UDC 515.145 + 515.146

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KERNEL OF MAP OF A SHIFT ALONG THE ORBITS OF CONTINUOUS FLOWS*

ЯДРО ВІДОБРАЖЕННЯ ЗСУВУ ВЗДОВЖ ОРБІТ НЕПЕРЕРВНИХ ПОТОКІВ

Let $\mathbf{F}\colon M \times \mathbb{R} \to M$ be a continuous flow on a topological manifold M. For every subset $V \subset M$ we denote by $P(V)$ the set of all continuous functions $\xi\colon V \to \mathbb{R}$ such that $\mathbf{F}(x, \xi(x)) = x$ for all $x \in V$. These functions vanish at non-periodic points of the flow, while their values at periodic points are integer multiples of the corresponding periods (in general not minimal). In this paper, the structure of $P(V)$ is described for arbitrary connected open subset $V \subset M$.

Нехай $\mathbf{F}\colon M \times \mathbb{R} \to M$ — неперервний потік на топологічному многовиді M. Для кожної підмножини $V \subset M$ позначимо через $P(V)$ множину всіх неперервних функцій $\xi\colon V \to \mathbb{R}$, що задовольняють умову $\mathbf{F}(x, \xi(x)) = x$ для всіх $x \in V$. Такі функції набувають нульового значення в неперіодичних точках потоку, а в періодичних точках їх значення є цілими кратними відповідних періодів (в загальному не мінімальними). В статті описано структуру $P(V)$ для довільної відкритої зв'язної підмножини $V \subset M$.

1. Introduction. Let $\mathbf{F}\colon M \times \mathbb{R} \to M$ be a continuous flow on a topological finite-dimensional manifold M. For $x \in M$ we will denote by o_x the orbit of x. If x is periodic, then $\mathrm{Per}(x)$ is the period of x. The set of fixed points of $\mathbf{F}$ will be denoted by Σ.

For each subset $V \subset M$ define the following map

$$\varphi_V\colon\ \mathcal{C}(M, \mathbb{R}) \to \mathcal{C}(V, M), \qquad \varphi_V(\alpha)(x) = \mathbf{F}(x, \alpha(x)),$$

for $\alpha \in \mathcal{C}(V, \mathbb{R})$ and $x \in V$. We will call φ_V the *shift map* along the orbits of $\mathbf{F}$. It was used by the author for study of homotopy types of certain infinite-dimensional functional spaces, see, e.g., [1 – 6].

Let $i_V\colon V \subset M$ be the inclusion map. Then the following set

$$P(V) = \varphi_V^{-1}(i_V)$$

will be called the *kernel* of φ_V.

Thus a continuous function $\xi\colon V \to \mathbb{R}$ belongs to $P(V)$ iff

$$\mathbf{F}(x, \xi(x)) = x \quad \forall x \in V. \tag{1.1}$$

In this case we will say that ξ is a *period* function or simply a *P-function* for $\mathbf{F}$ on V.

The aim of this paper is to give a description of $P(V)$ for open connected subsets $V \subset M$ with respect to a continuous flow on a topological manifold M (Theorem 1.1). Such a description was given in [1] (Theorem 12) for $\mathcal{C}^\infty$ flows. It turns out that both descriptions almost coincide. Our methods are based on well-known theorems of M. Newman about actions of finite groups.

The following easy lemma explains the term *P-function*. The proof is the same as in [1] (Lemmas 5 and 7) and we leave it for the reader.

*This research is done within the program of National Academy of Sciences of Ukraine "Modern methods of investigation of mathematical models in the problems of natural sciences", research No. 0107U002333.

Lemma 1.1 [1] (Lemmas 5 and 7)**.** *For any subset $V \subset M$ the set $P(V)$ is a group with respect to the point-wise addition.*

Let $x \in V$ and $\xi \in P(V)$. Then ξ is locally constant on $o_x \cap V$. In particular, if x is non-periodic, then $\xi|_{o_x \cap V} = 0$. Suppose x is periodic, and let ω be some path component of $o_x \cap V$. Then $\xi = n_\omega \cdot \mathrm{Per}(o_x)$ for some $n_\omega \in \mathbb{Z}$ depending on ω.

