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SOME NOTES CONCERNING RIEMANNIAN EXTENSIONS*
ДЕЯКІ ЗАУВАЖЕННЯ ЩОДО РІМАНОВИХ РОЗШИРЕНЬ

In this paper we investigate some properties of Riemannian extensions in the cotangent bundle using the adapted frames.

Досліджено деякі властивості ріманових розширень у кодотичному розшаруванні з використанням адаптованих реперів.

1. Introduction. Let M_n be an n-dimensional differentiable manifold of class C^∞, ${}^C T(M_n)$ its cotangent bundle, and π the natural projection ${}^C T(M_n) \to M_n$. A system of local coordinates $(U;\, x^i)$, $i = 1, \ldots, n$, in M_n induces on ${}^C T(M_n)$ a system of local coordinates $\left(\pi^{-1}(U);\, x^i,\, x^{\bar{i}} = p_i\right)$, $i = 1, \ldots, n$, $\bar{i} = n + i = n + 1, \ldots, 2n$, where $x^{\bar{i}} = p_i$ is the cartesian coordinates of covectors p in each cotangent space ${}^C T_x(M_n)$, $x \in U$ with respect to the natural coframe $\left\{dx^i\right\}$.

We denote by $\Im^r_s(M_n)$ $\left(\Im^r_s\left({}^C T(M_n)\right)\right)$ the modul over $F(M_n)$ $\left(F\left({}^C T(M_n)\right)\right)$ of C^∞ tensor fields of type (r, s), where $F(M_n)$ $\left(F\left({}^C T(M_n)\right)\right)$ is the ring of real-valued C^∞ functions on $M_n\left({}^C T(M_n)\right)$. The so-called Einsteins summation convention is used.

Let $X = X^i \dfrac{\partial}{\partial x^i}$ and $\omega = \omega^i dx^i$ be the local expressions in $U \subset M_n$ of a vector field $X \in \Im^0_1(M_n)$, and 1-form $\omega \in \Im^0_1(M_n)$ respectively. Then the horizontal lift ${}^H X \in \Im^1_0\left({}^C T(M_n)\right)$ of X and the vertical lift ${}^V\omega \in \Im^1_0\left({}^C T(M_n)\right)$ of ω are given, respectively, by

$$ {}^H X = X^i \frac{\partial}{\partial x^i} + \sum_i p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}} \tag{1}$$

and

$$ {}^V\omega = \sum_i \omega_i \frac{\partial}{\partial x^{\bar{i}}} \tag{2}$$

with respect to the natural frame $\left\{\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^{\bar{i}}}\right\}$, where Γ^h_{ij} are components of a symmetric (torsion-free) affine connection ∇ on M_n.

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We now consider a tensor field ${}^R\nabla \in \Im_2^0\left({}^C T(M_n)\right)$, whose components in $\pi^{-1}(U)$ are given by

$$ {}^R\nabla = \left({}^R\nabla_{JI}\right) = \begin{pmatrix} -2p_h\Gamma_{ji}^h & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix} \tag{3} $$

with respect to the natural frame, where δ_j^i denotes the Kronecker delta. The indices $I, J, K, \ldots = 1, \ldots, 2n$ indicate the indices with respect to the natural frame $\left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\right\}$. This tensor field defines a pseudo-Riemannian metric in ${}^C T(M_n)$ and the line element of pseudo-Riemannian metric ${}^R\nabla$ is given by

$$ ds^2 = 2dx^i\,\delta p_i, $$

where

$$ \delta p_i = dp_i - p_h\,\Gamma_{ji}^h\,dx^i. $$

This metric is called the Riemannian extension of the symmetric affine connection ∇ [1, 2]. A number of results referring to the applications of the Riemannian extension are contained in [3, 4].

The complete lift of vector field $X \in \Im_0^1(M_n)$ to cotangent bundle ${}^C T(M_n)$ is defined by

$$ {}^C X = X^i \frac{\partial}{\partial x^i} - \sum_i p_h\,\partial_i\,X^h \frac{\partial}{\partial x^{\bar{i}}}. \tag{4} $$

Using (3) and (4), we easily see that

$$ {}^R\nabla\left({}^C X,\ {}^C Y\right) = -\gamma\left(\nabla_X Y + \nabla_Y X\right), \tag{5} $$

where

$$ \gamma(\nabla_X Y + \nabla_Y X) = p_h\left(X^i \nabla_i Y^h + Y^i \nabla_i X^h\right). $$

Since the tensor field ${}^R\nabla \in \Im_2^0\left({}^C T(M_n)\right)$ is completely determined by its action on vector fields of type ${}^C X$ and ${}^C Y$ (see Proposition 4.2 of [2, p. 237]), we have an alternative definition of ${}^R\nabla$: The tensor field ${}^R\nabla$ is completely determined by the condition (5).

On the other hand, the vector fields ${}^H X$ and ${}^V\omega$ span the module $\Im_0^1\left({}^C T(M_n)\right)$. Hence tensor field ${}^R\nabla$ is also determined by its action of ${}^H X$ and ${}^V\omega$.

