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ON ss-QUASINORMAL AND WEAKLY s-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS *

ПРО ss-КВАЗІНОРМАЛЬНІ ТА СЛАБКО s-ДОПОВНЮВАНІ ПІДГРУПИ СКІНЧЕННИХ ГРУП

Suppose that G is a finite group and H is a subgroup of G. H is called ss-quasinormal in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B; H is called weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-quasinormal in G. In this paper we investigate the influence of ss-quasinormal and weakly s-supplemented subgroups on the structure of finite groups. Some recent results are generalized and unified.

Нехай G — скінченна група, а H — підгрупа G. Підгрупа H називається ss-квазінормальною в G, якщо існує така підгрупа B групи G, що $G = HB$ і H є переставною з кожною силовською підгрупою підгрупи B; H називається слабко s-доповнюваною в G, якщо існує така підгрупа T групи G, що $G = HT$ і $H \cap T \leq H_{sG}$, де H_{sG} — підгрупа H, що породжена усіма підгрупами H, які є s-квазінормальними в G. У даній роботі досліджено вплив ss-квазінормальних та слабко s-доповнюваних підгруп на структуру скінченних груп. Узагальнено та уніфіковано деякі нещодавні результати.

1. Introduction. All groups considered in this paper will be finite. We use conventional notions and notation, as in Huppert [1]. G denotes a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$ and G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$. Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (ii) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a saturated formation [1, p. 713] (Satz 8.6).

Two subgroups H and K of G are said to be permutable if $HK = KH$. A subgroup H of G is said to be s-quasinormal (or s-permutable, π-quasinormal) in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel in [2]. In 2008, Shirong Li [3] introduced the concept of ss-quasinormal subgroup: A subgroup H of a group G is said to be an ss-quasinormal subgroup of G if there is a subgroup B such that $G = HB$ and H permutes with every Sylow subgroup of B. Groups with certain ss-quasinormal subgroups of prime power order were studied in [3]. Obviously, s-quasinormal subgroup is ss-quasinormal subgroup. As another generalization of s-quasinormal subgroups, A. N. Skiba [4] introduced the following concept: A subgroup H of a group G is called weakly s-supplemented in G if there is a subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-quasinormal in G. In fact, this concept is also a generalization of c-normal and c-supplemented subgroups given in [5] and [6]. A. N. Skiba proposed in [4] two open questions related to weakly s-supplemented subgroups. In this paper, we prove some theorems which show that in most cases (for maximal and minimal subgroups) the question 6.4 in [4] has positive solution.

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There are examples to show that weakly s-supplemented subgroups are not ss-quasinormal subgroups and in general the converse is also false. The aim of this article is to unify and improve some earlier results using ss-quasinormal and weakly s-supplemented subgroups. The main result of the paper is Theorem 3.4.

2. Preliminaries.

Lemma 2.1 ([3], Lemma 2.1)**.** *Let H be an ss-quasinormal subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is ss-quasinormal in L.*

(2) *If $N \trianglelefteq G$, then HN/N is ss-quasinormal in G/N.*

Lemma 2.2 ([3], Lemma 2.2)**.** *Let H be a nilpotent subgroup of G. Then the following statements are equivalent*:

(1) *H is s-quasinormal in G.*

(2) *$H \leq F(G)$ and H is ss-quasinormal in G.*

Lemma 2.3 ([3], Lemma 2.5)**.** *If a p-subgroup P of G is ss-quasinormal* (p *a prime*), *then P permutes with every Sylow q-subgroup of G with $q \neq p$.*

Lemma 2.4. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|,\ p-1) = 1$. If every maximal subgroup of P is ss-quasinormal in G, then G is p-nilpotent.*

Proof. This is a corollary of the proof of [3] (Theorem 1.1) by [25] (Lemma 2.6).

