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ARE THE DEGREES OF BEST (CO)CONVEX AND UNCONSTRAINED POLYNOMIAL APPROXIMATION THE SAME? II

ЧИ ОДНАКОВІ ПОРЯДКИ НАЙКРАЩОГО (КО)ОПУКЛОГО НАБЛИЖЕННЯ ТА ПОЛІНОМІАЛЬНОГО НАБЛИЖЕННЯ БЕЗ ОБМЕЖЕНЬ? II

In Part I of this paper, we proved that for every $\alpha > 0$ and a continuous function f, which is either convex $(s = 0)$ or changes convexity at a finite collection $Y_s = \{y_i\}_{i=1}^s$ of points $y_i \in (-1, 1)$, one has

$$\sup\{n^\alpha E_n^{(2)}(f, Y_s):\ n \ge \mathcal{N}^*\} \le c(\alpha, s) \sup\{n^\alpha E_n(f):\ n \ge 1\},$$

where $E_n(f)$ and $E_n^{(2)}(f, Y_s)$ denote, respectively, the degrees of best unconstrained and (co)convex approximation, and $c(\alpha, s)$ is a constant depending only on α and s. Moreover, we showed that $\mathcal{N}^*$ may be chosen to be 1 if $s = 0$ or $s = 1$, $\alpha \ne 4$, and that it has to depend on Y_s and α if $s = 1$, $\alpha = 4$ or $s \ge 2$.

In this Part II, we show that a more general inequality

$$\sup\{n^\alpha E_n^{(2)}(f, Y_s):\ n \ge \mathcal{N}^*\} \le c(\alpha, \mathcal{N}, s) \sup\{n^\alpha E_n(f):\ n \ge \mathcal{N}\},$$

is valid, where, depending on the triple $(\alpha, \mathcal{N}, s)$, $\mathcal{N}^*$ may or may not depend on α, $\mathcal{N}$, Y_s and f.

У частині I цієї статті доведено, що для кожного $\alpha > 0$ та неперервної функції f, яка або опукла $(s = 0)$ або змінює опуклість у скінченному наборі $Y_s = \{y_i\}_{i=1}^s$ точок $y_i \in (-1, 1)$,

$$\sup\{n^\alpha E_n^{(2)}(f, Y_s):\ n \ge \mathcal{N}^*\} \le c(\alpha, s) \sup\{n^\alpha E_n(f):\ n \ge 1\},$$

де $E_n(f)$ та $E_n^{(2)}(f, Y_s)$ означають відповідно порядок найкращого наближення без обмежень та (ко)опуклого наближення, $c(\alpha, s)$ є сталою, що залежить лише від α і s. Більш того, було показано, що $\mathcal{N}^*$ можна вибрати рівним одиниці, якщо $s = 0$ або $s = 1$, $\alpha \ne 4$, і що воно повинно залежати від Y_s і α, якщо $s = 1$, $\alpha = 4$ або $s \ge 2$.

У частині II показано, що виконується більш загальна нерівність

$$\sup\{n^\alpha E_n^{(2)}(f, Y_s):\ n \ge \mathcal{N}^*\} \le c(\alpha, \mathcal{N}, s) \sup\{n^\alpha E_n(f):\ n \ge \mathcal{N}\},$$

де в залежності від трійки $(\alpha, \mathcal{N}, s)$ число $\mathcal{N}^*$ може залежати або ні від α, $\mathcal{N}$, Y_s та f.

1. Introduction and main results. Let $\mathbb{C}[-1, 1]$ be the space of continuous functions on $[-1, 1]$ equipped with the uniform norm $\|\cdot\|$, and let $\mathbb{Y}_s$, $s \in \mathbb{N}$, be the set of all collections $Y_s := \{y_i\}_{i=1}^s$ of points y_i, such that $y_{s+1} := -1 < y_s < \ldots < y_1 < 1 =: y_0$. For $Y_s \in \mathbb{Y}_s$ denote by $\Delta^2(Y_s)$ the set of all piecewise convex functions $f \in \mathbb{C}[-1, 1]$, that change convexity at the points Y_s, and are convex on $[y_1, 1]$. In particular, $\mathbb{Y}_0 = \{\varnothing\}$, and $\Delta^2 = \Delta^2(Y_0)$ denotes the set of all convex continuous functions. If f is twice continuously differentiable in $(-1, 1)$, then $f \in \Delta^2(Y_s)$ if and only if $f''(x)\Pi(x; Y_s) \ge 0$, $x \in (-1, 1)$, where $\Pi(x; Y_s) := \prod_{i=1}^s (x - y_i)$, $(\Pi(x, Y_0) :\equiv 1)$.

We also denote by

$$E_n(f) := \inf\left\{\|f - P_n\|:\ P_n \in \mathbb{P}_n\right\}$$

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and

$$E_n^{(2)}(f, Y_s) := \inf \left\{ \|f - P_n\| : \; P_n \in \mathbb{P}_n \cap \Delta^2(Y_s) \right\}$$

the degrees of best unconstrained and coconvex approximation of a function f by polynomials from $\mathbb{P}_n$, the space of algebraic polynomials of degree $< n$. In particular,

$$E_n^{(2)}(f) := E_n^{(2)}(f, Y_0) = \inf \left\{ \|f - P_n\| : \; P_n \in \mathbb{P}_n \cap \Delta^2 \right\}$$

is the degree of best convex approximation of f.

While it is obvious that $E_n(f) \le E_n^{(2)}(f)$, Lorentz and Zeller [1] showed that the inverse inequality $E_n^{(2)}(f) \le cE_n(f)$, is invalid even if a constant c is allowed to depend on the function $f \in \Delta^2$. There are many examples showing that the same is true for piecewise convex functions from $\Delta^2(Y_s)$. The existence of counterexamples notwithstanding, we recently have proved the following result.

Theorem A [2]. *For each $\alpha > 0$ and integer $s \ge 0$ there is a constant $c(\alpha, s)$, such that for every collection $Y_s \in \mathbb{Y}_s$ and a function $f \in \Delta^2(Y_s)$ we have*

$$\sup \left\{ n^\alpha E_n^{(2)}(f, Y_s) \colon n \ge \mathcal{N}^* \right\} \le c(\alpha, s) \sup \left\{ n^\alpha E_n(f) \colon n \ge 1 \right\}, \tag{1.1}$$

where $\mathcal{N}^* = 1$, if either $s = 0$, or $s = 1$ and $\alpha \ne 4$, and $\mathcal{N}^* = \mathcal{N}^*(\alpha, Y_s)$ — a constant, depending only on α and Y_s, if either $s \ge 2$, or $s = 1$ and $\alpha = 4$.

We also have shown that Theorem A cannot be improved, that is, if either $s \ge 2$, or $s = 1$ and $\alpha = 4$, then the constant $\mathcal{N}^*$ cannot be made independent of Y_s.

Theorem B [2]. *Let $s \ge 2$. Then for every $\alpha > 0$ and $m \in \mathbb{N}$, there exist a collection $Y_s \in \mathbb{Y}_s$ and a function $f \in \Delta^2(Y_s)$, such that*

$$m^\alpha E_m^{(2)}(f, Y_s) \ge c(\alpha, s) m^{\alpha + 1 - \lceil \alpha \rceil} \sup \left\{ n^\alpha E_n(f) \colon n \ge 1 \right\}, \tag{1.2}$$

where $c(\alpha, s)$ is a positive constant and $\lceil \alpha \rceil$ is the ceiling function (i.e., the smallest integer not less than α).

Theorem C [2]. *For every $Y_1 \in \mathbb{Y}_1$ there exists a function $f \in \Delta^2(Y_1)$, satisfying*

$$\sup \left\{ n^4 E_n(f) \colon n \in \mathbb{N} \right\} = 1,$$

such that for each $m \in \mathbb{N}$, we have

$$m^4 E_m^{(2)}(f, Y_1) \ge \left(c \ln \frac{m}{1 + m^2 \varphi(y_1)} - 1 \right), \tag{1.3}$$

and

$$\sup \left\{ n^4 E_n^{(2)}(f, Y_1) \colon n \in \mathbb{N} \right\} \ge c \left| \ln \varphi(y_1) \right|, \tag{1.4}$$

where $\varphi(y) := \sqrt{1 - y^2}$ and c is an absolute positive constant.

Everywhere below, we denote by $c(\ldots)$ positive real constants that depend only on the parameters, sets, functions in the parentheses and which may vary from one occurrence to another even when they appear in the same line. In particular, c denote absolute positive constants. Similarly, $\mathcal{N}(\ldots)$ denote natural numbers that depend only on the quantities in the parentheses. For instance, $\mathcal{N}(\alpha, Y_s)$ denotes a natural number that depends only on α and Y_s and nothing else.