From (1), (2) and (3) we have

$$ {}^R\nabla\left({}^V\omega, {}^V\theta\right) = 0, \tag{6} $$

$$ {}^R\nabla\left({}^V\omega, {}^H X\right) = {}^V(\omega(X)) = (\omega(X)) \circ \pi, \tag{7} $$

$$ {}^R\nabla\left({}^H X, {}^H Y\right) = 0 \tag{8} $$

for any $X, Y \in \Im_0^1(M_n)$ and $\omega, \theta \in \Im_1^0(M_n)$. Thus ${}^R\nabla$ is completely determined by the conditions (6), (7), (8) because of the above stated reasons.

In this paper we shall develop the Riemannian extension ${}^R\nabla$ using the conditions (6) – (8). Moreover, we find it more convenient to refer equations (6) – (8) to the adapted frame.

2. Adapted frames. Let ∇ be a torsion-free affine connection on M_n. In $U \subset M_n$, we put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \qquad \theta^{(i)} = dx^i, \quad i = 1, \ldots, n.$$

Then from (1) and (2) we see that ${}^H X_{(i)}$ and ${}^V\theta^{(i)}$ have respectively local expressions of the form

$$ {}^H X_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar h}}, \tag{9}$$

$$ {}^V\theta^{(i)} = \frac{\partial}{\partial x^{\bar i}}. \tag{10}$$

We call the set $\left\{{}^H X_{(i)}, {}^V\theta^{(i)}\right\} = \left\{\tilde e_{(i)}, \tilde e_{(\bar i)}\right\} = \left\{\tilde e_{(\alpha)}\right\}$ the frame adapted to the affine connection ∇. The indices $\alpha, \beta, \gamma, \ldots = 1, \ldots, 2n$ indicate the indices with respect to the adapted frame.

We now from equations (1), (2) and (9), (10) see that the lifts ${}^H X$ and ${}^V\omega$ have respectively components

$$ {}^H X = X^i \tilde e_{(i)}, \qquad {}^H X = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{11}$$

$$ {}^V\omega = \sum_i \omega_i \tilde e_{(\bar i)}, \qquad {}^V\omega = \begin{pmatrix} 0 \\ \omega^i \end{pmatrix} \tag{12}$$

with respect to the adapted frame $\left\{\tilde e_{(\alpha)}\right\}$, where $X \in \Im_0^1(M_n)$, $\omega \in \Im_1^0(M_n)$, X^i and ω_i being local components of X and ω, respectively. Also from (6) – (8) we see that

$$ {}^R\nabla\left({}^V\omega^{(i)}, {}^V\theta^{(j)}\right) = {}^R\nabla\left(\tilde e_{(\bar i)}, \tilde e_{(\bar j)}\right) = {}^R\tilde\nabla_{\bar i\bar j} = 0,$$

$$ {}^R\nabla\left({}^H X_{(i)}, {}^H Y_{(j)}\right) = {}^R\nabla\left(\tilde e_{(i)}, \tilde e_{(j)}\right) = {}^R\tilde\nabla_{ij} = 0,$$

$$ {}^R\nabla\left({}^V\omega^{(i)}, {}^H X^{(j)}\right) = {}^R\nabla\left(\tilde e_{(\bar i)}, \tilde e_{(j)}\right) = {}^R\tilde\nabla_{\bar i j} = {}^R\tilde\nabla_{j\bar i} = \left(dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = \delta^i_j,$$

$$ {}^R\nabla\left({}^H X_{(i)}, {}^V\omega^{(j)}\right) = {}^R\nabla\left(\tilde e_{(i)}, \tilde e_{(\bar j)}\right) = {}^R\tilde\nabla_{i\bar j} = {}^R\tilde\nabla_{\bar j i} = \left(dx^j\right)\left(\frac{\partial}{\partial x^i}\right) = \delta^j_i,$$

i.e., ${}^R\nabla$ has components

$$ {}^R\nabla = \left({}^R\tilde\nabla_{\beta\alpha}\right) = \begin{pmatrix} {}^R\tilde\nabla_{ji} & {}^R\tilde\nabla_{j\bar i} \\ {}^R\tilde\nabla_{\bar j i} & {}^R\tilde\nabla_{\bar j\bar i} \end{pmatrix} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix} \tag{13}$$

with respect to the adapted frame $\left\{\tilde e_{(\alpha)}\right\}$.

Using (9), (10), we now consider local vector fields $\tilde{e}_\beta$ and 1-forms $\tilde{\omega}^\alpha$ in $\pi^{-1}(U)$ defined by

$$\tilde{e}_\beta = A_\beta{}^A \partial_A, \qquad \tilde{\omega}^\alpha = \bar{A}^\alpha{}_B dx^B,$$

where

$$A = \left(A_\beta{}^A\right) = \begin{pmatrix} A_j{}^i & A_{\bar{j}}{}^i \\ A_j{}^{\bar{i}} & A_{\bar{j}}{}^{\bar{i}} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ p_a \Gamma_{ij}^a & \delta_i^j \end{pmatrix}, \tag{14}$$

$$A^{-1} = \left(\bar{A}^\alpha{}_B\right) = \begin{pmatrix} \bar{A}^i{}_j & \bar{A}^i{}_{\bar{j}} \\ \bar{A}^{\bar{i}}{}_j & \bar{A}^{\bar{i}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta_j^i & 0 \\ -p_a \Gamma_{ij}^a & \delta_i^j \end{pmatrix}. \tag{15}$$

We easily see that the set $\left\{\tilde{\omega}^\alpha\right\}$ is the coframe dual to the adapted frame $\left\{\tilde{e}_\beta\right\}$, i.e., $\tilde{\omega}^\alpha \tilde{e}_\beta = \bar{A}^\alpha{}_B A_\beta{}^B = \delta_\beta^\alpha$.