Lemma 2.5 ([4], Lemma 2.10)**.** *Let H be a weakly s-supplemented subgroup of a group G.*

(1) *If $H \leq L \leq G$, then H is weakly s-supplemented in L.*

(2) *If $N \trianglelefteq G$ and $N \leq H \leq G$, then H/N is weakly s-supplemented in G/N.*

(3) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-supplemented in G/N.*

Lemma 2.6 ([7], A, 1.2)**.** *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent*:

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.7 ([8], Lemma 2.2.)**.** *If P is an s-quasinormal p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 2.8 ([11], Lemma 2.6)**.** *Let H be a solvable normal subgroup of a group G $(H \neq 1)$. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.9 ([4], Lemma 2.16)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. Suppose that G is a group with a normal subgroup N such that $G/N \in \mathcal{F}$. If N is cyclic, then $G \in \mathcal{F}$.*

Lemma 2.10 ([20], Lemma 2.3)**.** *Let G be a group and $N \leq G$.*

(1) *If $N \trianglelefteq G$, then $F^*(N) \leq F^*(G)$.*

(2) *If $G \neq 1$, then $F^*(G) \neq 1$. In fact, $F^*(G)/F(G) = \mathrm{Soc}(F(G)C_G(F(G))/F(G))$.*

(3) *$F^*(F^*(G)) = F^*(G) \geq F(G)$. If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

3. Main results.

Theorem 3.1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|,\ p-1) = 1$. If every maximal subgroup of P is either ss-quasinormal or weakly s-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G has a unique minimal normal subgroup N and G/N is p-nilpotent. Moreover $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Consider G/N. We will show that G/N satisfies the hypothesis of the theorem. Let M/N be a maximal subgroup of PN/N. It is easy to see that $M = P_1N$ for some maximal subgroup P_1 of P. It follows that $P_1 \cap N = P \cap N$ is a Sylow p-subgroup of N. If P_1 is ss-quasinormal in G, then M/N is ss-quasinormal in G/N by Lemma 2.1. If P_1 is weakly s-supplemented in G, then there is a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. So $G/N = M/N \cdot TN/N = P_1N/N \cdot TN/N$. Since

$$(|N\colon P_1 \cap N|, |N\colon T \cap N|) = 1,$$

we have

$$(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap P_1T.$$

By Lemma 2.6, $(P_1N) \cap (TN) = (P_1 \cap T)N$. It follows that

$$(P_1N/N) \cap (TN/N) = (P_1N \cap TN)/N =$$

$$= (P_1 \cap T)N/N \leq (P_1)_{sG}N/N \leq (P_1N/N)_{sG}.$$

Hence M/N is weakly s-supplemented in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Consequently the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by step (1). Since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

is p-nilpotent, G is p-nilpotent, a contradiction.

(3) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (1) yields $N \leq O_p(G)$ and $\Phi(O_p(G)) \leq \Phi(G) = 1$. Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore $P \cap M < P$, thus there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence $P = NP_1$. By the hypothesis, P_1 is either ss-quasinormal or weakly s-quasinormal in G. If we assume that P_1 is ss-quasinormal in G, then P_1M_q is a group for $q \neq p$ by Lemma 2.3. Hence

$$P_1\big\langle M_p, M_q | q \in \pi(M), q \neq p\big\rangle = P_1M$$

is a group. Then $P_1M = M$ or G by maximality of M. If $P_1M = G$, then $P = P \cap P_1M = P_1(P \cap M) = P_1$, a contradiction. If $P_1M = M$, then $P_1 \leq M$. Therefore, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Therefore we may assume that P_1 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. From Lemma 2.7 we have $O^p(G) \leq N_G((P_1)_{sG})$. Since $(P_1)_{sG}$ is subnormal in G, we obtain

$$P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = N.$$

Thus $(P_1)_{sG} \leq P_1 \cap N$ and

$$(P_1)_{sG} \leq ((P_1)_{sG})^G = ((P_1)_{sG})^{O^p(G)P} = ((P_1)_{sG})^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N.$$

It follows that $((P_1)_{sG})^G = 1$ or $((P_1)_{sG})^G = P_1 \cap N = N$. If $((P_1)_{sG})^G = P_1 \cap N = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. If $((P_1)_{sG})^G = 1$, then $P_1 \cap T = 1$ and so $|T|_p = p$. Hence T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we may suppose that M has a normal Hall p'-subgroup $M_{p'}$ and $M \leq N_G(M_{p'}) \leq G$. The maximality of M implies that $M = N_G(M_{p'})$ or $N_G(M_{p'}) = G$. If the latter holds, then $M_{p'} \trianglelefteq G$, and $M_{p'}$ is actually the normal p-complement of G, which is contrary to the choice of G. Hence we may assume that $M = N_G(M_{p'})$. By applying a deep result of Gross ([9], main theorem) and Feit–Thompson's theorem, there exists $g \in G$ such that $T_{p'}^g = M_{p'}$. Hence $T^g \leq N_G(T_{p'}^g) = N_G(M_{p'}) = M$. However, $T_{p'}$ is normalized by T, so g can be considered as an element of P_1. Thus $G = P_1 T^g = P_1 M$ and $P = P_1(P \cap M) = P_1$, a contradiction.

