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**Yu. A. Drozd** (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv),
**R. V. Skuratovskii** (Kyiv Nat. Taras Shevchenko Univ.)

# CUBIC RINGS AND THEIR IDEALS
# КУБІЧНІ КІЛЬЦЯ ТА ЇХ ІДЕАЛИ

We give an explicit description of cubic rings over a discrete valuation ring, as well as a description of all ideals of such rings.

Наведено повний опис кубічних кілець над дискретно нормованим кільцем, а також опис усіх ідеалів таких кілець.

**Introduction.** Ideals of local rings have been studied by a lot of authors from quite different viewpoints. One of the questions that arise with this respect is on the *number of parameters* $\mathrm{par}(\mathbf{C})$ defining the ideals of such a ring $\mathbf{C}$ up to isomorphism, especially when it is reduced and of Krull dimension 1. Certainly, it makes sense if the residue field $\mathbf{k}$ is infinite. In [1] it was shown that $\mathrm{par}(\mathbf{C}) = 0$, i.e., $\mathbf{C}$ has a finite number of ideals (up to isomorphism), if and only if $\mathbf{C}$ is *Cohen – Macaulay finite*, i.e., has a finite number of indecomposable non-isomorphic Cohen – Macaulay modules (in the 1-dimensional reduced case they coincide with torsion free modules). Then Schappert [2] proved that a plane curve singularity has at most 1-parameter families of ideals if and only if it dominates one of the *strictly unimodal* plane curve singularities in the sense of [3], or, the same, *unimodal* and *bimodal* plane curve singularities in the sense of [4]. In [5] this result was generalized to all curve singularities. Note that this time $\mathrm{par}(\mathbf{C}) = 1$ does not imply that $\mathbf{C}$ is *Cohen – Macaulay tame*, i.e., has at most 1-dimensional families of indecomposable Cohen – Macaulay modules. Tameness means that $\mathbf{C}$ dominates a singularity of type $T_{pq}$ [6]. The case $\mathrm{par}(\mathbf{C}) > 1$ had not been studied before the second author described the one branch singularities of type $W$ such that $\mathrm{par}(\mathbf{C}) \leq 2$ [7].

In this paper we study the *cubic rings*. We describe all such rings, their ideals and, in particular, establish the value $\mathrm{par}(\mathbf{C})$ for any cubic ring $\mathbf{C}$. As a consequence, we show that a cubic ring is Gorenstein if and only if it is a plane curve singularity (i.e., its embedding dimension equals 2).

**1. Generalities.** We denote by $\mathbf{D}$ a discrete valuation ring with the ring of fractions $\mathbf{K}$, the maximal ideal $\mathfrak{m} = t\mathbf{D}$ and the residue field $\mathbf{k} = \mathbf{D}/t\mathbf{D}$. A *cubic ring* over $\mathbf{D}$ is, by definition, a $\mathbf{D}$-subalgebra $\mathbf{C}$ in a 3-dimensional semisimple $\mathbf{K}$-algebra $\mathbf{L}$, which is a free $\mathbf{D}$-module of rank 3. We also denote $\mathbf{A}$ the integral closure of $\mathbf{D}$ in $\mathbf{L}$ and always suppose that $\mathbf{A}$ is finitely generated as $\mathbf{C}$-module. Equivalent condition (see, for instance, [8]): the $\mathfrak{m}$-adic completion $\hat{\mathbf{C}}$ of the ring $\mathbf{C}$ has no nilpotent elements. It is always the case if the algebra $\mathbf{L}$ is *separable*, for instance, if $\mathrm{char}\,\mathbf{K} = 0$. We also set $\mathbf{A}_m = t^m\mathbf{A} + \mathbf{D}$ and $\mathbf{J}_m = t\mathbf{A}_{m-1} = \mathrm{rad}\,\mathbf{A}_m$, $m > 0$.

