_{(N-1)}(d(Tx,Sy))} \phi_N(s)ds\right) \leq$$

$$\leq \psi\left(\int_0^{I_{(N-1)}(M(x,y))} \phi_N(s)ds\right) - \varphi\left(\int_0^{I_{(N-1)}(M(x,y))} \phi_N(s)ds\right), \qquad (3.4)$$

where $\psi \in \Psi$ and $\varphi \in \Phi$. If other conditions of Theorem 2.1 *satisfy, then T, S, f and g have a unique common fixed point.*

Proof. Consider $\hat{\Psi} = \psi o I_N$ and $\hat{\Phi} = \varphi o I_N$. Then the above inequality becomes

$$\hat{\Psi}(d(Tx,Sy)) \leq \hat{\Psi}(M(x,y)) - \hat{\Phi}(M(x,y))).$$

Applying Theorem 2.1, we obtain the desired result (it is easy to verify that $\hat{\Psi} \in \Psi$ and $\hat{\Phi} \in \Phi$).

1. *Abbas M., Doric D.* Common fixed point theorem for four mappings satisfying generalized weak contractive condition // Filomat. – 2010. – **24**, № 2. – P. 1 – 10.
2. *Alber Ya. I., Guerre-Delabriere S.* Principle of weakly contractive maps in Hilbert spaces // New Results in Operator Theory: Adv. and Appl. – 1997. – **98**. – P. 7 – 22.
3. *Altun I., Turkoglu D., Rhoades B. E.* Fixed points of weakly compatible maps satisfying a general contractive condition of integral type // Fixed Point Theory and Appl. – 2007. – Article ID 17301.
4. *Azam A., Shakeel M.* Weakly contractive maps and common fixed points // Mat. Vesnik. – 2008. – **60**. – P. 101 – 106.
5. *Beg I., Abbas M.* Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition // Fixed Point Theory and Appl. – 2006. – Art. ID 74503. – P. 1 – 7.
6. *Branciari A.* A fixed point theorem for mapping satisfying a general contractive condition of general type // Int. J. Math. and Math. Sci. – 2002. – **29**, №. 9. – P. 531 – 536.
7. *Doric D.* Common fixed point for generalized (ψ, φ)-weak contractions // Appl. Math. Lett. – 2009. – **22**. – P. 1896 – 1900.
8. *Dutta P. N., Choudhury B. S.* A generalization of contraction principle in metric spaces // Fixed Point Theory and Appl. – 2008. – Article ID 406368.
9. *Jungck G., Rhoades B. E.* Fixed point for set valued functions without continuity // Indian J. Pure and Appl. Math. – 1998. – **29**, №. 3. – P. 227 – 238.
10. *Khan M. S., Swalel M., Sessa S.* Fixed point theorems by altering distances between the points // Bull. Aust. Math. Soc. – 1984. – **30**. – P. 1 – 9.
11. *Luong N. V., Thuan N. X.* A fixed point theorem for $\psi_{\int \varphi}$-weakly contractive mapping in metric spaces // Int. J. Math. Anal. – 2010. – **4**, №. 5. – P. 233 – 242.
12. *Moradi S., Fathi Z., Analouee E.* Common fixed point of single valued generalized φ_f-weak contractive mappings // Appl. Math. Lett. – 2011. – **24**, №. 5. – P. 771 – 776.
13. *Radenovic S., Kadelburg Z* Generalized weak contractions in partially ordered metric spaces // Comput. and Math. Appl. – 2010. – **60**. – P. 1776 – 1783.
14. *Rhoades B. E.* Some theorems on weakly contractive maps // Nonlinear Anal.. – 2001. – **47**. – P. 2683 – 2693.
15. *Zhang Q., Song Y.* Fixed point theory for generalized φ-weak contractions // Appl. Math. Lett. – 2009. – **22**. – P. 75 – 78.

Received 24.05.11