The main goal in this paper is to answer the following questions:

What happens if we replace $n \geq 1$ in (1.1) by $n \geq \mathcal{N}$, where $\mathcal{N} \in \mathbb{N}$? Is Theorem A still valid? What can be said about the dependence of $\mathcal{N}^*$ on α, $\mathcal{N}$, Y_s and f?

Our first result is the following generalization of Theorem A.

Theorem 1.1. *For each* $\alpha > 0$, $\mathcal{N} \in \mathbb{N}$, $s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $Y_s \in \mathbb{Y}_s$ *and* $f \in \Delta^2(Y_s)$, *there exists an* $\mathcal{N}^* \in \mathbb{N}$, *such that*

$$\sup\left\{n^\alpha E_n^{(2)}(f, Y_s) \colon n \geq \mathcal{N}^*\right\} \leq c(\alpha, \mathcal{N}, s) \sup\left\{n^\alpha E_n(f) \colon n \geq \mathcal{N}\right\}. \tag{1.5}$$

Note that $\mathcal{N}^* \in \mathbb{N}$ in the statement of Theorem 1.1 may or may not depend on α, $\mathcal{N}$, Y_s and f. Our Theorem 1.2 below provides a complete answer to when and how this dependence occurs.

It is rather easy to see that the assertion of Theorem 1.1 in the case $\mathcal{N} = 2$ immediately follows from Theorem A. Namely,

if $\mathcal{N} = 2$, then Theorem 1.1 is valid with $\mathcal{N}^* = 2$, if either $s = 0$, or $s = 1$ and $\alpha \neq 4$, and $\mathcal{N}^* = \mathcal{N}^*(\alpha, Y_s)$ if either $s \geq 2$, or $s = 1$ and $\alpha = 4$.

Indeed, noting that the function $g := f - p_2$, where $p_2 := \arg\inf_{p \in \mathbb{P}_2} \|f - p\|$, satisfies $E_n(g) = E_n(f)$, $E_n^{(2)}(g, Y_s) = E_n^{(2)}(f, Y_s)$ for all $n \geq 2$, and $E_1(g) \leq \|g\| = E_2(f)$, we have

$$\sup\left\{n^\alpha E_n^{(2)}(f, Y_s) \colon n \geq \mathcal{N}^*\right\} = \sup\left\{n^\alpha E_n^{(2)}(g, Y_s) \colon n \geq \mathcal{N}^*\right\} \leq$$

$$\leq c(\alpha, s) \sup\left\{n^\alpha E_n(g) \colon n \geq 1\right\} = c(\alpha, s) \sup\left\{n^\alpha E_n(f) \colon n \geq 2\right\}.$$

Moreover, Theorems B and C imply that

if $\mathcal{N}^* = 2$, then $\mathcal{N}^*$ cannot be made independent of Y_s if either $s \geq 2$, or $s = 1$ and $\alpha = 4$.

We now emphasize that, except when $3 \leq \mathcal{N} \leq s+2$, $\mathcal{N}^*$ cannot be smaller than $\mathcal{N}$. Indeed, to see this it suffices to consider any function $f_s \in \Delta^2(Y_s)$ which is a polynomial of degree exactly $\mathcal{N} - 1$, for instance, such that $f_s''(x) := (x+2)^{\mathcal{N}-s-3}\Pi(x; Y_s)$ if $\mathcal{N} \geq s+3$, and $f_s(x) := x$ if $\mathcal{N} = 2$. Then, $E_n(f_s) = 0$ for all $n \geq \mathcal{N}$, and one immediately gets a contradiction assuming that $\mathcal{N}^*$ in (1.5) is strictly smaller than $\mathcal{N}$. If $3 \leq \mathcal{N} \leq s+2$, then $\mathbb{P}_\mathcal{N} \cap \Delta^2(Y_s) = \mathbb{P}_2 \cap \Delta^2(Y_s)$ (any polynomial of degree $\leq s+1$ which has s convexity changes must be linear), and so $E_\mathcal{N}^{(2)}(f, Y_s) = E_2^{(2)}(f, Y_s) = E_2(f)$, i.e., if (1.5) is valid with $\mathcal{N}^* = \mathcal{N}$, then it is also valid with $\mathcal{N}^* = 2$.

Also, by Theorem B one may not expect, for $s \geq 2$, that $\mathcal{N}^*$ be independent of Y_s.

Given a triple $(\alpha, \mathcal{N}, s)$, we want to determine the exact dependence of $\mathcal{N}^*$ on all the quantities appearing in the statement of Theorem 1.1 so that (1.5) is satisfied.

We will show that there are three different types of behavior of $\mathcal{N}^*$, and in order to describe them we introduce the following notations.

Definition. *Let* $(\alpha, \mathcal{N}, s) \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N}_0$.

1. *We write* $(\alpha, \mathcal{N}, s) \in$ "$+$", *if Theorem* 1.1 *holds with* $\mathcal{N}^* = \mathcal{N}$.

2. *We write* $(\alpha, \mathcal{N}, s) \in$ "$\oplus$", *if*

(a) *Theorem* 1.1 *holds with* $\mathcal{N}^* = \mathcal{N}^*(\alpha, \mathcal{N}, Y_s)$, *and*

(b) *Theorem* 1.1 *is not valid with* $\mathcal{N}^*$ *which is independent of* Y_s, *that is, for each* $A > 0$ *and* $M \in \mathbb{N}$ *there are a number* $m > M$, *a collection* $Y_s \in \mathbb{Y}_s$, *and a function* $f \in \Delta^2(Y_s)$, *such that*

$$m^\alpha E_m^{(2)}(f, Y_s) \geq A \sup\left\{n^\alpha E_n(f) \colon n \geq \mathcal{N}\right\}. \tag{1.6}$$

3. *We write* $(\alpha, \mathcal{N}, s) \in$ "$\ominus$", *if*

(a) *Theorem* 1.1 *holds with* $\mathcal{N}^* = \mathcal{N}^*(\alpha, \mathcal{N}, Y_s, f)$, *and*

(b) *Theorem* 1.1 *is not valid with* $\mathcal{N}^*$ *which is independent of* f, *that is, for each* $A > 0$, $M \in \mathbb{N}$, *and* $Y_s \in \mathbb{Y}_s$, *there are* $m > M$ *and* $f \in \Delta^2(Y_s)$, *such that* (1.6) *holds.*

It turns out that $\mathcal{N}^*$ depends on

$$\overline{\alpha} := \lceil \alpha/2 \rceil \tag{1.7}$$

rather than on α itself with the only exception in the case $\overline{\alpha} = 2$, $\mathcal{N} \leq 2$ and $s = 1$, which has already been discussed above.

Theorem 1.2. *Let* $(\alpha, \mathcal{N}, s) \in \mathbb{R}_+ \times \mathbb{N} \times \mathbb{N}_0$. *Then*

(i) $(\alpha, \mathcal{N}, s) \in$ " $+$ "] *if*

$s = 0$, $\overline{\alpha} \leq 2$ *and* $\mathcal{N} \leq 3$;

$s = 0$, $\overline{\alpha} \geq 3$ *and* $\mathcal{N} \in \mathbb{N}$;

$s = 1$, $\overline{\alpha} = 1$ *and* $\mathcal{N} \leq 2$;

$s = 1$, $\overline{\alpha} = 2$, $\alpha \neq 4$ *and* $\mathcal{N} \leq 2$;

$s = 1$, $\overline{\alpha} = 3$ *and* $\mathcal{N} \leq 4$;

$s = 1$, $\overline{\alpha} \geq 4$ *and* $\mathcal{N} \in \mathbb{N}$.

(ii) $(\alpha, \mathcal{N}, s) \in$ " $\ominus$ "] *if*

$s \geq 0$, $\overline{\alpha} \leq 2$ *and* $\mathcal{N} \geq s + 4$;

$s \geq 1$, $\overline{\alpha} = 1$ *and* $\mathcal{N} = s + 3$.

(iii) $(\alpha, \mathcal{N}, s) \in$ " $\oplus$ " *in all other cases, except perhaps the case* $s \geq 3$, $\overline{\alpha} = 2$ *and* $\mathcal{N} = s + 3$.

We recall that the cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$ in this theorem follow from Theorems A – C and the discussion following the statement of Theorem 1.1.

In order to make it easier to see and remember what Theorem 1.2 establishes, and to recognize the patterns of behavior of the triples $(\alpha, \mathcal{N}, s)$, we summarize the results in tables relating $\mathcal{N}$ and $\overline{\alpha}$, for the various values of s.