Since the adapted frame $\left\{\tilde{e}_\beta\right\}$ is nonholonomic, we put

$$\left[\tilde{e}_\gamma, \tilde{e}_\beta\right] = \Omega_{\gamma\beta}{}^\alpha \tilde{e}_\alpha$$

from which we have

$$\Omega_{\gamma\beta}{}^\alpha = \left(\tilde{e}_\gamma A_\beta{}^A - \tilde{e}_\beta A_\gamma{}^A\right) \bar{A}^\alpha{}_A .$$

According to (9), (10), (14) and (15), the components of nonholonomic object $\Omega_{\gamma\beta}{}^\alpha$ are given by

$$\begin{aligned} \Omega_{l\bar{j}}{}^{\bar{i}} &= -\Omega_{\bar{j}l}{}^{\bar{i}} = -\Gamma_{li}^j, \\ \Omega_{lj}{}^{\bar{i}} &= p_a R_{lji}{}^a \end{aligned} \tag{16}$$

all the others being zero, where $R_{ljk}{}^h$ being local components of the curvature tensor R of ∇.

Let ${}^C\nabla$ be the Levi-Civita connection determined by the Riemannian extension ${}^R\nabla$. We call ${}^C\nabla$ the complete lift of the symmetric affine connection ∇ to ${}^C T(M_n)$. We put

$${}^C\nabla_{\tilde{e}_\gamma} \tilde{e}_\beta = {}^C\Gamma_{\gamma\beta}^\alpha \tilde{e}_\alpha .$$

From the equation ${}^C\nabla_X Y - {}^C\nabla_Y X = [X, Y] \quad \forall X, Y \in \Im_0^1\left({}^C T(M_n)\right)$ we have

$${}^C\Gamma_{\gamma\beta}^\alpha - {}^C\Gamma_{\beta\gamma}^\alpha = \Omega_{\gamma\beta}{}^\alpha . \tag{17}$$

The equation $\left({}^C\nabla_X {}^R\nabla\right)(Y, Z) = 0$ has form

$$\tilde{e}_\delta {}^R\nabla_{\gamma\beta} - {}^C\Gamma_{\delta\gamma}^\varepsilon {}^R\nabla_{\varepsilon\beta} - {}^C\Gamma_{\delta\beta}^\varepsilon {}^R\nabla_{\gamma\varepsilon} = 0 \tag{18}$$

with respect to the adapted frame $\left\{\tilde{e}_\beta\right\}$. We have from (17) and (18)

$$ {}^C\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2}\,{}^R\nabla^{\alpha\varepsilon}\Big(\tilde{e}_{\gamma}\,{}^R\nabla_{\varepsilon\beta} + \tilde{e}_{\beta}{}^R\nabla_{\gamma\varepsilon} - \tilde{e}_{\varepsilon}\,{}^R\nabla_{\gamma\beta}\Big) + \frac{1}{2}\Big(\Omega_{\gamma\beta}{}^{\alpha} + \Omega^{\alpha}{}_{\gamma\beta} + \Omega^{\alpha}{}_{\beta\gamma}\Big), $$

where $\Omega^{\alpha}{}_{\gamma\beta} = {}^R\nabla^{\alpha\varepsilon}\,{}^R\nabla_{\delta\beta}\Omega_{\varepsilon\gamma}{}^{\delta}$ and $\left({}^R\nabla^{\alpha\varepsilon}\right) = \begin{pmatrix} 0 & \delta^i_m \\ \delta^m_i & 0 \end{pmatrix}$.

Taking account of (9), (10), (13) and (16) we obtain

$$ \begin{aligned} {}^C\Gamma^{i}_{\bar{k}j} &= {}^C\Gamma^{i}_{k\bar{j}} = {}^C\Gamma^{i}_{\bar{k}\bar{j}} = {}^C\Gamma^{\bar{i}}_{\bar{k}\bar{j}} = {}^C\Gamma^{\bar{i}}_{\bar{k}j} = 0, \\ {}^C\Gamma^{i}_{kj} &= \Gamma^{i}_{kj}, \quad {}^C\Gamma^{\bar{i}}_{k\bar{j}} = -\Gamma^{j}_{ki}, \\ {}^C\Gamma^{\bar{i}}_{kj} &= \frac{1}{2}\,p_a\Big(R_{kji}{}^{a} - R_{jik}{}^{a} + R_{ikj}{}^{a}\Big). \end{aligned} \tag{19} $$

Let $X \in \Im^1_0\left({}^C T(M_n)\right)$ and $X = \tilde{X}^{\alpha}\tilde{e}_{\alpha} = \tilde{X}^{i}\tilde{e}_{(i)} + \tilde{X}^{\bar{i}}\,\tilde{e}_{(\bar{i})}$. Then the covariant derivative ${}^C\nabla X$ has components

$$ {}^C\nabla_{\gamma}\tilde{X}^{\alpha} = \tilde{e}_{\gamma}\tilde{X}^{\alpha} + {}^C\Gamma^{\alpha}_{\gamma\beta}\tilde{X}^{\beta}. $$