In what follows, an *ideal* means a *fractional* $\mathbf{C}$*-ideal* in $\mathbf{K}$, i.e., a finitely generated $\mathbf{C}$-submodule $M \subseteq \mathbf{K}$ such that $\mathbf{K}M = \mathbf{K}$. Then $M$ is a free $\mathbf{D}$-module of rank 3. We are going to describe all ideals of cubic rings up to isomorphism. It is known (see, for instance, [9]) that there is a one-to-one correspondence between $\mathbf{C}$-ideals and $\hat{\mathbf{C}}$-ideals, mapping $M$ to its $\mathfrak{m}$-adic completion. This correspondence *reflects isomorphisms*, i.e., maps non-isomorphic ideals to non-isomorphic. So, in what follows we may (and will) suppose that $\mathbf{D}$ is *complete* with respect to the $\mathfrak{m}$-adic topology.

Recall also that the *embedding dimension* $\operatorname{edim}\mathbf{C}$ of a local noetherian ring $\mathbf{C}$ with the maximal ideal $\mathbf{J}$ and the residue filed $\mathbf{k}$ is defined as $\dim_{\mathbf{k}}\mathbf{J}/\mathbf{J}^2$. If $\mathbf{C}$ is of Krull dimension 1 and $\operatorname{edim}\mathbf{C} = 2$, $\mathbf{C}$ is called a *plane curve singularity*. In the geometric case, when $\mathbf{C}$ contains a subfield of representatives of $\mathbf{k}$, it actually means that there is a plane curve $C$ such that $\mathbf{C}$ is the completion of the local ring of a singular point $x \in C$.

From the general theory of ramification in finite extensions we see that the following cases can happen:

**One branch, ramified case:** $\mathbf{L}$ is a field, the maximal ideal of $\mathbf{A}$ equals $\tau\mathbf{A}$, $\mathbf{A}/\tau\mathbf{A} \simeq \mathbf{k}$ and $t\mathbf{A} = \tau^3\mathbf{A}$.

**One branch, non-ramified case:** $\mathbf{L}$ is a field, the maximal ideal of $\mathbf{A}$ equals $t\mathbf{A}$ and $\mathbf{A}/t\mathbf{A} = \mathbf{k}[\bar\theta]$ is a cubic extension of the field $\mathbf{k}$, where $\bar\theta$ is a root of an irreducible cubic polynomial $f(x) \in \mathbf{k}[x]$.

**Two branches, ramified case:** $\mathbf{L} = \mathbf{K}_1 \times \mathbf{K}$, where $\mathbf{K}_1$ is a quadratic extension of $\mathbf{K}$, $\mathbf{A} = \mathbf{D}_1 \times \mathbf{D}$, the maximal ideal of $\mathbf{D}_1$ is $\tau\mathbf{D}_1$, $\mathbf{D}_1/\tau\mathbf{D}_1 \simeq \mathbf{k}$ and $t\mathbf{D}_1 = \tau^2\mathbf{D}_1$.

**Two branches, non-ramified case:** $\mathbf{L} = \mathbf{K}_1 \times \mathbf{K}$, where $\mathbf{K}_1$ is a quadratic extension of $\mathbf{K}$, $\mathbf{A} = \mathbf{D}_1 \times \mathbf{D}$, the maximal ideal of $\mathbf{D}_1$ is $t\mathbf{D}_1$ and $\mathbf{D}_1/\tau\mathbf{D}_1 = \mathbf{k}[\bar\theta]$ is a quadratic extension of the field $\mathbf{k}$, where $\bar\theta$ is a root of an irreducible quadratic polynomial $f(x) \in \mathbf{k}[x]$.

**Three branches case:** $\mathbf{L} = \mathbf{K}^3$, $\mathbf{A} = \mathbf{D}^3$.

We recall [10, 11] that, for any cubic ring $\mathbf{C}$, every ideal of $\mathbf{C}$ is isomorphic either to an *over-ring* of $\mathbf{C}$, i.e., a cubic ring $\mathbf{B}$ such that $\mathbf{C} \subseteq \mathbf{B} \subset \mathbf{L}$, or to the *dual ideal* $\mathbf{B}^* = \operatorname{Hom}_{\mathbf{D}}(\mathbf{B}, \mathbf{D})$ of such an over-ring. Hence, to describe all ideals of $\mathbf{C}$, we only need to describe over-rings of $\mathbf{C}$. Obviously, any cubic ring in $\mathbf{L}$ contains some $\mathbf{A}_m$. Therefore, to describe all cubic rings (so their ideals as well), we have to describe the over-rings of $\mathbf{A}_m$. If $\mathbf{B}$ is an over-ring of $\mathbf{C}$, they also say that $\mathbf{B}$ *dominates* $\mathbf{C}$.