The symbol "$\overset{\circ}{+}$" in the positions $(\overline{\alpha}, \mathcal{N}) = (2, 1)$ and $(2, 2)$ for $s = 1$ (the exceptional case) means that $(\alpha, \mathcal{N}, s) \in$ "$+$" if $\alpha \neq 4$ (i.e., $2 < \alpha < 4$), and $(\alpha, \mathcal{N}, s) \in$ "$\oplus$" if $\alpha = 4$.

We also write "?" in the position $(\overline{\alpha}, \mathcal{N}) = (2, s+3)$ for $s \geq 3$ since we do not know exactly what happens in this case. We do know, however, that $(\alpha, \mathcal{N}, s) \in$ "$\ominus$" or "$\oplus$", when $s \geq 3$, $2 < \alpha \leq 4$ and $\mathcal{N} = s + 3$ (see Theorem B and case 11 in Section 4.2).

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$+$	$+$	$+$	$+$	$+$	$\cdots$
3	$+$	$+$	$+$	$+$	$+$	$\cdots$
2	$+$	$+$	$+$	$\ominus$	$\ominus$	$\cdots$
1	$+$	$+$	$+$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	$\mathcal{N}$

$s = 0$

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
5	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
4	$+$	$+$	$+$	$+$	$+$	$+$	$\cdots$
3	$+$	$+$	$+$	$+$	$\oplus$	$\oplus$	$\cdots$
2	$\overset{\circ}{+}$	$\overset{\circ}{+}$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
1	$+$	$+$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	6	$\mathcal{N}$

$s = 1$

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\cdots$
1	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	3	4	5	6	7	$\mathcal{N}$

$s = 2$

$\overline{\alpha}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\iddots$
4	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
3	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\oplus$	$\cdots$
2	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	?	$\ominus$	$\ominus$	$\cdots$
1	$\oplus$	$\oplus$	$\cdots$	$\oplus$	$\oplus$	$\ominus$	$\ominus$	$\ominus$	$\cdots$
	1	2	$\cdots$	$s+1$	$s+2$	$s+3$	$s+4$	$s+5$	$\mathcal{N}$

$s \geq 3$

2. Proofs of the negative results. We first state the following well known result (see, e.g., [3, p. 418], Theorem 7.5.2).

Lemma 2.1. *Let $r \in \mathbb{N}$ and $G_r(x) = (x+1)^r \ln(x+1)$, $G_r(-1) := 0$. Then*

$$E_n(G_r) \leq c(r)n^{-2r}, \quad n \in \mathbb{N}. \tag{2.1}$$

Next we prove the following lemma.

Lemma 2.2. *For every $A > 0$ and $m \in \mathbb{N}$, there are points $y_1 \in (-1,1)$ and $\tilde{y}_1 \in (-1,1)$, and functions $f \in \Delta^2(Y_1)$ and $\tilde{f} \in \Delta^2(\tilde{Y}_1)$, where $Y_1 := \{y_1\}$ and $\tilde{Y}_1 := \{\tilde{y}_1\}$, such that*

$$n^4 E_n(f) \leq 1, \quad n \geq 3, \quad \text{and} \quad n^6 E_n(\tilde{f}) \leq 1, \quad n \geq 5, \tag{2.2}$$

while

$$E_m^{(2)}(f, Y_1) \geq A \quad \textit{and} \quad E_m^{(2)}(\tilde{f}, \tilde{Y}_1) \geq A.$$

Proof. Given $A > 0$ and $m \in \mathbb{N}$, in the proof of [4] (Theorem 2.4), we have constructed functions $g_4 \in \Delta^2(Y_1)$ and $g_6 \in \Delta^2(\tilde{Y}_1)$, for some $-1 < y_1 < 1$ and $-1 < \tilde{y}_1 < 1$, such that

$$E_m^{(2)}(g_4, Y_1) \geq A \quad \text{and} \quad E_m^{(2)}(g_6, \tilde{Y}_1) \geq A. \tag{2.3}$$

The functions had the representation $g_{2r} = P_{2r-1} + c_r G_r$, $r = 2, 3$, where $P_{2r-1} \in \mathbb{P}_{2r-1}$ and c_r is an absolute constant. By virtue of (2.1) we therefore conclude that

$$n^{2r} E_n(g_{2r}) \leq c, \quad n \geq 2r - 1,$$

and the proof is complete.

***Remark* 2.1.** Note that Lemma 2.2 readily implies that if $s = 1$, then for $\overline{\alpha} = 1, 2$ and all $\mathcal{N} \geq 3$ as well as for $\overline{\alpha} = 3$ and all $\mathcal{N} \geq 5$, there cannot be "+" in the position $(\overline{\alpha}, \mathcal{N})$.

Our next result is valid for arbitrary $s \in \mathbb{N}_0$.

Lemma 2.3. *Let $s \in \mathbb{N}_0$ and $Y_s \in \mathbb{Y}_s$. For every $A > 0$ and $m \in \mathbb{N}$, there is a function $f \in \Delta^2(Y_s)$, such that*

$$n^4 E_n(f) \leq 1, \quad n \geq s + 4,$$

while

$$E_m^{(2)}(f, Y_s) \geq A.$$

Proof. Following [4], for each $b \in (-1, 0)$, we denote

$$f_b(x) := \int_0^x (x - t)\Pi(t; Y_s) \left(\int_b^t \frac{t - u}{(u + 1)^2} du \right) dt.$$

Clearly, $f_b''(x)\Pi(x; Y_s) \geq 0$, $x \in (-1, 1)$, so that $f_b \in \Delta^2(Y_s)$. Straightforward computations using the Taylor expansion of $\Pi(x; Y_s)$ about $t = -1$, yield,

$$f_b = P_{s+4} - \sum_{r=0}^{s} \frac{\Pi^{(r)}(-1; Y_s)}{(r + 2)!} G_{r+2},$$

where $P_{s+4} \in \mathbb{P}_{s+4}$. Hence, by virtue of Lemma 2.1, we obtain,

$$n^4 E_n(f_b) \leq c(s), \quad n \geq s + 4, \tag{2.4}$$

since $\left\| \Pi^{(r)}(\cdot; Y_s) \right\| \leq c(s)$, $0 \leq r \leq s$.

The polynomial

$$p_{s+4}(x) := \int_0^x (x - t)\Pi(t; Y_s) \left(\int_b^1 \frac{t - u}{(u + 1)^2} du \right) dt,$$

belongs to $\mathbb{P}_{s+4}$ and satisfies $\Pi(-1; Y_s) p_{s+4}''(-1) = \Pi^2(-1; Y_s) \ln \dfrac{b + 1}{2}$. Hence, for each polynomial $P_m \in \mathbb{P}_m \cap \Delta^2(Y_s)$, $m \geq s + 4$, we have

$$-\Pi^2(-1;Y_s)\ln\frac{b+1}{2} = -\Pi(-1;Y_s)p''_{s+4}(-1) \le$$
$$\le \Pi(-1;Y_s)\left(P''_m(-1) - p''_{s+4}(-1)\right) \le$$
$$\le m^4|\Pi(-1;Y_s)|\|P_m - p_{s+4}\|, \tag{2.5}$$

where we used Markov's inequality. Also

$$p_{s+4}(x) - f_b(x) = \int_0^x (x-t)\Pi(t;Y_s)\left(\int_t^1 \frac{t-u}{(u+1)^2}du\right)dt,$$

which is independent of b. Hence, by (2.5),

$$m^{-4}\big|\Pi(-1;Y_s)\big|\ln\frac{2}{b+1} \le \|P_m - f_b\| + \|f_b - p_{s+4}\| \le \|P_m - f_b\| + c(s).$$

Thus,

$$E_m^{(2)}(f_b,Y_s) \ge m^{-4}\big|\Pi(-1;Y_s)\big|\ln\frac{2}{b+1} - c(s),$$

and taking $f := cf_b$ with suitable $c = c(s)$ and b concludes the proof of the lemma.

Remark **2.2.** Lemma 2.3 implies that if $\overline{\alpha} = 1$ or 2, then for all $s \ge 0$ and $\mathcal{N} \ge s+4$, there cannot be "+" or "$\oplus$" in the position $(\overline{\alpha},\mathcal{N})$ (and so the best we can hope for is that there is "$\ominus$" in those positions which, as will be shown below, is indeed the case).

Finally, for $s \ge 1$, we have the following lemma.