If $X = {}^H X$ and $X = {}^V\omega$, then using (9), (10), (11), (12) and (19) we see that covariant derivatives ${}^C\nabla{}^H X$ and ${}^C\nabla\,{}^V\omega$ have ω respectively components

$$ \left({}^C\nabla_{\gamma}{}^H\tilde{X}^{\alpha}\right) = \begin{pmatrix} \nabla_k X^i & 0 \\ \frac{1}{2}\,p_a\Big(R_{kji}{}^{a} - R_{jik}{}^{a} + R_{ikj}{}^{a}\Big)X^i & 0 \end{pmatrix}, \tag{20} $$

$$ \left({}^C\nabla_{\gamma}{}^V\tilde{\omega}^{\alpha}\right) = \begin{pmatrix} 0 & 0 \\ \nabla_k\omega_i & 0 \end{pmatrix} \tag{21} $$

with respect to the adapted frame $\{\tilde{e}_{\alpha}\}$.

Taking account (4), (9) and (10), we find

$$ {}^C X = X^i\tilde{e}_{(i)} + \sum_i\Big(-p_h\nabla_i X^h\Big)\tilde{e}_{(\bar{i})} \tag{22} $$

for any $X \in \Im^1_0(M_n)$.

Using now (19) and (22), by similar devices we can prove

$$ \left({}^C\nabla_{\gamma}{}^C\tilde{X}^{\alpha}\right) = \begin{pmatrix} \nabla_k X^i & 0 \\ -p_h\nabla_k\nabla_i X^h + \frac{1}{2}\,p_a\Big(R_{kji}{}^{a} - R_{jik}{}^{a} + R_{ikj}{}^{a}\Big)X^j & -\nabla_i X^k \end{pmatrix}. \tag{23} $$

From (21) we have the following theorem.

Theorem 1. *The vertical lift of covector field* $\omega \in \Im^0_1(M_n)$ *to* ${}^C T(M_n)$ *with metric* ${}^R\nabla$ *is parallel if and only if the given covector field* ω *is parallel with respect to* ∇.

If M_n has pseudo-Riemannian metric g, then by virtue of

$$ p_a R_{kji}{}^{a}X^j = p_a X^j\Big(R_{kjis}g^{sa}\Big) = $$

$$ = p_a X^j\Big(R_{iskj}g^{sa}\Big) = p_a X^j\Big(-R_{isjk}g^{sa}\Big) = $$

$$= p_a X^j \left(-R_{isj}{}^t g_{tk} g^{sa}\right) = -p_a g_{tk} g^{sa} \nabla_{[i} \nabla_{s]} X^t, \tag{24}$$

we have from (20) and (23) the following theorem.

Theorem 2. *When M_n has pseudo-Riemannian metric g and the Levi-Civita connection ∇ of g and ${}^C T(M_n)$ has the Riemannian extension ${}^R\nabla$ as its metric, the horizontal and the complete lifts of a vector field $X \in \Im_0^1(M_n)$ to ${}^C T(M_n)$ with the metric ${}^R\nabla$ are parallel if and only if the given vector field X is parallel with respect to the Levi-Civita connection ∇.*

3. The metric connection of ${}^R\nabla$. In Introduction and Section 2, we have given to the cotangent bundle ${}^C T(M_n)$ the metric ${}^R\nabla$ and considered the Levi-Civita connection ${}^C\nabla$ of ${}^R\nabla$. This is the unique connection which satisfies ${}^C\nabla\left({}^R\nabla\right) = 0$, and has no torsion. But there exists another connection which satisfies $\tilde{\nabla}\left({}^R\nabla\right) = 0$, and has nontrivial torsion tensor. We call this connection the metric connection of ${}^R\nabla$.

The horizontal lift ${}^H\nabla$ of the non-torsion connection ∇ to the cotangent bundle ${}^C T(M_n)$ defined by

$$\begin{gathered} {}^H\nabla_{{}^V\theta} {}^V\omega = 0, \qquad {}^H\nabla_{{}^V\theta} {}^H Y = 0, \\ {}^H\nabla_{{}^H X} {}^V\omega = {}^V(\nabla_X \omega), \qquad {}^H\nabla_{{}^H X} {}^H Y = {}^H(\nabla_X Y) \end{gathered} \tag{25}$$

for any $X, Y \in \Im_0^1(M_n)$ and $\omega, \theta \in \Im_1^0(M_n)$.

We now put ${}^H\nabla_\alpha = {}^H\nabla_{\tilde{e}_{(\alpha)}}$, where $\left\{\tilde{e}_{(\alpha)}\right\} = \left\{\tilde{e}_{(i)}, \tilde{e}_{(\bar{i})}\right\}$-adapted frame. Then taking account of ${}^C\nabla_\alpha \tilde{e}_{(\beta)} = {}^H\Gamma_{\alpha\beta}^\gamma \tilde{e}_{(\gamma)}$ and writing ${}^H\tilde{\Gamma}_{\alpha\beta}^\gamma$ for the different indices, from (25) we have

$$\begin{gathered} {}^H\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k, \qquad {}^H\tilde{\Gamma}_{i\bar{j}}^{\bar{k}} = -\Gamma_{ik}^j, \\ {}^H\tilde{\Gamma}_{\bar{i}j}^k = {}^H\tilde{\Gamma}_{i\bar{j}}^k = {}^H\tilde{\Gamma}_{\bar{i}\bar{j}}^k = {}^H\tilde{\Gamma}_{\bar{i}j}^{\bar{k}} = {}^H\tilde{\Gamma}_{ij}^{\bar{k}} = {}^H\tilde{\Gamma}_{\bar{i}\bar{j}}^{\bar{k}} = 0. \end{gathered} \tag{26}$$