Since the unique (up to isomorphism) $\mathbf{A}$-ideal is $\mathbf{A}$ itself, we proceed by induction: supposing that all over-rings of $\mathbf{A}_m$ are known, we find all over-rings of $\mathbf{A}_{m+1}$. If $\mathbf{C}$ is an over-ring of $\mathbf{A}_{m+1}$, then $\mathbf{B} = \mathbf{C}\mathbf{A}_m$ is an over-ring of $\mathbf{A}_m$, $t\mathbf{B} \subset \mathbf{C}$ and $\mathbf{C}/t\mathbf{B}$ is a $\mathbf{k}$-subalgebra in $\mathbf{B}/t\mathbf{B}$. If $\mathbf{B} \supseteq \mathbf{A}_{m-1}$, then $t\mathbf{B} \supseteq \mathbf{J}_m$, hence, $\mathbf{C} \supseteq \mathbf{J}_m + \mathbf{D} = \mathbf{A}_m$. Therefore, the following procedure gives all over-rings of $\mathbf{A}_{m+1}$ which are not over-rings of $\mathbf{A}_m$:

**Procedure.**

For every over-ring $\mathbf{B}$ of $\mathbf{A}_m$, which is not an over-ring of $\mathbf{A}_{m-1}$, calculate $\bar{\mathbf{B}} = \mathbf{B}/t\mathbf{B}$. Set $\bar{\mathbf{A}} = (\mathbf{A}_m + t\mathbf{B})/t\mathbf{B} \subseteq \bar{\mathbf{B}}$.

Find all proper subalgebras $\mathbf{S} \subset \bar{\mathbf{B}}$ such that $\bar{\mathbf{A}}\mathbf{S} = \bar{\mathbf{B}}$. Equivalently, the natural map $\mathbf{S} \to \mathbf{B}/\mathbf{B}\mathbf{J}_m$ must be surjective.

For each such $\mathbf{S}$ take its preimage in $\mathbf{B}$.

**2. Calculations.** ***2.1. One branch, ramified case.*** We set

$$\mathbf{C}_{2r}(\alpha) = \mathbf{D} + t^r\alpha\mathbf{D} + t^{2r}\mathbf{A}, \quad \text{where} \quad v(\alpha) = 1,$$

$$\mathbf{C}_{2r+1}(\alpha) = \mathbf{D} + t^r\alpha\mathbf{D} + t^{2r+1}\mathbf{A}, \quad \text{where} \quad v(\alpha) = 2,$$

where $v$ is the discrete valuation related to the ring $\mathbf{A}$, i.e., $v(\alpha) = k$ means that $\alpha \in \tau^k\mathbf{D} \setminus \tau^{k+1}\mathbf{D}$. Note that $\mathbf{C}_0(\alpha) = \mathbf{A}$. Obviously, $\alpha$ can be uniquely chosen as $\tau + a\tau^2$ for $\mathbf{C}_{2r}$ and as $\tau^2 + at\tau$ for $\mathbf{C}_{2r+1}$, where $a \in \mathbf{D}$ is defined modulo $t^r$.

**Theorem 2.1.** *Every over-ring of* $\mathbf{A}_m$ *coincides with* $t^k\mathbf{C}_r(\alpha)+\mathbf{D}$ *for some* $k$, $r$ *such that* $r+k\le m$ *and some* $\alpha$. *The rings* $\mathbf{C}_r(\alpha)$ *are just all plane curve singularities in this case.*

***Proof.*** For $m=1$ it is easy and known [1, 12]. So, we use the Procedure for $m>1$, setting $\mathbf{B}=t^k\mathbf{C}_r(\alpha)+\mathbf{D}$, where $k+r=m$. Then the basis of $\bar{\mathbf{B}}$ consists of the classes of the elements $\{1, t^h\alpha, t^m\tau^s\}$, where $h=k+[r/2]$, $s\in\{1,2\}$ and $s\equiv r \pmod 2$. Since $t^h\alpha\notin\mathbf{J}_m$, the subalgebra $\mathbf{S}$ necessarily contains the class of $t^h\alpha+ct^m\tau^s$ for some $c\in\mathbf{D}$. If $k=0$, then $m=r$ and $v(t^m\tau^s)=2v(t^h\alpha)$. Therefore, $\bar{\mathbf{B}}$ has no proper subalgebra containing the class of $t^h\alpha+ct^m\tau^s$. If $k>0$, the preimage of $\mathbf{S}$ is $\mathbf{D}+(t^h\alpha+ct^m\tau^s)\mathbf{D}+t^{m+1}\mathbf{A}$. It coincides with $t^{k-1}\mathbf{C}_{r+2}(\alpha')+\mathbf{D}$ where $\alpha'=\alpha+ct^{m-h}\tau^s$.