Lemma 2.4. *Let $s \in \mathbb{N}$ and $Y_s \in \mathbb{Y}_s$. For each $A > 0$ and $m \in \mathbb{N}$, there is a function $f \in \Delta^2(Y_s)$, such that*

$$n^2E_n(f) \le 1, \quad n \ge s+3,$$

and

$$E_m^{(2)}(f,Y_s) \ge A.$$

Proof. Denote $D_j(x) := x^j\ln|x|$ $(D_j(0) := 0)$. It is well known (and is easy to check) that for $j \ge 1$, $D_j^{(j-1)}$ belongs to the Zygmund class, i.e., $\omega_2(D_j^{(j-1)},t) \le c(j)t$. Thus, for $j \ge 2$, $E_n(D_j) \le c(j)n^{-j} \le c(j)n^{-2}$, $n \ge 1$. Hence, for $D_{j,\gamma}(x) :=$ $:= D_j(x+\gamma)$, $-1 < \gamma < 1$, $j \ge 2$, it follows that

$$E_n(D_{j,\gamma}) \le \frac{c(j)}{n^2} \quad n \ge 1. \tag{2.6}$$

Take $0 < b < \dfrac{1}{2}\min\{y_1 - y_2, 1 - y_1\}$, and let

$$\tilde{l}_b(x) := \frac{x}{b} - 1 + \ln b. \tag{2.7}$$

(Note that $y = l_b(x)$ is the tangent to the function $\ln|x|$ at the point $x = b$.) Further, let b^* be the other (clearly negative) root of the equation $\tilde{l}_b(x) = \ln|x|$. Clearly,

$$|b^*| = -b^* < b, \tag{2.8}$$

and $(x-b^*)\big(\tilde{l}_b(x)-\ln|x|\big)\ge 0$, $x\ne 0$, so that for

$$l_b(x) := \tilde{l}_b(x+b^*), \tag{2.9}$$

we have

$$x(l_b(x)-\ln|x+b^*|)\ge 0, \quad x\ne |b^*|. \tag{2.10}$$

Denote

$$\Pi_1(x) := \Pi_{i=2}^{s}(x-y_i)$$

($\Pi_1 :\equiv 1$ if $s=1$), and let

$$L_b(x) := \int\limits_0^x (x-u)\Pi_1(u)l_b(u-y_1)du,$$

and

$$g_b(x) := \int\limits_0^x (x-u)\Pi_1(u)\ln|u+b^*-y_1|du.$$

Finally, write

$$f_b := L_b - g_b.$$

Integration by parts yields

$$\int\limits_0^x (x-u)\ln|u+b^*-y_1|du = \frac{1}{2}D_2(x+b^*-y_1)+p_3(x),$$

where $p_3\in\mathbb{P}_3$. Similarly,

$$g_b(x) = \sum_{r=0}^{s-1}\frac{\Pi_1^{(r)}(y_1-b^*)}{(r+2)!}D_{r+2}(x+b^*-y_1)+p_{s+2}(x), \tag{2.11}$$

where $p_{s+2}\in\mathbb{P}_{s+2}$, and since $L_b\in\mathbb{P}_{s+3}$, (2.6) yields

$$E_n(f_b)\le\frac{c(s)}{n^2}, \quad n\ge s+3. \tag{2.12}$$

At the same time, it follows by (2.10) that $f_b\in\Delta^2(Y_s)$.

On the other hand, given $P_m\in\mathbb{P}_m\cap\Delta^2(Y_s)$, we conclude by (2.7) through (2.9) that

$$0<\Pi_1(y_1)\ln\frac{1}{b}<\Pi_1(y_1)\left(\ln\frac{1}{b}+1-\frac{b^*}{b}\right)=$$

$$=-L_b''(y_1)=P_m''(y_1)-L_b''(y_1)\le c(s,y_1)m^2\|P_m-L_b\|,$$

where we applied Bernstein's inequality. Since

$$\|g_b\|\le 2\|\Pi_1\|\int\limits_0^1|\ln x|dx = 2\|\Pi_1\|\le 2^s,$$

then

$$0 < \Pi_1(y_1) \ln \frac{1}{b} \le c(s, y_1) m^2 \big(\|P_m - f_b\| + \|g_b\|\big) \le c(Y_s) m^2 (\|P_m - f_b\| + 1).$$

Hence

$$E_m^{(2)}(f_b, Y_s) \ge \frac{c(Y_s)}{m^2} \ln \frac{1}{b} - 1,$$

and this combined with (2.12) implies the statement of the lemma for $f := cf_b$ with suitable $c = c(s)$ and b.

Remark **2.3.** Lemma 2.4 implies that if $\overline{\alpha} = 1$ and $s \ge 1$, then for all $\mathcal{N} \ge s + 3$, there cannot be "+" or "⊕" in the position $(\overline{\alpha}, \mathcal{N})$ (and so the best we can hope for is that there is "⊖" in those positions which is indeed the case, see below).

3. Auxiliary results. Recall that $\varphi(x) = \sqrt{1 - x^2}$, and let $\mathbb{C}_\varphi^r$, $r \ge 1$, be the space of functions $f \in \mathbb{C}^r(-1, 1) \cap \mathbb{C}[-1, 1]$ such that

$$\lim_{x \to \pm 1} \varphi^r(x) f^{(r)}(x) = 0,$$

and $\mathbb{C}_\varphi^0 := \mathbb{C}[-1, 1]$.

If

$$\Delta_\delta^k(g, x) := \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} g\left(x - \frac{k\delta}{2} + i\delta\right),$$

denotes the k-th symmetric difference of a function g with a step δ, then the Ditzian–Totik type modulus of smoothness of the rth derivative of a function $f \in \mathbb{C}_\varphi^r$, is defined by

$$\omega_{k,r}^{\varphi}(f^{(r)}, t) := \sup_{h \in [0,t]} \sup_{x : |x| + (kh)\varphi(x)/2 < 1} W^r\left(x, \frac{kh}{2}\right) \left|\Delta_{h\varphi(x)}^k (f^{(r)}, x)\right|, \qquad (3.1)$$

with the weight

$$W(x, \mu) := \varphi\big(|x| + \mu\varphi(x)\big), \quad |x| + \mu\varphi(x) < 1. \qquad (3.2)$$

If $r = 0$, then

$$\omega_k^{\varphi}(f, t) := \omega_{k,0}^{\varphi}(f, t)$$

is the (usual) Ditzian–Totik modulus of smoothness. Finally, let $\|f\|_{C[a,b]}$ denote the uniform norm of a function $f \in \mathbb{C}[a, b]$ (in particular, $\|f\|_{C[-1,1]} = \|f\|$) and recall that the ordinary k-th modulus of smoothness of $f \in \mathbb{C}[a, b]$ is

$$\omega_k\big(f, t, [a, b]\big) := \sup_{h \in [0,t]} \big\|\Delta_h^k(f, \cdot)\big\|_{\mathbb{C}[a + kh/2, b - kh/2]},$$

and denote $\omega_k(f, t) := \omega_k(f, t, [-1, 1])$.

The following results are so-called inverse theorems. They characterize the smoothness (i.e., describe the class) of functions that have the prescribed order of polynomial approximation.

First we formulate a corollary of the classical Dzyadyk–Timan–Lebed–Brudnyi inverse theorem (see, e.g., [3], Theorem 7.1.2).

Theorem 3.1. *Let* $2r < \alpha < 2k + 2r$, *and* $f \in \mathbb{C}[-1,1]$. *If*

$$n^\alpha E_n(f) \le 1, \quad n \ge k + r,$$

then $f \in \mathbb{C}^r[-1,1]$ *and*

$$\omega_k(f^{(r)}, t^2) \le c(\alpha, k, r) t^{\alpha - 2r}. \tag{3.3}$$

For the Ditzian – Totik type moduli of smoothness we need the following result which is a generalization of [5] (Theorem 7.2.4) in the case $p = \infty$.

Denote by Φ the set of nondecreasing functions ϕ: $[0,\infty) \to [0,\infty)$, satisfying $\phi(0+) = 0$.

Theorem 3.2. *Given* $k \in \mathbb{N}$, $r \in \mathbb{N}_0$, $N \in \mathbb{N}$, *and* $\phi \in \Phi$ *such that*

$$\int_0^1 \frac{r\phi(u)}{u^{r+1}} du < +\infty.$$

If

$$E_n(f) \le \phi\left(\frac{1}{n}\right), \quad \textit{for all} \quad n \ge N,$$

then $f \in \mathbb{C}^r_\varphi$, *and*

$$\omega^\varphi_{k,r}(f^{(r)}, t) \le c(k,r) \int_0^t \frac{r\phi(u)}{u^{r+1}} du + c(k,r) t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du +$$

$$+ c(k,r,N) t^k E_{k+r}(f), \quad t \in [0, 1/2].$$

If, in addition, $N \le k + r$, *then the following Bari – Stechkin type estimate holds*:

$$\omega^\varphi_{k,r}(f^{(r)}, t) \le c(k,r) \int_0^t \frac{r\phi(u)}{u^{r+1}} du + c(k,r) t^k \int_t^1 \frac{\phi(u)}{u^{k+r+1}} du, \quad t \in [0, 1/2].$$

For readers' sake, we provide a proof of this theorem in the appendix.