Let T be the torsion tensor of the horizontal lift ${}^H\nabla$. Then T is the skew-symmetric tensor field of type (1, 2) in ${}^C T(M_n)$ determined by [2, p. 287]

$$\mathrm{T}\left({}^V\omega, {}^V\theta\right) = 0, \qquad \mathrm{T}\left({}^H X, {}^V\theta\right) = 0, \qquad \mathrm{T}\left({}^H X, {}^H Y\right) = -\gamma R(X, Y),$$

where R is curvature tensor of ∇ and $\gamma R(X, Y) = \sum_i p_h R_{kli}^h X^k Y^l \dfrac{\partial}{\partial x^{\bar{i}}}$. Thus the connection ${}^H\nabla$ has nontrivial torsion even for Levi-Civita connection ∇ determined by g, unless g is locally flat.

Using (6) – (8) and (25), we have

$$\left({}^H\nabla_{{}^V\omega} {}^R\nabla\right)\left({}^V\theta, {}^V\varepsilon\right) = 0,$$

$$\left({}^H\nabla_{{}^H X} {}^R\nabla\right)\left({}^V\theta, {}^V\varepsilon\right) = -{}^R g\left({}^V(\nabla_X \theta), {}^V\varepsilon\right) = 0,$$

$$\left({}^H\nabla_{{}^V\omega}{}^R\nabla\right)\left({}^V\theta,\ {}^HZ\right) = {}^V\omega^V(\theta(Z)) = 0,$$

$$\left({}^H\nabla_{{}^HX}{}^R\nabla\right)\left({}^V\theta,\ {}^HZ\right) = {}^HX^V(\theta(Z)) - {}^Rg\left({}^V\left({}^H\nabla_X\theta\right),\ {}^HZ\right) -$$

$$- {}^Rg\left({}^V\theta,\ {}^H\left({}^H\nabla_XZ\right)\right) = {}^V\left(X\theta(Z) - (\nabla_X\theta)Z - \theta\nabla_XZ\right) = 0,$$

$$\left({}^H\nabla_{{}^V\omega}{}^R\nabla\right)\left({}^HY,\ {}^V\varepsilon\right) = {}^V\omega^V(\varepsilon(Y)) = 0,$$

$$\left({}^H\nabla_{{}^HX}{}^R\nabla\right)\left({}^HY,\ {}^V\varepsilon\right) = {}^HX^V(\varepsilon(Y)) - {}^Rg\left({}^V\left({}^H\nabla_XY\right),\ {}^V\varepsilon\right) -$$

$$- {}^Rg\left({}^HY,\ {}^V(\nabla_X\varepsilon)\right) = {}^V\left(X\varepsilon(Y) - \varepsilon(\nabla_XY) - (\nabla_X\varepsilon)Y\right) = 0,$$

$$\left({}^H\nabla_{{}^V\omega}{}^R\nabla\right)\left({}^HY,\ {}^HZ\right) = 0,$$

$$\left({}^H\nabla_{{}^HX}{}^R\nabla\right)\left({}^HY,\ {}^HZ\right) = 0$$

for any $X, Y, Z \in \Im^1_0(M_n)$ and $\omega, \theta, \varepsilon \in \Im^0_1(M_n)$.

Let now ${}^HR$ be a curvature tensor field of ${}^H\nabla$. The curvature tensor ${}^HR$ of the metric connection ${}^H\nabla$ of ${}^R\nabla$ has components

$${}^H\tilde{R}_{\delta\gamma\beta}{}^{\alpha} = \tilde{e}_{(\delta)}{}^H\tilde{\Gamma}^{\alpha}_{\gamma\beta} - \tilde{e}_{(\gamma)}{}^H\tilde{\Gamma}^{\alpha}_{\delta\beta} + {}^H\tilde{\Gamma}^{\alpha}_{\delta\varepsilon}{}^H\tilde{\Gamma}^{\varepsilon}_{\gamma\beta} - {}^H\tilde{\Gamma}^{\alpha}_{\gamma\varepsilon}{}^H\tilde{\Gamma}^{\varepsilon}_{\delta\beta} - \Omega_{\delta\gamma}{}^{\varepsilon H}\tilde{\Gamma}^{\alpha}_{\varepsilon\beta} \tag{27}$$

with respect to the adapted frame.

Using (9), (10), (16), (26), (27) and computing components of the contracted curvature tensor field (Ricci tensor field) ${}^H\tilde{R}_{\gamma\beta} = {}^H\tilde{R}_{\alpha\gamma\beta}{}^{\alpha}$, we obtain

$$\begin{gathered} {}^H\tilde{R}_{kj} = {}^H\tilde{R}_{\alpha kj}{}^{\alpha} = {}^H\tilde{R}_{ikj}{}^{i} + {}^H\tilde{R}_{\bar{i}kj}{}^{\bar{i}} = R_{ikj}{}^{i} = R_{kj}, \\ {}^H\tilde{R}_{\bar{k}j} = 0, \quad {}^H\tilde{R}_{k\bar{j}} = 0, \quad {}^H\tilde{R}_{\bar{k}\bar{j}} = 0, \end{gathered} \tag{28}$$

where R_{kj} is the Ricci tensor field of ∇ in M_n.