Now one easily checks that $\operatorname{edim}\mathbf{C}_r(\alpha)=2$, while $\operatorname{edim}\mathbf{C}=3$ for all other rings.

The theorem is proved.

***2.2. One branch, non-ramified case.*** We set $\mathbf{C}_r(\alpha)=\mathbf{D}+t^r\alpha\mathbf{D}+t^{2r}\mathbf{A}_0$, where $\alpha\in\mathbf{A}^\times\setminus\mathbf{D}$. Again $\mathbf{C}_0(\alpha)=\mathbf{A}_0$. Note that $\alpha$ can be uniquely chosen as $\theta+a\theta^2$, where $\theta$ is a fixed preimage of $\bar\theta$ in $\mathbf{D}_1$ and $a\in\mathbf{D}$ is defined modulo $t^r$.

**Theorem 2.2.** *Every over-ring of* $\mathbf{A}_m$ *coincides with* $t^k\mathbf{C}_r(\alpha)+\mathbf{D}$ *for some* $k, r$ *and* $\alpha$ *with* $2r+k\le m$. *The rings* $\mathbf{C}_r(\alpha)$ *are just all plane curve singularities in this case.*

***Proof.*** For $m=1$ it is obvious. So, using the Procedure for $m>1$, we set $\mathbf{B}=t^k\mathbf{C}_r(\alpha)+\mathbf{D}$ with $2r+k=m$. Then a basis of $\bar{\mathbf{B}}$ consists of the classes of elements $\{1, t^{r+k}\alpha, t^m\alpha^2\}$ for some $\alpha^2\in\mathbf{A}^\times\setminus(\mathbf{D}+\alpha\mathbf{D})$. Since $t^{r+k}\alpha\notin\mathbf{J}_m$, $\mathbf{S}$ must contain the class of an element $t^{r+k}\alpha'=t^{r+k}\alpha+ct^m\alpha^2$ for some $c\in\mathbf{D}$. As above, it is impossible if $k=0$. If $k>0$, then the preimage of $\mathbf{S}$ is $\mathbf{D}+t^{r+k}\alpha'+t^{m+1}\mathbf{A}=t^{k-1}\mathbf{C}_{r+1}(\alpha')+\mathbf{D}$.

Now one easily checks that $\operatorname{edim}\mathbf{C}_r(\alpha)=2$, while $\operatorname{edim}\mathbf{C}=3$ for all other rings.

The theorem is proved.

***2.3. Two branches, ramified case.*** We denote by $v$ the valuation defined by the ring $\mathbf{D}_1$, by $e$ the idempotent in $\mathbf{A}$ such that $e\mathbf{A}=\mathbf{D}_1$ and set

$$\mathbf{C}_{l,q}(\alpha)=\mathbf{D}+t^l(e+t^q\alpha)\mathbf{D}+t^r\mathbf{A},\quad \text{where}\quad r=2l+q,$$

$$\mathbf{C}_r(\alpha)=\mathbf{D}+t^r\alpha\mathbf{D}+t^{2r+1}\mathbf{A}.$$

In both cases $\alpha\in\mathbf{D}_1$ and $v(\alpha)=1$, where $v$ is the valuation defined by the ring $\mathbf{D}_1$. Obviously, $\alpha$ can be uniquely chosen as $a\tau$, where $a\in\mathbf{D}$ is defined modulo $r$. Note that $\mathbf{C}_{0,q}(\alpha)=\mathbf{D}+e\mathbf{D}+t^q\mathbf{A}$ are just all decomposable rings in this case and $\mathbf{C}_{0,0}(\alpha)=\mathbf{A}$.