In fact, we only need the following theorem which is an immediate consequence of Theorem 3.2 ($\phi(u) := u^\alpha$), but is of special interest in the context of this paper.

Theorem 3.3. *Let* $r \in \mathbb{N}_0$, $k \in \mathbb{N}$ *and* $\alpha > 0$, *be such that* $r < \alpha < k + r$, *and let* $f \in \mathbb{C}[-1,1]$. *If*

$$n^\alpha E_n(f) \le 1, \quad \textit{for all} \quad n \ge N,$$

where $N \ge k + r$, *then* $f \in \mathbb{C}^r_\varphi$ *and*

$$\omega^\varphi_{k,r}(f^{(r)}, t) \le c(\alpha, k, r) t^{\alpha - r} + c(N, k, r) t^k E_{k+r}(f).$$

In particular, if $N = k + r$, *then*

$$\omega^\varphi_{k,r}(f^{(r)}, t) \le c(\alpha, k, r) t^{\alpha - r}.$$

Lemma 3.1 ([6], [4] (Theorems 2.7, 2.8 and 2.11), [2] (Lemma 2.8), [7] (Theorem 3.1)).

I. *Let* $f \in \Delta^2$. *If* $f \in \mathbb{C}[-1,1]$, *then*

$$E_n^{(2)}(f) \le c\omega_4^{\varphi}\left(f, \frac{1}{n}\right) + cn^{-6}\|f\|, \quad n \ge 3. \tag{3.4}$$

Moreover, if $f \in \mathbb{C}_{\varphi}^2 \cap \mathbb{C}^1[-1,1]$, *then*

$$E_n^{(2)}(f) \le c(k)n^{-2}\omega_{k,2}^{\varphi}\left(f'', \frac{1}{n}\right) + c(k)n^{-2}\omega_2\left(f', \frac{1}{n^2}\right), \quad n \ge 3. \tag{3.5}$$

Furthermore, if $f \in \mathbb{C}_{\varphi}^2 \cap \mathbb{C}^2[-1,1]$, *and* $k, l \in \mathbb{N}$, *then, for* $n \ge l+2$, *we have*

$$E_n^{(2)}(f) \le c(k,l)n^{-2}\omega_{k,2}^{\varphi}\left(f'', \frac{1}{n}\right) + c(k,l)n^{-4}\omega_l\left(f'', \frac{1}{n^2}\right). \tag{3.6}$$

II. *Let* $f \in \Delta^2(Y_1)$. *If* $f \in \mathbb{C}[-1,1]$, *then*

$$E_n^{(2)}(f, Y_1) \le c\omega_3^{\varphi}\left(f, \frac{1}{n}\right) + c\omega_2\left(f, \frac{1}{n^2}\right), \quad n \ge 2. \tag{3.7}$$

If, in addition, $f \in \mathbb{C}_{\varphi}^2 \cap \mathbb{C}^1[-1,1]$, *then*

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^{\varphi}\left(f'', \frac{1}{n}\right) + cn^{-2}\omega_1\left(f', \frac{1}{n^2}\right), \quad n \ge 2, \tag{3.8}$$

and

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^{\varphi}\left(f'', \frac{1}{n}\right) + cn^{-2}\omega_2\left(f', \frac{1}{n^2}\right), \quad n\varphi(y_1) > 1. \tag{3.9}$$

If $f \in \mathbb{C}_{\varphi}^2$, *then*

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^{\varphi}\left(f'', \frac{1}{n}\right) + cn^{-4}\omega_{2,2}^{\varphi}\left(f'', \frac{1}{n}\right), \quad n \ge N(Y_1), \tag{3.10}$$

and

$$E_n^{(2)}(f, Y_1) \le cn^{-2}\omega_{3,2}^{\varphi}\left(f'', \frac{1}{n}\right), \quad n \ge N(f). \tag{3.11}$$

Moreover, if we actually have $f \in \mathbb{C}_{\varphi}^3 \cap \mathbb{C}^2[-1,1]$, *then for any* $k \in \mathbb{N}$,

$$E_n^{(2)}(f, Y_1) \le c(k)n^{-3}\omega_{k,3}^{\varphi}\left(f^{(3)}, \frac{1}{n}\right) + c(k)n^{-4}\omega_2\left(f'', \frac{1}{n^2}\right), \quad n \ge 4. \tag{3.12}$$

Furthermore, if $f \in \mathbb{C}^3[-1,1]$, *then*

$$E_n^{(2)}(f, Y_1) \le c(k)n^{-3}\omega_{k,3}^{\varphi}\left(f^{(3)}, \frac{1}{n}\right) + c(k)n^{-6}\omega_k\left(f^{(3)}, \frac{1}{n^2}\right), \quad n \ge k+3. \tag{3.13}$$

III. *Let* $f \in \Delta^2(Y_s)$, $s \in \mathbb{N}$. *If* $f \in \mathbb{C}[-1,1]$, *then*

$$E_n^{(2)}(f, Y_s) \le c(s)\omega_3^{\varphi}\left(f, \frac{1}{n}\right), \quad n \ge N(Y_s). \tag{3.14}$$

Moreover, if $f \in \mathbb{C}_{\varphi}^3 \cap \mathbb{C}^2[-1,1]$, $s \in \mathbb{N}$, *and* $k, l \in \mathbb{N}$, *then there exists* $N(Y_s, k, l)$ *such that, for all* $n \ge N(Y_s, k, l)$,

$$E_n^{(2)}(f, Y_s) \le c(k,l,s) n^{-3} \omega_{k,3}^{\varphi}\left(f^{(3)}, \frac{1}{n}\right) + c(k,l,s) n^{-4} \omega_l\left(f'', \frac{1}{n^2}\right). \tag{3.15}$$

Also, if $s \ge 2$ and $f \in \mathbb{C}_{\varphi}^2$, then

$$E_n^{(2)}(f, Y_s) \le c(s) n^{-2} \omega_{3,2}^{\varphi}\left(f'', \frac{1}{n}\right), \quad n \ge N(Y_s). \tag{3.16}$$

***Remark* 3.1.** Estimate (3.13) was not proved in [2]. However, its proof is very similar to those in [2], and it is based upon the fact that if $f \in \mathbb{C}^3[a,b]$ is such that f is concave on $[a, y_1]$ and convex on $[y_1, b]$ (i.e., $f''(x)(x - y_1) \ge 0$, $a \le x \le b$), and p_k is such that $p_k \ge f^{(3)}$ on $[a,b]$ and $\|f^{(3)} - p_k\| \le c(k)\omega_k(f^{(3)}, b-a, [a,b])$ (for example, $p_k := \arg\inf_{p \in \mathbb{P}_k} \|f^{(3)} - p\|_{C[a,b]} + \inf_{p \in \mathbb{P}_k} \|f^{(3)} - p\|_{C[a,b]}$), then

$$P(x) := \int_a^x \int_a^t \int_{y_1}^s p_k(v)\, dv\, ds\, dt + f(a) + f'(a)(x-a)$$

is a polynomial from $\mathbb{P}_{k+3}$ that is coconvex with f on $[a,b]$ and satisfies $P(a) = f(a)$ and

$$\|f - P\|_{C[a,b]} \le c(k)(b-a)^3 \omega_k\big(f^{(3)}, b-a, [a,b]\big). \tag{3.17}$$

We omit the details.

4. Proofs of the positive results. Since the cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$ have already been discussed we assume that $\mathcal{N} \ge 3$. Given $\alpha > 0$, integers $\mathcal{N} \ge 3$, $s \ge 0$, a collection $Y_s \in \mathbb{Y}_s$, and a function $f \in \Delta^2(Y_s)$, assume without loss of generality that

$$n^{\alpha} E_n(f) \le 1, \quad \text{for all} \quad n \ge \mathcal{N}. \tag{4.1}$$

Then we have to prove the inequality

$$n^{\alpha} E_n^{(2)}(f, Y_s) \le c(\alpha, \mathcal{N}, s), \quad n \ge \mathcal{N}^*, \tag{4.2}$$

with a proper $\mathcal{N}^*$.

4.1. Convex approximation: $s = 0$.

1. $\mathcal{N} = 3$, $0 < \alpha < 3$ ("+").

Theorem 3.3 (with $r = 0$ and $k = 3$), inequality (4.1), and the estimate $E_n^{(2)}(f) \le c\omega_3^{\varphi}(f, 1/n)$, $n \ge 3$, proved in [8], yield $E_n^{(2)}(f) \le c\omega_3^{\varphi}(f, 1/n) \le cn^{-\alpha}$, for $n \ge 3 =: \mathcal{N}^*$.