(4) G has Hall p'-subgroups and any two subgroups of G are conjugate in G.

If every maximal subgroup of P is ss-quasinormal in G, then G is p-nilpotent by Lemma 2.4, a contradiction. Thus there is a maximal subgroup P_0 of P such that P_0 is weakly s-supplemented in G. Then there exists a subgroup T of G such that $G = P_0 T$ and

$$P_0 \cap T \leq (P_0)_{sG} \leq O_p(G) = 1.$$

By [10] (Theorem 2.2), G is not simple and G has Hall p'-subgroups. A new application of the result of Gross ([9], main theorem) and Feit–Thompson's theorem yields that any two Hall p'-subgroups of G are conjugate in G.

(5) The final contradiction.

If we suppose that $NP < G$, then NP satisfies the hypothesis of the theorem. The choise of G yields that N is p-nilpotent, a contradiction with steps (2) and (3). Therefore we may assume that $G = NP$. By step (4), G has Hall p'-subgroups. Then we may suppose that N has a Hall p'-subgroup $N_{p'}$. By Frattini's argument,

$$G = NN_G(N_{p'}) = (P \cap N)N_{p'}N_G(N_{p'}) = (P \cap N)N_G(N_{p'})$$

and so

$$P = P \cap G = P \cap (P \cap N)N_G(N_{p'}) = (P \cap N)(P \cap N_G(N_{p'})).$$

Since $N_G(N_{p'}) < G$, it follows that $P \cap N_G(N_{p'}) < P$. Consider a maximal subgroup P_1 of P such that $P \cap N_G(N_{p'}) \leq P_1$. Then $P = (P \cap N)P_1$. By the hypothesis, P_1 is either ss-quasinormal or weakly s-supplemented in G. If P_1 is ss-quasinormal in G, then $P_1 N_G(N_{p'}) = P_1 N_{p'}$ forms a group by Lemma 2.3. Since $|G\colon P_1 N_{p'}| = p$ and $(|G|, p-1) = 1$, we have $P_1 N_{p'} \trianglelefteq G$ by [25] (Lemma 2.6). By Frattini's argument again, $G = P_1 N_{p'} N_G(N_{p'}) = P_1 N_G(N_{p'}) < G$, a contradiction. Now assume that P_1 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_1 T$ and

$$P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = 1.$$

Since $|T|_p = p$, we have that T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T, then $T_{p'}$ is a Hall p'-subgroup of G. By step (4), $T_{p'}$ and $N_{p'}$ are conjugate in G. Since $T_{p'}$ is normalized by T, there exists $g \in P_1$ such that $T_{p'}^g = N_{p'}$. Hence

$$G = (P_1T)^g = P_1T^g = P_1N_G(T_{p'}^g) = P_1N_G(N_{p'})$$

and

$$P = P \cap G = P \cap P_1N_G(N_{p'}) = P_1(P \cap N_G(N_{p'})) \leq P_1,$$

a contradiction.

***Corollary* 3.1.** *Let p be a prime dividing the order of a group G, where p is the smallest prime divisor of $|G|$ and H a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every maximal subgroup of P is either ss-quasinormal or weakly s-supplemented in G, then G is p-nilpotent.*

Proof. By Lemmas 2.1 and 2.5, every maximal subgroup of P is either ss-quasinormal or weakly s-supplemented in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \trianglelefteq G$. Assume that $H_{p'} \neq 1$ and consider $G/H_{p'}$. Applying Lemmas 2.1 and 2.5 it is easy to see that $G/H_{p'}$ satisfies the hypotheses for the normal subgroup $H/H_{p'}$. Therefore by induction $G/H_{p'}$ is p-nilpotent and so G is p-nilpotent. Hence we may assume $H_{p'} = 1$ and therefore $H = P$ is a p-group. Since G/H is p-nilpotent, we can consider K/H be the normal p-complement of G/H. By Schur–Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. A new application of Theorem 3.1 yields that K is p-nilpotent and so $K = H \times K_{p'}$. Hence $K_{p'}$ is a normal p-complement of G. This completes the proof.