**Theorem 2.3.** *Every over-ring of* $\mathbf{A}_m$ *coincides with either* $t^k\mathbf{C}_{l,r}(\alpha)+\mathbf{D}$ *or* $t^k\mathbf{C}_r(\alpha)+\mathbf{D}$, *where* $k+r\le m$. *The rings* $\mathbf{C}_{l,q}(\alpha)$ *and* $\mathbf{C}_r(\alpha)$ *are just all plane curve singularities in this case.*

***Proof.*** The case $m=1$ is obvious. So, using the Procedure, we suppose that $m>1$ and $k+r=m$. If $\mathbf{B}=t^k\mathbf{C}_{l,q}(\alpha)+\mathbf{D}$, a basis of $\bar{\mathbf{B}}$ consists of the classes of $\left\{1, t^{k+l}(e+t^q\alpha), t^m\tau\right\}$. Since $t^{k+l}(e+t^q\alpha)\notin\mathbf{J}_m$, the subalgebra $\mathbf{S}$ must contain the classe of $t^{k+l}(e+t^q\alpha')$ for some $\alpha'\in\mathbf{D}_1$ with $v(\alpha')=1$. Again the case $k=0$ is impossible. If $k>0$, the preimage of $\mathbf{S}$ coincides with $t^{k-1}\mathbf{C}_{l+1,q}+\mathbf{D}$. If $\mathbf{B}=$ $=t^k\mathbf{C}_r(\alpha)+\mathbf{D}$, the calculations are quite similar.

Now one easily checks that $\operatorname{edim}\mathbf{C}_{l,q}(\alpha)=\operatorname{edim}\mathbf{C}_r(\alpha)=2$, while $\operatorname{edim}\mathbf{C}=3$ for all other rings.

The theorem is proved.

***2.4. Two branches, non-ramified case.*** We set

$$\mathbf{C}_{l,q}(\alpha)=\mathbf{D}+t^l(e_1+t^q\alpha)\mathbf{D}+t^r\mathbf{A},$$

$$\text{where}\quad r=2l+q,\ \text{ and }\ \alpha\in\mathbf{D}_1\setminus(e_1\mathbf{D}+t\mathbf{D}).$$

Then $\alpha$ can be chosen as $a\theta$, where $\theta$ is a fixed preimage of $\bar\theta$ in $\mathbf{D}_1$ and $a\in\mathbf{D}$ is uniquely defined modulo $t^l$. Again $\mathbf{C}_{0,q}(\alpha)=\mathbf{D}+e_1\mathbf{D}+t^q\mathbf{A}$ are just all decomposable rings in this case. Especially, $\mathbf{C}_{0,0}(\alpha)=\mathbf{A}$.

**Theorem 2.4.** *Every over-ring of $\mathbf{A}_m$ coincides with one of the rings $t^k\mathbf{C}_{l,q}(\alpha)+ \mathbf{D}$, where $k+r\le m$. The rings $\mathbf{C}_{l,q}(\alpha)$ are just all plane curve singularities in this case.*

We omit the proof in this case, since it practically repeats the calculations in the other cases.

***2.5. Three branches case.*** We set

$$\mathbf{C}_{l,q}(\alpha)=\mathbf{D}+t^l\alpha\mathbf{D}+t^r\mathbf{A},$$

where $\alpha=e+t^qae'$, $e\ne e'$ are primitive idempotent in $\mathbf{A}$, $r=2l+q$, $a\in\mathbf{D}^\times$ and $a\not\equiv1\,(\operatorname{mod}t)$ if $q=0$. Obviously, $a$ is unique modulo $t^l$. Again $\mathbf{C}_{0,q}(\alpha)= \mathbf{D}+e\mathbf{D}+t^q\mathbf{A}$ are just all decomposable rings in this case and $\mathbf{C}_{0,0}=\mathbf{A}$. Note also that if $\mathbf{C}=\mathbf{D}+t^l\alpha\mathbf{D}+t^r\mathbf{A}$, where $\alpha=e+ae'$ as above with $a\equiv1\,(\operatorname{mod}t)$, then, for $a\equiv1\,(\operatorname{mod}t^l)$, $\mathbf{C}=t^l\mathbf{C}_{0,q}(1-e-e')+\mathbf{D}$, and for $a\equiv1\,(\operatorname{mod}t^q)$ with $0<q<l$, $\mathbf{C}=\mathbf{C}_{l,q}(\alpha')$ for some $\alpha'$.