For the scalar curvature of ${}^CT(M_n)$ with the metric connection ${}^H\nabla$, we have

$$\tilde{R} = {}^R\tilde{\nabla}^{\gamma\beta}\,{}^H\tilde{R}_{\gamma\beta} = 0$$

by means of (28) and

$$\left({}^R\tilde{\nabla}^{\gamma\beta}\right) = \begin{pmatrix} 0 & \delta^k_j \\ \delta^j_k & 0 \end{pmatrix}.$$

Thus we have the following theorem.

Theorem 3. *The cotangent bundle* ${}^CT(M_n)$ *with the metric connection* ${}^H\nabla$ *has vanishing scalar curvature with respect to the metric* ${}^R\nabla$.

4. Killing vector fields in $\left({}^CT(M_n),\ {}^R\nabla\right)$. In a manifold with a pseudo-Riemannian metric g, a vector field is called a Killing vector field (or, an infinitesimal isometry) if $L_Xg = 0$, where L_X is the Lie derivative.

The condition $L_X g = 0$ can be rewritten as

$$(L_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0 \tag{29}$$

for any $Y, Z \in \Im_0^1(M_n)$, where ∇ is the Levi-Civita connection of g.

We now compute the Lie derivative of the metric ${}^R\nabla$. In view of the adapted frame $\{\tilde{e}_{(\alpha)}\}$, from (29) we obtain

$${}^R\nabla\left(\left({}^C\nabla_\beta \tilde{X}^\sigma\right)\tilde{e}_{(\sigma)}, \tilde{e}_{(\gamma)}\right) + {}^R\nabla\left(\left({}^C\nabla_\gamma \tilde{X}^\sigma\right)\tilde{e}_{(\sigma)}, \tilde{e}_{(\beta)}\right) = 0$$

or

$${}^C\nabla_\beta \tilde{X}_\gamma + {}^C\nabla_\gamma \tilde{X}_\beta = 0, \tag{30}$$

where $\left(\tilde{X}_\gamma\right)$ is an associated covector field of a vector field $\left(\tilde{X}^\sigma\right)$ is given by

$$\left(\tilde{X}_\gamma\right) = \left({}^R\tilde{\nabla}_{\gamma\sigma}\tilde{X}^\sigma\right).$$

The associated covector fields of the vertical, horizontal and complete lifts to ${}^C T(M_n)$ with the metric ${}^R\nabla$, with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, are given respectively by

$$\left({}^V\tilde{X}_\gamma\right) = \left({}^R\tilde{\nabla}_{\gamma\sigma}{}^V\tilde{\omega}^\sigma\right) = (\omega_k, 0),$$

$$\left({}^H\tilde{X}_\gamma\right) = \left({}^R\tilde{\nabla}_{\gamma\sigma}{}^H\tilde{X}^\sigma\right) = (0, X_k),$$

$$\left({}^C\tilde{X}_\gamma\right) = \left({}^R\tilde{\nabla}_{\gamma\sigma}{}^C\tilde{X}^\sigma\right) = \left(-p_h\nabla_k X^h, X^k\right),$$

because of (11), (12), (13) and (22).

Using (21) and (30) we see that the Lie derivative of ${}^R\nabla$ with respect to ${}^V\omega$ has components

$$\left(L_{{}^V\omega}{}^R\nabla\right)_{\beta\gamma} = {}^C\nabla_\beta{}^V\tilde{\omega}_\gamma + {}^C\nabla_\gamma{}^V\tilde{\omega}_\beta = \begin{pmatrix} \nabla_j\omega_k + \nabla_k\omega_j & 0 \\ 0 & 0 \end{pmatrix} \tag{31}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$. We put $\omega_i = g_{ij}X^j$ for any $X \in \Im_0^1(M_n)$. Then from (31) we have the following theorem.

Theorem 4. *A necessary and sufficient condition for a vector field ${}^V\omega$ in cotangent bundle with metric ${}^R\nabla$ to be a Killing vector field is that an associated vector field is $X^i = g^{ij}\omega_j$ is Killing vector field.*

Also, using (20), (23) and (30), we see that $L_{{}^H X}{}^R\nabla$ and $L_{{}^C X}{}^R\nabla$ have respectively components

$$\left(L_{{}^H X}{}^R\nabla\right)_{\beta\gamma} = \begin{pmatrix} p_a\left(R_{ksj}{}^a + R_{jsk}{}^a\right)X^s & \nabla_k X^j + \nabla_j X^k \\ 0 & 0 \end{pmatrix},$$

$$\left(L_{{}^C X}{}^R\nabla\right)_{\beta\gamma} =$$

$$= \begin{pmatrix} -2p_h\left(\nabla_k\nabla_j X^h + \nabla_j\nabla_k X^h\right) + p_a\left(R_{ksj}{}^a + R_{jsk}{}^a\right)X^s & \nabla_k X^j + \nabla_j X^k \\ -\nabla_k X^j - \nabla_j X^k & 0 \end{pmatrix}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$. From these equations and (24) we have the following theorem.