***Corollary* 3.2** ([13], Theorem 3.4)**.** *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-normal in G, then G is p-nilpotent.*

***Corollary* 3.3** ([14], Theorem 3.4)**.** *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are c-supplemented in G, then G is p-nilpotent.*

***Corollary* 3.4** ([24], Theorem 3.2)**.** *Let p be a prime dividing the order of a group G with $(|G|,\ p-1) = 1$. If there exists a Sylow p-subgroup P of G such that every maximal subgroup of P is weakly s-permutable in G, then G is p-nilpotent.*

***Corollary* 3.5.** *Suppose that every maximal subgroup of any Sylow subgroup of a group G is either ss-quasinormal or weakly s-supplemented in G, then G is a Sylow tower group of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. Then every maximal subgroup of P is either ss-quasinormal or weakly s-supplemented in G. By Theorem 3.1, G is p-nilpotent. Let U be the normal p-complement of G. By Lemmas 2.1 and 2.5, U satisfies the hypothesis of the corollary. Therefore it follows by induction that U, and hence G is a Sylow tower group of supersolvable type.

Theorem 3.2. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is either ss-quasinormal or weakly s-supplemented in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order.

(1) By Lemmas 2.1 and 2.5, every maximal subgroup of any Sylow subgroup of H is either ss-quasinormal or weakly s-supplemented in H. By Corollary 3.4, H is a Sylow tower group of supersolvable type. Let p be the largest prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in G. Let N be a minimal normal subgroup of G contained in P. We consider G/N. It is easy to see that $(G/N, H/N)$ satisfies the hypothesis of the theorem. By the minimality of G, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. By Lemma 2.8, it follows that $P = F(P) = N$.

(2) Since $N \trianglelefteq G$, we may take a maximal subgroup N_1 of N such that $N_1 \trianglelefteq G_p$, where G_p is a Sylow p-subgroup of G. Then N_1 is either ss-quasinormal or weakly s-supplemented in G. If N_1 is weakly group s-supplemented in G, then there is a subgroup T of G such that $G = N_1T$ and $N_1 \cap T \leq (N_1)_{sG}$. Thus $G = NT$ and $N = N \cap N_1T = N_1(N \cap T)$. This implies that $N \cap T \neq 1$. But since $N \cap T$ is normal in G and N is minimal normal in G, $N \cap T = N$. It follows that $T = G$ and so $N_1 = (N_1)_{sG}$ is s-quasinormal in G. By Lemma 2.7, $O^p(G) \leq N_G(N_1)$. Thus $N_1 \trianglelefteq G_pO^p(G) = G$. It follows that $N_1 = 1$ and so $|N| = p$. By Lemma 2.9, $G \in \mathcal{F}$, a contradiction. If N_1 is ss-quasinormal in G, then N_1 is s-quasinormal in G by Lemma 2.2 and it follows the same contradiction.

***Corollary* 3.6** ([14], Theorem 4.2)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. If there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is c-supplemented in G, then $G \in \mathcal{F}$.*

***Corollary* 3.7** ([3], Theorem 1.5)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. If there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is ss-quasinormal in G, then $G \in \mathcal{F}$.*

***Corollary* 3.8** ([11], Theorem 3.3)**.** *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is c-normal in G, then G is supersolvable.*

***Corollary* 3.9** ([6], Theorem 3.3)**.** *Let H be a normal subgroup of a group G such that G/H is supersolvable. If every maximal subgroup of any Sylow subgroup of H is c-supplemented in G, then G is supersolvable.*

***Corollary* 3.10** ([5], Theorem 4.1)**.** *If every maximal subgroup of any Sylow subgroup of a group G is c-normal in G, then G is supersolvable.*

Theorem 3.3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. A group $G \in \mathcal{F}$ if and only if there is a normal subgroup E of G such that $G/E \in \mathcal{F}$ and every cyclic subgroup $\langle x \rangle$ of any Sylow subgroup of E with prime order or order 4* (*if the Sylow 2-subgroups are non-abelian*) *is either ss-quasinormal or weakly s-supplemented in G.*

Proof. We need only to prove the sufficiency part since the necessity part is evident. Suppose that the assertion is false and let G be a counterexample of minimal order. Then

(1) E is solvable.