**Theorem 2.5.** *Every over-ring of $\mathbf{A}_m$ coincides with $t^k\mathbf{C}_{l,q}(\alpha)+\mathbf{D}$ for some $\alpha$ and some $l,q$ with $k+r\le m$. The rings $\mathbf{C}_{l,q}(\alpha)$ are just all plane curve singularities in this case.*

We also omit the proof in this case, since it practically repeats the calculations in the other cases.

***2.6. Table of plane curve cubic singularities.*** We present in Table 1 below all plane curve cubic singularities. In this table $s$ is the number of branches, * marks the unramified cases (related to the residue field extensions, hence impossible if $\mathbf{k}$ is algebraically closed); $x,y$ are generators of the maximal ideal, $v(a)$ denotes the *multivaluation* of an element $a\in\mathbf{A}$, i.e., the vector of valuations of its components with respect to the decomposition of $\mathbf{A}$ into the product of discrete valuation rings. The column "type" shows the correspondence with the Arnold's classification [4] (§ 15). If $\operatorname{char}\mathbf{k}=0$ and $\mathbf{A}$ is ramified, it actually shows the place of the rings in this classification. If $\operatorname{char}\mathbf{k}=0$ and $\mathbf{C}$ is non-ramified, it shows the place of the ring in this classification after the natural extension of the field $\mathbf{k}$. The validation of this column is given in [5] (Section 2.3). Note that, following [5], we denote by $E_{l,q}$ the singularities $J_{l,q}$ in the sense of [4]. Such notations seem more uniform. Note also that the singularities of types $E_1$ and $E_2$ are actually not cubic, but quadratic, and coincide with those of types $A_1$ and $A_2$ of [4]. Finally, the last column, "par" shows the number of parameters $p$ from the residue field $\mathbf{k}$ which define a unique ring of this type. We will consider this value in the last section. It does not coincide with the *modality* in the sense of [4]; the latter equals $p-1$.

Table 1

| $s$ | Name | $v(x)$ | $v(y)$ | Type | Par |
|---|---|---|---|---|---|
| 1 | $\mathbf{C}_{2r}(\alpha)$ | $(3)$ | $(3r+1)$ | $E_{6r}$ | $r$ |
| | $\mathbf{C}_{2r+1}(\alpha)$ | $(3)$ | $(3r+2)$ | $E_{6r+2}$ | $r$ |
| $1^*$ | $\mathbf{C}_r(\alpha)$ | $(1)$ | $(r)$ | $E^*_{r,0}$ | $r$ |
| 2 | $\mathbf{C}_r(\alpha)$ | $(2,1)$ | $(2r+1,\infty)$ | $E_{6r+1}$ | $r$ |
| | $\mathbf{C}_{l,q}(\alpha)$ | $(2,1)$ | $(2l,\infty)$ | $E_{l,2q+1}$ | $l$ |
| $2^*$ | $\mathbf{C}_{l,q}(\alpha)$ | $(1,1)$ | $(l,\infty)$ | $E^*_{l,2q}$ | $l$ |
| 3 | $\mathbf{C}_{l,q}(\alpha)$ | $(1,1,1)$ | $(l,l+q,\infty)$ | $E_{l,2q}$ | $l$ |

***Remark.*** The *tame* cubic plane curve singularities $T_{3,q},\ q\ge 6$ [6, 13], coincide with those of types $E_{2,q-6}$.

**3. Ideals.** As we have mentioned above, every ideal of a cubic ring $\mathbf{C}$ is isomorphic either to an over-ring $\mathbf{B}\supseteq\mathbf{C}$ or to its dual $\mathbf{B}^*=\mathrm{Hom}_{\mathbf{D}}(\mathbf{B},\mathbf{D})$. If $\mathbf{C}$ is *Gorenstein* (for instance, if it is a plane cubic singularity) [14], then $\mathbf{C}^*\simeq\mathbf{C}$, thus $\mathbf{B}^*\simeq\mathrm{Hom}_{\mathbf{C}}(\mathbf{B},\mathbf{C})$. Therefore, to calculate $\mathbf{B}^*$, one has to choose a Gorenstein subring $\mathbf{C}\subseteq\mathbf{B}$ and to calculate

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{B},\mathbf{C})\simeq\{\lambda\in\mathbf{L}\,|\,\lambda\mathbf{B}\subseteq\mathbf{C}\}=\{\lambda\in\mathbf{C}\,|\,\lambda\mathbf{B}\subseteq\mathbf{C}\}$$

(the latter equality holds since $1\in\mathbf{B}$). This remark easily leads to the following result.