2. $\mathcal{N} = 3$, $3 \le \alpha \le 4$ ("+").

Theorem 3.3 (with $r = 2$ and $k = 3$), Theorem 3.1 (with $r = 1$ and $k = 2$), and inequality (4.1), imply that $f \in \mathbb{C}_{\varphi}^2 \cap \mathbb{C}^1[-1,1]$, $\omega_{3,2}^{\varphi}(f'', t) \le c(\alpha) t^{\alpha-2}$, and $\omega_2(f', t^2) \le c(\alpha) t^{\alpha-2}$. Inequality (3.5) now yields $E_n^{(2)}(f) \le c(\alpha) n^{-\alpha}$ for $n \ge 3 =: \mathcal{N}^*$.

3. $\alpha > 4$, $\mathcal{N} > \alpha$ ("+").

Theorem 3.3 (with $r = 2$ and $k = \mathcal{N} - 2$), Theorem 3.1 (with $r = 2$ and $k = \mathcal{N} - 2$), and inequality (4.1), imply that $f \in \mathbb{C}_{\varphi}^2 \cap \mathbb{C}^2[-1,1]$, $\omega_{\mathcal{N}-2,2}^{\varphi}(f'', t) \le c(\alpha, \mathcal{N}) t^{\alpha-2}$, and $\omega_{\mathcal{N}-2}(f'', t^2) \le c(\alpha, \mathcal{N}) t^{\alpha-4}$. Therefore, (3.6) (with $k = l = \mathcal{N} - 2$) yields (4.2) with $\mathcal{N}^* = \mathcal{N}$.

4. $\alpha > 4$, $4 < \mathcal{N} \le \alpha$ ("+").

Let $\mathcal{N}_1 := \lfloor \alpha \rfloor + 1$ and note that $\mathcal{N}_1 > \alpha \ge \mathcal{N}$. Since (4.1) is satisfied with $\mathcal{N}_1$ instead of $\mathcal{N}$, it follows from case 3 that $n^\alpha E_n^{(2)}(f) \le c(\alpha)$, $n \ge \mathcal{N}_1$. Now, let $\alpha_1 := \mathcal{N}/2 + 2$ and note that $4 < \alpha_1 < \mathcal{N}$. It follows from (4.1) that $n^{\alpha_1} E_n(f) \le 1$, for all $n \ge \mathcal{N}$, and using case 3 again we get $n^{\alpha_1} E_n^{(2)}(f) \le c(\mathcal{N})$, $n \ge \mathcal{N}$. Therefore, for $\mathcal{N} \le n < \mathcal{N}_1$, we have $n^\alpha E_n^{(2)}(f) \le c(\mathcal{N}) n^{\alpha-\alpha_1} \le c(\mathcal{N}) \mathcal{N}_1^{\alpha-\alpha_1} \le c(\alpha, \mathcal{N})$, which verifies (4.2) with $\mathcal{N}^* = \mathcal{N}$.

5. $\mathcal{N} = 3$, $\alpha > 4$ ("+").

It follows from cases 3 and 4 that (4.2) is valid for $n \ge 5$. We note that the polynomial of best approximation of degree ≤ 2 to a convex function f has to be convex (this follows, for example, from the Chebyshev Equioscillation Theorem), and so $E_3^{(2)}(f) = E_3(f)$. Hence, for $n = 3$ and 4, we have

$$E_n^{(2)}(f) \le E_3^{(2)}(f) = E_3(f) \le 1 \le 4^\alpha n^{-\alpha},$$

and so (4.2) with $\mathcal{N}^* = 3$ follows.

6. $\mathcal{N} = 4$, $\alpha > 4$ ("+").

As in case 5, it follows from cases 3 and 4 that (4.2) is valid for $n \ge 5$ and so we only need to show that $E_4^{(2)}(f) \le c(\alpha)$. Since (4.1) implies that $n^{\alpha_1} E_n(f) \le 1$, $n \ge 4$, where $\alpha_1 := \min\{\alpha, 5\}$, it follows from Theorem 3.1 (with $r = 2$ and $k = 2$) that $f \in \mathbb{C}^2[-1,1]$ and $\omega_2(f'', t^2) \le c(\alpha) t^{\alpha_1 - 4}$ and, in particular, $\omega_2(f'', 1) \le c(\alpha)$. Therefore, $E_2(f'') \le c\omega_2(f'', 1) \le c(\alpha)$. Now, since the inequality $E_4^{(2)}(f) \le 2E_2(f'')$ holds for each $f \in \mathbb{C}^2[-1,1] \cap \Delta^2$, we conclude that $E_4^{(2)}(f) \le c(\alpha)$ as needed.

7. $\mathcal{N} \ge 4$, $0 < \alpha < 4$ ("$\ominus$").

Theorem 3.3 (with $k = 4$ and $N = \mathcal{N}$), and inequalities (4.1) and (3.4), yield

$$E_n^{(2)}(f) \le c(\alpha) n^{-\alpha} + c(\mathcal{N}) n^{-4} \|f\| \le c(\alpha) n^{-\alpha},$$

for all $n \ge \max\left\{3, c(\alpha, \mathcal{N}) \|f\|^{1/(4-\alpha)}\right\} =: \mathcal{N}^*$.

8. $\mathcal{N} \ge 4$, $\alpha = 4$ ("$\ominus$").

Theorem 3.3 (with $r = 2$ and $k = 3$), Theorem 3.3 (with $r = 1$ and $k = 3$), and inequality (4.1), imply that $f \in \mathbb{C}_\varphi^2 \cap \mathbb{C}^1[-1,1]$, $\omega_{3,2}^\varphi(f'', t) \le ct^2$, and $\omega_3(f', t^2) \le ct^2$. By the Marchaud classical inequality (see, e.g., [5], (4.3.1)) the latter estimate implies $\omega_2(f', t) \le ct + ct^2 \|f'\|$. Inequality (3.5) (with $k = 3$) now yields $E_n^{(2)}(f) \le cn^{-4} + cn^{-6} \|f'\|$, $n \ge 3$, and hence (4.2) follows with $\mathcal{N}^* := \max\left\{3, c\|f'\|^{1/2}\right\}$.

4.2. Coconvex approximation: the case $s \ge 1$. For some cases below we need the fact (see [9]), that for any $f \in \Delta^2(Y_s)$, $s \ge 1$,

$$E_2(f) \le c(Y_s) E_{s+2}(f). \tag{4.3}$$

***Remark* 4.1.** For the reader's convenience, we list in each case below, the full range of α's for which that particular proof is suitable. Hence, the same triple $(\alpha, \mathcal{N}, s)$ may be covered by more than one case.

1. $s = 1$, $4 < \alpha < 8$, $\mathcal{N} = 4$ ("+").

Theorem 3.3 (with $r = 3$ and $k = 5$), Theorem 3.3 (with $r = k = 2$), and inequality (4.1), imply that $f \in \mathbb{C}_\varphi^3 \cap \mathbb{C}^2[-1,1]$, $\omega_{5,3}^\varphi(f^{(3)}, t) \le c(\alpha) t^{\alpha-3}$, and $\omega_2(f'', t^2) \le c(\alpha) t^{\alpha-4}$. Therefore (3.12) (with $k = 5$), yields (4.2) with $\mathcal{N}^* = 4$.

2. $s = 1$, $4 < \alpha < 8$, $\mathcal{N} = 3$ ("+").

It follows from case 1 that (4.2) is satisfied for $n \geq 4$. Thus, in order to show that $\mathcal{N}^* = 3$ we only need to verify that $E_3^{(2)}(f, Y_1) \leq c(\alpha)$. Indeed, since (4.1) is satisfied with $\alpha_1 := \min\{\alpha, 5\}$, it follows from Theorem 3.2 (with $r = 2$ and $k = 1$), that $f \in \mathbb{C}^2[-1, 1]$ (so that $f''(y_1) = 0$) and $\omega_1(f'', t^2) \leq c(\alpha)t^{\alpha_1 - 4}$ and, in particular, $\omega_1(f'', 1) \leq c(\alpha)$. Now take $p_2(x) := f(y_1) + f'(y_1)(x - y_1)$, and we have

$$E_3^{(2)}(f, Y_1) = E_2^{(2)}(f, Y_1) = E_2(f) \leq \|f - p_2\| =$$

$$= \left\| \int_{y_1}^{x} \int_{y_1}^{u} (f''(s) - f''(y_1))\, ds\, du \right\| \leq c\omega_1(f'', 1) \leq c(\alpha).$$

3. $s = 1$, $\alpha > 6$, $\mathcal{N} > \alpha$ ("+").