Let K be any proper subgroup of E. Then $|K| < |G|$ and $K/K \in \mathcal{U}$. Let $\langle x \rangle$ be any cyclic subgroup of any Sylow subgroup of K with prime order or order 4 (if the

Sylow 2-subgroups are non-abelian). It is clear that $\langle x\rangle$ is also a cyclic subgroup of a Sylow subgroup of E with prime order or order 4. By the hypothesis, $\langle x\rangle$ is either ss-quasinormal or weakly s-supplemented in G. By Lemmas 2.1 and 2.5, $\langle x\rangle$ is either ss-quasinormal or weakly s-supplemented in K. This shows that the hypothesis still holds for $(\mathcal{U}, K)$. By the choice of G, K is supersoluble. By [12] (Theorem 3.11.9), E is solvable.

(2) $G^{\mathcal{F}}$ is a p-group, where $G^{\mathcal{F}}$ is the $\mathcal{F}$-residual of G. Moreover $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G and $\exp(G^{\mathcal{F}}) = p$ or $\exp(G^{\mathcal{F}}) = 4$ (if $p = 2$ and $G^{\mathcal{F}}$ is non-abelian).

Since $G/E \in \mathcal{F}$, $G^{\mathcal{F}} \leq E$. Let M be a maximal subgroup of G such that $G^{\mathcal{F}} \nsubseteq M$ (that is, M is an $\mathcal{F}$-abnormal maximal subgroup of G). Then $G = ME$. We claim that the hypothesis holds for $(\mathcal{F}, M)$. In fact,

$$M/M \cap H \cong ME/E = G/E \in \mathcal{F}$$

and by the similar argument as above, we can prove that the hypothesis holds for $(\mathcal{F}, M)$. By the choice of G, $M \in \mathcal{F}$. Thus (2) holds by [12] (Theorem 3.4.2).

(3) $\langle x\rangle$ is s-quasinormal in G for any element $x \in G^{\mathcal{F}}$.

Let $x \in G^{\mathcal{F}}$. Then the order of x is p or 4 by step (2). By the hypothesis, $\langle x\rangle$ is either ss-quasinormal or weakly s-supplemented in G. If $\langle x\rangle$ is ss-quasinormal in G, then $\langle x\rangle$ is s-quasinormal in G by Lemma 2.2 since $\langle x\rangle \leq G^{\mathcal{F}} \leq O_p(G) \leq F(G)$. If $\langle x\rangle$ is weakly s-supplemented in G, then there is a subgroup T of G such that $G = \langle x\rangle T$ and $\langle x\rangle \cap T \leq \langle x\rangle_{sG}$. Hence

$$G^{\mathcal{F}} = G^{\mathcal{F}} \cap G = G^{\mathcal{F}} \cap \langle x\rangle T = \langle x\rangle (G^{\mathcal{F}} \cap T).$$

Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is abelian, we have

$$(G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}})/\Phi(G^{\mathcal{F}}) \trianglelefteq G/\Phi(G^{\mathcal{F}}).$$

Since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G, $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$ or $G^{\mathcal{F}} = (G^{\mathcal{F}} \cap T)\Phi(G^{\mathcal{F}}) = G^{\mathcal{F}} \cap T$. If $G^{\mathcal{F}} \cap T \leq \Phi(G^{\mathcal{F}})$, then $\langle x\rangle = G^{\mathcal{F}} \trianglelefteq G$. In this case, $\langle x\rangle$ is s-quasinormal in G. If $G^{\mathcal{F}} = G^{\mathcal{F}} \cap T$, then $T = G$ and so $\langle x\rangle = \langle x\rangle_{sG}$ is s-quasinormal in G.

(4) $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

Assume that $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| \neq p$ and let $L/\Phi(G^{\mathcal{F}})$ be any cyclic subgroup of $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$. Let $x \in L\backslash\Phi(G^{\mathcal{F}})$. Then $L = \langle x\rangle\Phi(G^{\mathcal{F}})$. Since $\langle x\rangle$ is s-quasinormal in G by step (3), $L/\Phi(G^{\mathcal{F}})$ is s-quasinormal in $G/\Phi(G^{\mathcal{F}})$. It follows from [4] (Lemma 2.11) that $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ has a maximal subgroup which is normal in $G/\Phi(G^{\mathcal{F}})$. But this is impossible since $G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})$ is a chief factor of G. Thus $|G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})| = p$.

(5) The final contradiction.