**Theorem 3.1.** *The duals to the cubic rings are as follows*:

**One branch ramified case:** *If* $\mathbf{B}=\mathbf{D}+t^k\mathbf{C}_r(\alpha)$, *then* $\mathbf{B}^*\simeq\mathbf{D}+t^{[r/2]}\alpha\mathbf{D}+ t^{k+r}\mathbf{A}$.

**One branch non-ramified case:** *If* $\mathbf{B}=\mathbf{D}+t^k\mathbf{C}_r(\alpha)$, *then* $\mathbf{B}^*\simeq\mathbf{D}+t^{r}\alpha\mathbf{D}+ t^{k+2r}\mathbf{A}$.

**Two branches ramified case:**

1. *If* $\mathbf{B}=\mathbf{D}+t^k\mathbf{C}_{l,q}(\alpha)$, *then* $\mathbf{B}^*\simeq\mathbf{D}+t^{l}(e+t^q\alpha)\mathbf{D}+t^{k+2l+q}\mathbf{A}$.
2. *If* $\mathbf{B}=\mathbf{D}+t^k\mathbf{C}_r(\alpha)$, *then* $\mathbf{B}^*\simeq\mathbf{D}+t^{r}\alpha\mathbf{D}+t^{k+2r+1}\mathbf{A}$.

**Two branches non-ramified case:** *If* $\mathbf{B}=\mathbf{D}+t^k\mathbf{C}_{l,q}(\alpha)$, *then* $\mathbf{B}^*\simeq\mathbf{D}+t^{l}(e+ t^q\alpha)\mathbf{D}+t^{k+2l+q}\mathbf{A}$.

**Three branches case:** *If* $\mathbf{B}=\mathbf{D}+t^k\mathbf{C}_{l,q}(\alpha)$, *then* $\mathbf{B}^*\simeq\mathbf{D}+t^{l}\alpha\mathbf{D}+t^{k+2l+q}\mathbf{A}$.

***Proof.*** The proof is immediate if we choose for a Gorenstein subring $\mathbf{C}\subseteq\mathbf{B}$ the plane curve singularity $\mathbf{C}=\mathbf{C}_{k+r}(\alpha)$ or $\mathbf{C}_{k+l,q}(\alpha)$ depending on the shape of $\mathbf{B}$. For instance, in two branches ramified case, if $\mathbf{B}=\mathbf{D}+t^k\mathbf{C}_{l,q}(\alpha)$ and $\mathbf{C}=\mathbf{C}_{l+k,q}(\alpha)$, then

$$\mathbf{B}^*\simeq\{\lambda\in\mathbf{C}\,|\,\lambda\mathbf{B}\subseteq\mathbf{C}\}=t^k\mathbf{D}+t^{k+l}(e+t^q\alpha)\mathbf{D}+t^{2k+2l+q}\mathbf{A}\simeq$$

$$\simeq\mathbf{D}+t^{l}(e+t^q\alpha)\mathbf{D}+t^{k+2l+q}\mathbf{A}.$$

***Corollary* 3.1.** *If a cubic ring is Gorenstein, it is a plane curve singularity.*

Note that it is no more the case for the extensions of bigger degrees. For instance, the rings $P_{pq}$ from [6], which are quartic, are Gorenstein (they are complete intersections) but of embedding dimension 3.