Theorems 3.3 and 3.1 (with $r = 3$ and $k = \mathcal{N} - 3$), and inequality (4.1), imply that $f \in \mathbb{C}^3$ and $\omega^{\varphi}_{\mathcal{N}-3,3}(f^{(3)}, t) \leq c(\alpha, \mathcal{N})t^{\alpha-3}$ and $\omega_{\mathcal{N}-3}(f^{(3)}, t^2) \leq c(\alpha, \mathcal{N})t^{\alpha-6}$. Estimate (3.13) (with $k = \mathcal{N} - 3$) now yields (4.2) with $\mathcal{N}^* = \mathcal{N}$.

4. $s = 1$, $\alpha > 6$, $6 < \mathcal{N} \leq \alpha$ ("+").

Let $\mathcal{N}_1 := \lfloor \alpha \rfloor + 1$ and note that $\mathcal{N}_1 > \alpha \geq \mathcal{N}$. Since (4.1) is satisfied with $\mathcal{N}_1$ instead of $\mathcal{N}$, it follows from case 3 that $n^{\alpha} E_n^{(2)}(f, Y_1) \leq c(\alpha)$, $n \geq \mathcal{N}_1$. Now, let $\alpha_1 := (\mathcal{N} + 6)/2$ and note that $6 < \alpha_1 < \mathcal{N}$. It follows from (4.1) that $n^{\alpha_1} E_n(f) \leq 1$, for all $n \geq \mathcal{N}$, and using case 3 again we get $n^{\alpha_1} E_n^{(2)}(f, Y_1) \leq c(\mathcal{N})$, $n \geq \mathcal{N}$.

Therefore, for $\mathcal{N} \leq n < \mathcal{N}_1$, we have $n^{\alpha} E_n^{(2)}(f, Y_1) \leq c(\mathcal{N}) n^{\alpha - \alpha_1} \leq \leq c(\mathcal{N}) \mathcal{N}_1^{\alpha - \alpha_1} \leq c(\alpha, \mathcal{N})$, which verifies (4.2) with $\mathcal{N}^* = \mathcal{N}$.

5. $s = 1$, $\alpha > 6$, $\mathcal{N} = 3$ ("+").

It follows from cases 3 and 4, that (4.2) is valid with $n \geq 7$. Now, since (4.1) is obviously valid with, say, $\alpha = 5$, it follows from case 2 that $E_3^{(2)}(f, Y_1) \leq c$, and so, for $3 \leq n \leq 6$,

$$n^{\alpha} E_n^{(2)}(f, Y_1) \leq 6^{\alpha} E_3^{(2)}(f, Y_1) \leq c(\alpha),$$

and so (4.2) is valid with $\mathcal{N}^* = 3$.

6. $s = 1$, $\alpha > 6$, $\mathcal{N} = 4$ ("+").

The proof is completely analogous to the one in case 5 except that the fact that $E_4^{(2)}(f, Y_1) \leq c$ follows from case 1.

7. $s = 1$, $\alpha > 6$, $\mathcal{N} = 6$ ("+").

It follows from cases 3 and 4, that (4.2) is valid with $n \geq 7$. Hence, as in case 5, we only need to show that $E_6^{(2)}(f, Y_1) \leq c(\alpha)$. If $\alpha_1 := \min\{\alpha, 7\}$, it follows from (4.1) that $n^{\alpha_1} E_n(f) \leq 1$, for all $n \geq 6$, so that applying Theorem 3.2 (with $r = 3$ and $k = 3$), we conclude that $f \in \mathbb{C}^3[-1, 1]$ and $\omega_3(f^{(3)}, t^2) \leq c(\alpha)t^{\alpha_1 - 6}$ and, in particular, $\omega_3(f^{(3)}, 1) \leq c(\alpha)$. The inequality $E_6^{(2)}(f, Y_1) \leq c(\alpha)$ now follows from (3.17) (with $k = 3$ and $[a, b] = [-1, 1]$).

8. $s = 1$, $\alpha > 6$, $\mathcal{N} = 5$ ("+").

The argument is exactly the same as in the previous case with the only difference that $k = 2$ is used instead of $k = 3$.

9. $s \geq 1$, $0 < \alpha < 3$, $3 \leq \mathcal{N} \leq s + 2$ ("$\oplus$").

Theorem 3.189 (with $k = 3$ and $f - p_3$ in place of f where $p_3 := \arg \inf\limits_{p \in \mathbb{P}_3} \|f - p\|$), implies $\omega_3^{\varphi}(f, t) \leq c(\alpha)t^{\alpha} + c(s)t^3 E_3(f)$. Now, by (4.3) and (4.1),

$$E_3(f) \le E_2(f) \le c(Y_s)E_{s+2}(f) \le c(\alpha, Y_s).$$

Therefore, $\omega_3^\varphi(f, 1/n) \le c(\alpha)n^{-\alpha} + c(\alpha, Y_s)n^{-3} \le c(\alpha)n^{-\alpha}$, for $n \ge N(\alpha, Y_s)$. Inequality (4.2) now follows from (3.14).

10. $s \ge 2$, $2 < \alpha < 5$, $3 \le \mathcal{N} \le s + 2$ ("$\oplus$").

Theorem 3.3 (with $r = 2$, $k = 3$), implies that $f \in \mathbb{C}_\varphi^2$ and $\omega_{3,2}^\varphi(f'', t) \le$ $\le c(\alpha)t^{\alpha-2} + c(s)t^3 E_5(f)$, and by (4.3) and (4.1) we have

$$E_5(f) \le E_2(f) \le c(Y_s)E_{s+2}(f) \le c(\alpha, Y_s).$$

Hence, $\omega_{3,2}^\varphi(f'', 1/n) \le c(\alpha)n^{-\alpha+2} + c(\alpha, Y_s)n^{-3} \le c(\alpha)n^{-\alpha+2}$, for $n \ge N(\alpha, Y_s)$. Inequality (4.2) now follows from (3.16).

11. $s \ge 1$, $2 < \alpha < 5$, $\mathcal{N} \ge s + 3$ ("$\ominus$") (except all "$\oplus$" cases in these regions).

As in case 10, we can prove that $\omega_{3,2}^\varphi(f'', 1/n) \le c(\alpha)n^{-\alpha+2} + c(s)n^{-3}\|f\|$, so that $\omega_{3,2}^\varphi(f'', 1/n) \le c(\alpha)n^{-\alpha+2}$, for $n \ge N(\alpha, f)$. Hence, (4.2) with $\mathcal{N}^* = \mathcal{N}^*(\alpha, f)$ follows from (3.16) for $s \ge 2$, and from (3.11) for $s = 1$.

12. $s = 2$, $2 < \alpha < 5$, $\mathcal{N} = 5$ ("$\oplus$").

Theorem 3.3 (with $r = 2$ and $k = 3$) and (4.1), imply that $f \in \mathbb{C}_\varphi^2$ and $\omega_{3,2}^\varphi(f'', t) \le$ $\le c(\alpha)t^{\alpha-2}$. Now, (3.16) implies (4.2) with $\mathcal{N}^* = \mathcal{N}^*(\alpha, Y_s)$.

13. $s = 1$, $4 < \alpha \le 6$, $\mathcal{N} \ge 5$ and $s \ge 2$, $\alpha > 4$, $\mathcal{N} \ge 3$ ("$\oplus$").

If $\mathcal{N}_1 := \max\{\lfloor\alpha\rfloor + 1, \mathcal{N}\}$, then Theorem 3.3 (with $r = 3$ and $k = \mathcal{N}_1 - 3$), Theorem 3.3 (with $r = 2$ and $k = \mathcal{N}_1 - 2$), and (4.1), imply that $f \in \mathbb{C}_\varphi^3 \cap \mathbb{C}^2[-1, 1]$, $\omega_{\mathcal{N}_1-3,3}^\varphi(f^{(3)}, t) \le c(\alpha, \mathcal{N})t^{\alpha-3}$ and $\omega_{\mathcal{N}_1-2}(f'', t^2) \le c(\alpha, \mathcal{N})t^{\alpha-4}$. Therefore, (3.15) (with $k = \mathcal{N}_1 - 3$ and $l = \mathcal{N}_1 - 2$), yields (4.2) with $\mathcal{N}^* = \mathcal{N}^*(\alpha, \mathcal{N}, Y_s)$.

14. $s = 1$, $2 < \alpha < 5$, $\mathcal{N} = 3$ or 4 ("$\oplus$").