Theorem 5. *The horizontal and complete lifts of vector fields in* M_n *to* ${}^{C}T(M_n)$ *with metric* ${}^{R}\nabla$ *is Killing if the given vector field* $X \in \Im_0^1(M_n)$ *is parallel with respect to the Levi-Civita connection* ∇ *of the metric* g *in* M_n.

5. Norden structures in ${}^{C}\boldsymbol{T(M_n)}$ **with metric** ${}^{R}\boldsymbol{\nabla}$**.** Let (M_{2n}, φ) be an almost complex manifold with almost complex structure φ. A pseudo-Riemannian metric $g \in \Im_2^0(M_{2n})$ is a Norden metric with respect to structure φ if

$$g(\varphi X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Im_0^1(M_{2n})$. Metrics of this kind have been also studied under the names: pure, anti-Hermitian and B-metrics (see, for example, [5 – 10]). If (M_{2n}, φ) is an almost complex manifold with Norden metric g, we say that (M_{2n}, φ, g) is an almost Norden manifold. If φ is integrable, we say that (M_{2n}, φ, g) is a Norden manifold.

Let (M_{2n}, φ) be an almost complex manifold with almost complex structure φ. This structure is said to be integrable if the matrix $\varphi = (\varphi_j^i)$ is reduced to the constant form in a certain holonomic natural frame in a neighborhood U_x of every point $x \in M_{2n}$. In order that the almost complex structure φ be integrable, it is necessary and sufficient that it is possible to introduce a torsion-free affine connection ∇ with respect to which the structure tensor φ is covariantly constant, i.e., $\nabla\varphi = 0$. Also, we know that the integrability of φ is equivalent to the vanishing of the Nijenhuis tensor $N_\varphi \in \Im_2^1(M_{2n})$. If φ is integrable, then φ is a complex structure and moreover M_{2n} is a $\mathbb{C}$-holomorphic manifold $X_n(\mathbb{C})$ whose transition functions are holomorphic mappings.

Let $\overset{*}{t}$ be a complex tensor field on $X_n(\mathbb{C})$. The real model of such a tensor field is a tensor field on M_{2n} of the same order that is independent of whether its vector or covector arguments is subject to the action of the affinor structure φ. Such tensor fields are said to be pure with respect to φ. They were studied by many authors (see, e.g., [10 – 14]). In particular, being applied to a $(0, q)$-tensor field ω, the purity means that for any $X_1, \ldots, X_q \in \Im_0^1(M_{2n})$ the following conditions should hold:

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \ldots = \omega(X_1, X_2, \ldots, \varphi X_q).$$

We define an operator

$$\Phi_\varphi : \Im_q^0(M_{2n}) \to \Im_{q+1}^0(M_{2n})$$

applied to the pure tensor field ω by (see [15])

$$(\Phi_\varphi \omega)(X, Y_1, Y_2, \ldots, Y_q) = (\varphi X)\big(\omega(Y_1, Y_2, \ldots, Y_q)\big) -$$
$$- X\big(\omega(\varphi Y_1, Y_2, \ldots, Y_q)\big) + \omega\big((L_{Y_1}\varphi)X, Y_2, \ldots, Y_q\big) + \ldots$$
$$\ldots + \omega\big(Y_1, Y_2, \ldots, (L_{Y_q}\varphi)X\big),$$

where L_Y denotes the Lie differentiation with respect to Y.

When φ is a complex structure on M_{2n} and the tensor field $\Phi_\varphi\omega$ vanishes, the complex tensor field $\overset{*}{\omega}$ on $X_n(\mathbb{C})$ is said to be holomorphic (see [11, 15]). Thus a holomorphic tensor field $\overset{*}{\omega}$ on $X_n(\mathbb{C})$ is realized on M_{2n} in the form of a pure tensor field ω, such that

$$(\Phi_\varphi\omega)(X, Y_1, Y_2, \ldots, Y_q) = 0$$

for any $X, Y_1, \ldots, Y_q \in \Im_0^1(M_{2n})$. Therefore such a tensor field ω on M_{2n} is also called holomorphic tensor field. When φ is an almost complex structure on M_{2n}, a tensor field ω satisfying $\Phi_\varphi\omega = 0$ is said to be almost holomorphic.

In a Norden manifold a Norden metric g is called a holomorphic if

$$(\Phi_\varphi g)(X, Y, Z) = 0$$

for any $X, Y, Z \in \Im_0^1(M_{2n})$.

If (M_{2n}, φ, g) is a Norden manifold with holomorphic Norden metric g, we say that (M_{2n}, φ, g) is a holomorphic Norden manifold.

In some aspects, holomorphic Norden manifolds are similar to Kähler manifolds. The following theorem is analogue to the next known result: An almost Hermitian manifold is Kähler if and only if the almost complex structure is parallel with respect to the Levi-Civita connection.

Theorem 6 [6] (For paracomplex version see [9]). *For an almost complex manifold with Norden metric* g, *the condition* $\Phi_\varphi g = 0$ *is equivalent to* $\nabla\varphi = 0$, *where* ∇ *is the Levi-Civita connection of* g.