**4. Geometric case. Number of parameters.** In this section we suppose that our rings are of *geometric nature*, i.e., $\mathbf{D} = \mathbf{k}[[t]]$, where $\mathbf{k}$ is algebraically closed. Then one can consider the *number of parameters* $\operatorname{par}(\mathbf{C})$ defining $\mathbf{C}$-ideals (see [13], Section 2.2, or [15], Section 3, where it is denoted by $\operatorname{par}(1; \mathbf{C}, \mathbf{A})$). Actually, it coincides with the minimal possible number $p$ for which there is a finite set of *families of ideals* $\mathcal{I}_k$, $1 \le k \le m$, of dimensions at most $p$ such that every $\mathbf{C}$-ideal is isomorphic to one belonging to some family $\mathcal{I}_k$. Equivalently, it is the maximal possible $p$ such that there is a $p$-dimensional family of ideals $\mathcal{I}$ where every isomorphism class of ideals only occurs finitely many times. In [5] a criterion was established in order that $\operatorname{par}(\mathbf{C}) \le 1$. For cubic rings it means that $\mathbf{C}$ dominates a singularity of type $E_m$, $18 \le m \le 20$, or $E_{3,i}$. The following results give the exact value of $\operatorname{par}(\mathbf{C})$ for all cubic rings of geometric nature. (Note that no unramified case can occur for such rings.)

**Theorem 4.1.** *If $\mathbf{C}$ is a cubic ring of geometric nature,* $\operatorname{par}(\mathbf{C}) \le n$ *if and only if* $\mathbf{C}$ *dominates one of the singularities of type* $E_{12n+i}$, $6 \le i \le 8$, *or* $E_{2n+1,q}$, $q \ge 0$.

***Proof.*** Certainly, we have to prove that

(1) every ring of one of the listed types have at most $n$-parameter families of ideals;

(2) if $\mathbf{C}$ dominates no ring of the listed types, it has $(n+1)$-parameter families of ideals.

Since the calculations in all cases are similar, we only consider the one branch ramified case. Note first that the rings $\mathbf{C}_{2r}(\alpha)$ as well as $\mathbf{C}_{2r+1}(\alpha)$ form a $r$-parametric family. Indeed, we can choose in the first case $\alpha = \tau + a\tau^2$, and in the second one $\alpha = \tau^2 + a\tau^4$, where $a \in \mathbf{D}$ is defined modulo $t^r$, and such a presentation is unique. The same is true also for $t^k\mathbf{C}_{2r}(\alpha) + \mathbf{D}$ and $t^k\mathbf{C}_{2r+1}(\alpha) + \mathbf{D}$ for any $k$. Since $\mathbf{C}_{2r}(\alpha) \supseteq \mathbf{A}_{2r}$ for all $\alpha$, we get $\operatorname{par}(\mathbf{A}_{2r}) \ge r$.

Let $\mathbf{C}$ dominate neither a ring of type $E_{12n+6}$, i.e., $\mathbf{C}_{4n+2}(\alpha)$, nor a ring of type $E_{12n+8}$, i.e., $\mathbf{C}_{4n+3}(\alpha)$. Then it contains no element of valuation smaller than $6n+6$, so $\mathbf{C} \subseteq \mathbf{A}_{2n+2}$. Hence, $\operatorname{par}(\mathbf{C}) \ge n+1$.

On the other hand, consider the ring $\mathbf{C}_{2r+q}(\alpha)$, where $q \in \{0, 1\}$. Its over-rings are of the kind $\mathbf{D} + t^k\mathbf{C}_{2m+q}(\beta)$, where $k + m \le r$ and $k + 2m \le 2r$. Moreover, let $\alpha = \tau^{q+1} + a\tau^{2q+2}$ and $\beta = \tau^{q+1} + b\tau^{2q+2}$. Then $b$ is defined modulo $t^m$ and $b \equiv a \pmod{t^{r-m-k}}$. Therefore, the over-rings with the fixed $m, k$ form a $p$-parameter family, where $p = \min(m, r-m-k)$. Hence, $2p \le r$ and $p \le [r/2]$. If we set $r = 2n+1$, we get that $\operatorname{par}(\mathbf{C}_{4n+2}(\alpha)) \le n$ and $\operatorname{par}(\mathbf{C}_{4n+3}(\alpha)) \le n$ for all possible $\alpha$. It accomplishes the proof.

Obvious considerations give the number of parameters for special rings.

***Corollary* 4.1.**

$$\operatorname{par}(\mathbf{C}_r(\alpha)) = [r/2],$$

$$\operatorname{par}(\mathbf{C}_{l,q}(\alpha)) = [l/2],$$

$$\operatorname{par}(\mathbf{A}_m) = [m/2].$$

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