Theorem 3.3 (with $r = 2$ and $k = 3$) and (4.1), imply that $f \in \mathbb{C}_\varphi^2$ and $\omega_{3,2}^\varphi(f'', t) \le$ $c(\alpha)t^{\alpha-2}$. Setting $\alpha_1 := \min\{\alpha, 3\}$, Theorem 3.3 (with $r = k = 2$) implies that $\omega_{2,2}^\varphi(f'', t) \le c(\alpha)t^{\alpha_1-2}$. Therefore, it follows from (3.10) that

$$E_n^{(2)}(f, Y_1) \le cn^{-\alpha} + cn^{-\alpha_1-2} \le cn^{-\alpha}, \quad n \ge N(Y_1),$$

as required.

15. $s \ge 1$, $0 < \alpha < 3$, $\mathcal{N} \ge s + 3$ ("$\oplus$").

Theorem 3.3 (with $k = 3$ and $N = \mathcal{N}$), and inequalities (4.1) and (3.14), yield $E_n^{(2)}(f) \le c(\alpha)n^{-\alpha} + c(\mathcal{N})n^{-3}\|f\| \le c(\alpha)n^{-\alpha}$, for all sufficiently large n, $n \ge$ $\mathcal{N}^*(\alpha, \mathcal{N}, Y_s, f)$.

5. Appendix: proof of Theorem 3.2. We first give the proof for the case $r \ge 1$. Without any loss of generality assume that $N \ge k + r$. Set $m_j := N2^j$ and $\phi_j :=$ $:= \phi(m_j^{-1})$. We expand f into the telescopic series

$$f = P_{k+r} + (P_N - P_{k+r}) + \sum_{j=0}^{\infty}(P_{m_{j+1}} - P_{m_j}) =: P_{k+r} + Q + \sum_{j=0}^{\infty} Q_j, \tag{5.1}$$

where $P_n \in \mathbb{P}_n$ are the polynomials of best approximation of f, that is $\|f - P_n\| =$ $= E_n(f)$. Hence, the polynomials Q_j are of degree $< m_{j+1}$ and satisfy $\|Q_j\| \le \phi_{j+1} +$ $+ \phi_j \le 2\phi_j$. For a fixed $x \in (-1, 1)$ and $h \in [0, t]$, satisfying $kh\varphi(x)/2 < 1 - |x|$, set $x_* := |x| + kh\varphi(x)/2$ and note that if $u \in [-x_*, x_*] \supseteq [x - kh\varphi(x)/2, x + kh\varphi(x)/2] =:$ $=: A$, then $\varphi(u) \ge \varphi(x_*)$. Hence, for $u \in A$ and $l \in \mathbb{N}$, the Markov – Bernstein inequality implies,

$$\left|Q_j^{(l)}(u)\right| \le c(l) m_{j+1}^l \left(\frac{1}{m_{j+1}} + \varphi(u)\right)^{-l} \phi_j \le c(l) m_j^l \left(\frac{1}{m_j} + \varphi(x_*)\right)^{-l} \phi_j, \tag{5.2}$$

which in turn yields for $l = r$,

$$\left|\Delta^k_{h\varphi(x)}(Q_j^{(r)}, x)\right| \le 2^k \max_{u\in A} |Q_j^{(r)}(u)| \le c(r) 2^k \frac{m_j^r}{\varphi^r(x_*)} \phi_j.$$

Therefore, if we denote $J := \min\{j : 1/m_j \le h\}$, then we have

$$\varphi^r(x_*) \sum_{j=J+1}^{\infty} \left|\Delta^k_{h\varphi(x)}\big(Q_j^{(r)}, x\big)\right| \le c(r) 2^k \sum_{j=J+1}^{\infty} m_j^r \phi_j =$$

$$= c(k,r) \sum_{j=J+1}^{\infty} \int\limits_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi_j}{u^{r+1}} du \le c(k,r) \sum_{j=J+1}^{\infty} \int\limits_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi(u)}{u^{r+1}} du =$$

$$= c(k,r) \int\limits_0^{m_J^{-1}} \frac{\phi(u)}{u^{r+1}} du \le c(k,r) \int\limits_0^h \frac{\phi(u)}{u^{r+1}} du. \tag{5.3}$$

We also note that

$$\frac{\varphi(x) - \varphi(x_*)}{kh/2} = \frac{\varphi(x) - \varphi(x_*)}{x_* - |x|} \varphi(x) <$$

$$< \frac{\varphi(x) - \varphi(x_*)}{x_* - |x|} \big(\varphi(x) + \varphi(x_*)\big) = x_* + |x| < 2,$$

so that

$$\varphi(x) < kh + \varphi(x_*).$$

Hence, for $0 \le j \le J$, taking into account that $1/m_j > h/2$, we obtain by (5.2) with $l = r + k$,

$$\left|\Delta^k_{h\varphi(x)}(Q_j^{(r)}, x)\right| \le (h\varphi(x))^k \max_{u\in A} \left|Q_j^{(k+r)}(u)\right| \le$$

$$\le c(k,r) \frac{h^k m_j^{k+r} \varphi^k(x)}{(kh + \varphi(x_*))^{k+r}} \phi_j \le c(k,r) \frac{h^k m_j^{k+r}}{\varphi^r(x_*)} \phi_j \le$$

$$\le c(k,r) \frac{h^k}{\varphi^r(x_*)} \int\limits_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi_j}{u^{k+r+1}} du \le$$

$$\le c(k,r) \frac{h^k}{\varphi^r(x_*)} \int\limits_{m_j^{-1}}^{m_{j-1}^{-1}} \frac{\phi(u)}{u^{k+r+1}} du,$$

where $m_{-1} := N/2$. Hence, we get

$$\varphi^r(x_*)\sum_{j=0}^{J}\left|\Delta^k_{h\varphi(x)}(Q_j^{(r)},x)\right| \le c(k,r)h^k\sum_{j=0}^{J}\int\limits_{m_j^{-1}}^{m_{j-1}^{-1}}\frac{\phi(u)}{u^{k+r+1}}du =$$

$$= c(k,r)h^k\int\limits_{m_J^{-1}}^{2/N}\frac{\phi(u)}{u^{k+r+1}}du \le c(k,r)h^k\int\limits_{h/2}^{1}\frac{\phi(u)}{u^{k+r+1}}du \le$$

$$\le c(k,r)h^k\int_h^1\frac{\phi(u)}{u^{k+r+1}}du, \tag{5.4}$$

and note that

$$\int\limits_0^h\frac{\phi(u)}{u^{r+1}}du + h^k\int\limits_h^1\frac{\phi(u)}{u^{k+r+1}}du \le \int\limits_0^t\frac{\phi(u)}{u^{r+1}}du + t^k\int\limits_t^1\frac{\phi(u)}{u^{k+r+1}}du, \quad h\le t.$$

Finally, we have the estimate

$$\left|\Delta^k_{h\varphi(x)}(Q^{(r)},x)\right| \le h^k\left\|Q^{(k+r)}\right\| \le 2N^{2(k+r)}h^kE_{k+r}(f), \tag{5.5}$$

which follows by Markov's inequality. Note that if $N = k+r$, then $Q\equiv 0$, so that the left-hand side of (5.5) vanishes and no estimate is needed

Now, the observation that $\Delta^k_{h\varphi(x)}(P^{(r)}_{k+r},x) = 0$, combined with (5.3), (5.4), and (5.5), completes the proof of the theorem for $r\ge 1$.

For $r = 0$, we write

$$f = P_k + Q + \sum_{j=0}^{J}Q_j + (f - P_{m_{J+1}}),$$

where $Q := P_N - P_k$ and $Q_j := P_{m_{j+1}} - P_{m_j}$ (see (5.1)), and complete the proof as above, just applying (5.4), (5.5) and the inequality

$$h^k\int_h^1\frac{\phi(u)}{u^{k+1}}du \le 3t^k\int\limits_t^1\frac{\phi(u)}{u^{k+1}}du, \quad h\le t\le\frac{1}{2}.$$

Theorem 3.2 is proved.

***Remark* 5.1.** In the definition of the modulus $\omega^{\varphi}_{k,r}$ in this paper, we have used the weight $W(x,\mu)$ from (3.2) where $\mu = kh/2$. Note that we could also use the weights (see [4, 10])

$$W_1(x,\mu) := \left((1-\mu\varphi(x))^2 - x^2\right)^{1/2},$$

or (see [2])

$$W_2(x,\mu) := \left(\varphi^2(x) - \mu\varphi(x)(1+|x|)\right)^{1/2},$$

which would yield equivalent definitions of the modulus $\omega^{\varphi}_{k,r}$ since, for $\mu\in(0,1)$ and $x\colon |x|+\mu\varphi(x) < 1$,

$$\sqrt{\frac{1-\mu}{1+\mu}}\,W \le W_1 \le W_2 \le W.$$

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