Since

$$(G/\Phi(G^{\mathcal{F}}))/(G^{\mathcal{F}}/\Phi(G^{\mathcal{F}})) \cong G/G^{\mathcal{F}} \in \mathcal{F},$$

we have that $G/\Phi(G^{\mathcal{F}}) \in \mathcal{F}$ by Lemma 2.9. But $\Phi(G^{\mathcal{F}}) \leq \Phi(G)$ and $\mathcal{F}$ is a saturated formation, therefore $G \in \mathcal{F}$, the final contradiction.

***Corollary* 3.11** ([15], Theorem 4.2)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. If every cyclic subgroup of $G^{\mathcal{F}}$ of prime order or order 4 is c-normal in G, then $G \in \mathcal{F}$.*

***Corollary* 3.12** ([16], Theorem 4.1)**.** *If every cyclic subgroup of $G^{\mathcal{U}}$ of prime order or order 4 is c-supplemented in G, then G is supersolvable.*

***Corollary* 3.13** ([17], Theorem 3.4)**.** *If every cyclic subgroup of* G *of prime order or order* 4 *is* ss*-quasinormal in* G, *then* G *is supersolvable.*

***Corollary* 3.14** ([19], Theorem 3.2)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *A group* $G \in \mathcal{F}$ *if and only if there is a normal subgroup* E *of* G *such that* $G/E \in \mathcal{F}$ *and every subgroup of* E *of prime order or order* 4 *is weakly* s*-supplemented in* G.

Theorem 3.4. *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *Then* $G \in \mathcal{F}$ *if and only if one of the following conditions holds*:

(1) *every maximal subgroup of any Sylow subgroup of* $F^*(H)$ *is either* ss*-quasinormal or weakly* s*-supplemented in* G.

(2) *every cyclic subgroup of any Sylow subgroup of* $F^*(H)$ *of prime order or order* 4 *is either* ss*-quasinormal or weakly* s*-supplemented in* G.

Proof. We only need to prove the "if" part. If the condition (1) holds, then every maximal subgroup of any Sylow subgroup of $F^*(H)$ is either ss-quasinormal or weakly s-supplemented in $F^*(H)$ by Lemmas 2.1 and 2.5. From Theorem 3.2, we have that $F^*(H)$ is supersolvable. In particular, $F^*(H)$ is solvable. By Lemma 2.10, $F^*(H) = F(H)$. Since s-quasinormal subgroup is weakly s-supplemented subgroup, it follows that every maximal subgroup of any Sylow subgroup of $F^*(H)$ is weakly s-supplemented in G by Lemma 2.2. Applying [21] (Theorem A), $G \in \mathcal{F}$. If the condition (2) holds, then we have also $G \in \mathcal{F}$ using similar arguments as above.

***Corollary* 3.15** ([22], Theorem 3.4)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If all maximal subgroups of any Sylow subgroup of* $F^*(H)$ *are* s*-quasinormal in* G, *then* $G \in \mathcal{F}$.

***Corollary* 3.16** ([20], Theorem 3.1)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If all maximal subgroups of any Sylow subgroup of* $F^*(H)$ *are* c*-normal in* G, *then* $G \in \mathcal{F}$.

***Corollary* 3.17** ([23], Theorem 1.1)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If all maximal subgroups of any Sylow subgroup of* $F^*(H)$ *are* c*-supplemented in* G, *then* $G \in \mathcal{F}$.

***Corollary* 3.18** ([17], Theorem 3.3)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *Then* $G \in \mathcal{F}$ *if and only if every maximal subgroup of any Sylow subgroup of* $F^*(H)$ *is* ss*-quasinormal in* G.

***Corollary* 3.19** ([18], Theorem 3.3)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every cyclic subgroup of any Sylow subgroup of* $F^*(H)$ *of prime order or order* 4 *is* s*-quasinormal in* G, *then* $G \in \mathcal{F}$.

***Corollary* 3.20** ([20], Theorem 3.2)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every cyclic subgroup of any Sylow subgroup of* $F^*(H)$ *of prime order or order* 4 *is* c*-normal in* G, *then* $G \in \mathcal{F}$.

***Corollary* 3.21** ([23], Theorem 1.2)**.** *Let* $\mathcal{F}$ *be a saturated formation containing* $\mathcal{U}$. *Suppose that* G *is a group with a normal subgroup* H *such that* $G/H \in \mathcal{F}$. *If every* *cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is c-supplemented in G, then $G \in \mathcal{F}$.*

***Corollary* 3.22** ([17], Theorem 3.7)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is ss-quasinormal in G.*

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