It is not true that any P-function on any subset $V \subset M$ is constant on all of $o_x \cap V$ for each $x \in V$, see Example 1.1 below. Therefore we give the following definition.

Definition 1.1. *A P-function $\xi \colon V \to \mathbb{R}$ will be called regular if ξ is constant on $o_x \cap V$ for each $x \in V$.*

Denote by $RP(V)$ the set of all regular P-functions on V. Then $RP(V)$ is a subgroup of $P(V)$.

***Remark* 1.1.** If for any periodic orbit o the intersection $o \cap V$ is either empty or connected, e.g. in the case when V is $\mathbf{F}$-invariant, then any P-function on V is regular.

The following theorem extends [1] (Theorem 12) for continuous case.

Theorem 1.1. *Let M be a finite-dimensional topological manifold possibly non-compact and with or without boundary, $\mathbf{F} \colon M \times \mathbb{R} \to M$ be a flow, and $V \subset M$ be an open, connected set.*

(A) *If* $\mathrm{Int}(\Sigma) \cap V \neq \varnothing$, *then*

$$P(V) = \big\{\xi \in \mathcal{C}(V, \mathbb{R}) \colon \ \xi|_{V \setminus \mathrm{Int}(\Sigma)} = 0\big\}.$$

(B) *Suppose* $\mathrm{Int}(\Sigma) \cap V = \varnothing$. *Then one of the following possibilities is realized*: *either*

$$P(V) = \{0\}$$

or

$$P(V) = \{n\theta\}_{n \in \mathbb{Z}}$$

for some continuous function $\theta \colon V \to \mathbb{R}$ having the following properties:

(1) *$\theta > 0$ on $V \setminus \Sigma$, so this set consists of periodic points only.*

(2) *There exists an open and everywhere dense subset $Q \subset V$ such that $\theta(x) = \mathrm{Per}(x)$ for all $x \in Q$.*

(3) *θ is a regular P-function.*

(4) *Denote $U = \mathbf{F}(V \times \mathbb{R})$. Then θ extends to a P-function on U and there is a circle action $\mathbf{G} \colon U \times S^1 \to U$ defined by $\mathbf{G}(x, t) = \mathbf{F}(x, t\theta(x))$, $x \in U$, $t \in S^1 = \mathbb{R}/\mathbb{Z}$. The orbits of this action coincide with the ones of $\mathbf{F}$.*

In particular, in all the cases $RP(V) = P(V)$.

Theorem 1.1 will be proved in Section 3.

The following simple example illustrates necessity of conditions of Theorem 1.1. It shows that on non-open or disconnected sets $V \subset M$ there may exist non-regular P-functions and that P-functions for continuous flows may vanish at fixed points.

***Example* 1.1.** Let $\mathbf{F} \colon \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ be a continuous flow on the complex plane $\mathbb{C}$ defined by

$$\mathbf{F}(z, t) = \begin{cases} e^{2\pi i\, t/|z|^2}\, z, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

The orbits of $\mathbf{F}$ are the origin $0 \in \mathbb{C}$ and the concentric circles centered at 0. Then $\theta = |z|^2$ is a P-function on $\mathbb{C}$ and

$$RP(\mathbb{C}) = P(\mathbb{C}) = \{n\theta\}_{n\in\mathbb{Z}}.$$

Also notice that $\theta(0) = 0$ and $\theta > 0$ on $\mathbb{C} \setminus 0$. This agrees with (1) of Theorem 1.1 and shows that non-zero P-functions may vanish at fixed points of flows.

Let V_i, $i = 1, 2, 3$, be the corresponding subset in $\mathbb{C}$ shown in Fig. 1.1. Thus V_1 is an open segment, say $(-1, 1)$, on the real axis, V_2 is a union of two closed triangles with common vertex at the origin 0, and V_3 is union of a triangle with a segment $(-1, 0]$ of the real axis intersecting at the origin. In particular, $\mathrm{Int}(V_1) = \varnothing$, $\mathrm{Int}(V_2)$ is not