A Kähler – Norden manifold can be defined as a triple (M_{2n}, φ, g) which consists of a manifold M_{2n} endowed with an almost complex structure φ and a pseudo-Riemannian metric g such that $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g and the metric g is assumed to be Nordenian. Therefore, there exist a one-to-one correspondence between Kähler – Norden manifolds and Norden manifolds with a holomorphic metric. Recall that in such a manifold, the Riemannian curvature tensor is pure and holomorphic, also the curvature scalar is locally holomorphic function (see [6, 9]).

***Remark* 1.** We know that the integrability of the almost complex structure φ is equivalent to the existing a torsion-free affine connection with respect to which the equation $\nabla\varphi = 0$ holds. Since the Levi-Civita connection ∇ of g is a torsion-free affine connection, we have: If $\Phi_\varphi g = 0$, then φ is integrable. Thus, almost Norden manifold with conditions $\Phi_\varphi g = 0$ and $N_\varphi \neq 0$, i.e., almost holomorphic Norden manifolds does not exist.

***Remark* 2.** The Levi-Civita connection of Kähler – Norden metric g coincides with the Levi-Civita connection of twin metric $G = g \circ \varphi$ (nonuniquences of the metric for the Levi-Civita connection in Kähler – Norden manifolds).

We define the horizontal lift ${}^H\varphi \in \Im_1^1\left({}^C T(M_{2n})\right)$ by [2, p.281]

$$\begin{aligned} {}^H\varphi\, {}^V\omega &= {}^V(\omega \circ \varphi), \\ {}^H\varphi\, {}^H X &= {}^H(\varphi X) \end{aligned} \tag{32}$$

for any $X \in \Im_0^1(M_{2n})$ and $\omega \in \Im_1^0(M_{2n})$. We see from (9), (10) and (32) that, the horizontal lift ${}^H\varphi$ has components of the form

$$^H\varphi = \left(\tilde{\varphi}^{\alpha}_{\beta}\right) = \begin{pmatrix} \varphi^i_j & 0 \\ 0 & \varphi^j_i \end{pmatrix} \tag{33}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$, φ^i_j being local components of φ.

It is well known that if φ an almost complex structure in M_{2n} with torsion free connection ∇, then $^H\varphi$ is an almost complex structure in $^C T(M_n)$ [2, p. 283].

From (6), (7), (8) and (32), we easily verify that

$$^R\nabla\left(^H\varphi\tilde{X}, \tilde{Y}\right) = {}^R\nabla\left(\tilde{X}, {}^H\varphi\tilde{Y}\right)$$

for any $\tilde{X} = {}^H X$ or $^V\omega$ and $\tilde{Y} = {}^H Y$ or $^V\theta$, that is, $\left(\mathrm{T}(M_n), {}^R\nabla, {}^H\varphi\right)$ is an almost Norden manifold.

We now consider covariant derivative of the almost complex structure $^H F$ with respect to Levi-Civita connection $^C\nabla$ of $^R\nabla$. Taking account of (19) and (33), we find that

$$^C\nabla_i {}^H\tilde{\varphi}^k_j = \nabla_i\varphi^k_j, \quad {}^C\nabla_i {}^H\tilde{\varphi}^{\bar{k}}_{\bar{j}} = \nabla_i\varphi^j_k,$$

$$^C\nabla_i {}^H\tilde{\varphi}^{\bar{k}}_j = \tag{34}$$

$$= \frac{1}{2} p_a\left[\left(R_{imk}{}^a - R_{mki}{}^a + R_{kim}{}^a\right)\varphi^m_j - \left(R_{ijm}{}^a - R_{jmi}{}^a + R_{mij}{}^a\right)\varphi^m_k\right]$$

the other being all zero, with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}$.

If a torsion free affine connection ∇ preserving the structure φ ($\nabla\varphi = 0$) satisfies the condition $\nabla_{\varphi X} Y = \varphi(\nabla_X Y)$ $\forall X, Y \in \mathfrak{I}^1_0(M_{2n})$, then ∇ is called a holomorphic connection [14, p. 185]. The purity of the curvature tensor field of a connection ∇ ($R_{mjk}{}^s\varphi^m_i = R_{imk}{}^s\varphi^m_j = R_{ijm}{}^s\varphi^m_k = R_{ijk}{}^m\varphi^s_m$) is a necessary and sufficient condition for its holomorphy [11, 14]. Therefore, from (34) we have the following theorem.

Theorem 7. *The cotangent bundle* $^C T(M_n)$ *is a Kähler – Norden with respect to* $^R\nabla$ *and the almost complex structure* $^H\varphi$ *if the a torsion-free connection* ∇ *is a holomorphic connection with respect to the structure* φ.

On the other hand it is well known that in a Kähler – Norden manifold the curvature tensor of Norden-metric is pure [6]. Therefore, when M_{2n} has Kähler – Norden metric g and the Levi-Civita connection ∇ of g and $^C T(M_{2n})$ has the Riemannian extension $^R\nabla$ as its metric, we have the following theorem.

Theorem 8. *The cotangent bundle* $^C T(M_{2n})$ *of a pseudo-Riemannian manifold* M_{2n} *is a Kähler – Norden with respect to* $^R\nabla$ *and* $^H\varphi$, *if* (M_{2n}, g, φ) *is a Kähler – Norden.*

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