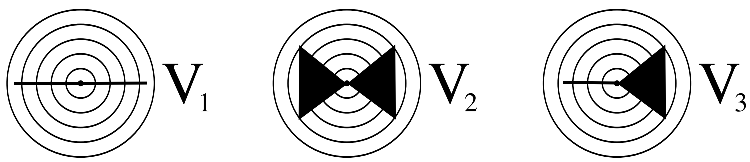

Fig. 1.1

connected, and $\mathrm{Int}(V_3)$ is not dense in V. For any pair $m, n \in \mathbb{Z}$ define the function $\xi_{m,n}\colon V_i \to \mathbb{R}$ by

$$\xi_{m,n}(z) = \begin{cases} -m|z|, & \Re(z) \le 0, \\ n|z|, & \Re(z) > 0. \end{cases}$$

Evidently, $P(V_i) = \{\xi_{m,n}\}_{m,n\in\mathbb{Z}}$, while $RP(V_i) = \{\xi_{m,m}\}_{m\in\mathbb{Z}}$. Thus not every P-function is regular.

Structure of the paper. In next section we describe certain properties of P-function for continuous flows: local uniqueness, local regularity, and continuity of extensions of regular P-functions. We also deduce from well-known M. Newman's theorem a sufficient condition for divisibility of regular P-functions by integers in $P(V)$. These results will be used in Section 3 for the proof of Theorem 1.1.

2. Properties of P-functions.

Lemma 2.1. *Let $z \in M$. Suppose there exists a sequence of periodic points $\{x_i\}_{i\in\mathbb{N}}$ converging to z and such that $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = 0$. Then $z \in \Sigma$.*

Proof. Suppose $z \notin \Sigma$, so there exists $\tau > 0$ such that $z \neq \mathbf{F}_\tau(z)$. Let U be a neighbourhood of z such that

$$U \cap \mathbf{F}_\tau(U) = \varnothing. \tag{2.1}$$

Since $\mathbf{F}(z, 0) = z$, there exists $\varepsilon > 0$ and a neighbourhood W of z such that $\mathbf{F}(W \times [0, \varepsilon]) \subset U$. Then we can find $x_i \in W$ with $\mathrm{Per}(x_i) < \varepsilon$. Hence

$$\mathbf{F}_\tau(x_i) \in \mathbf{F}_\tau(U).$$

On the other hand,

$$\mathbf{F}_\tau(x_i) \in o_{x_i} = \mathbf{F}(x_i, [0, \mathrm{Per}(x_i)]) \subset \mathbf{F}(W \times [0, \varepsilon]) \subset U,$$

which contradicts to (2.1).

Lemma 2.2 (Local uniqueness of P-functions, c.f. [1], Corollary 8)**.** *Let $V \subset M$ be any subset, $z \in V \setminus \Sigma$ and $\xi \in P(V)$. If $\xi(z) = 0$, then $\xi = 0$ on some neighbourhood of z in V.*

Proof. Suppose ξ is not identically zero on any neighbourhood of z in $V \setminus \Sigma$. Then there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$ converging to z and such that $\xi(x_i) \neq 0$. Hence every x_i is periodic and $\xi(x_i) = n_i \mathrm{Per}(x_i)$ for some $n_i \in \mathbb{Z} \setminus \{0\}$. By continuity of ξ we get

$$0 = \lim_{i\to\infty} \xi(x_i) = \lim_{i\to\infty} n_i \mathrm{Per}(x_i).$$

Since $|n_i| \geq 1$, it follows that $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = 0$, whence by Lemma 2.1 $z \in \Sigma$, which contradicts to the assumption.

Lemma 2.3 (Local regularity of P-functions on open sets). *Let $V \subset M$ be an open subset and $\xi \in P(V)$. Then for each $z \in V$ there exists a neighbourhood $W \subset V$ such that the restriction $\xi|_W$ is regular.*

Proof. Suppose ξ is not regular on arbitrary small neighbourhood of z. Then we can find two sequences $\{x_i\}_{i\in\mathbb{N}}$ and $\{y_i\}_{i\in\mathbb{N}}$ converging to z such that $y_i = \mathbf{F}(x_i, \tau_i)$ for some $\tau_i \in \mathbb{R}$ and $\xi(x_i) < \xi(y_i)$ for all $i \in \mathbb{N}$.

It follows that x_i and y_i are periodic. Otherwise, by Lemma 1.1, we would have $\xi(x_i) = \xi(y_i)$. Hence $0 < \xi(y_i) - \xi(x_i) = n_i \mathrm{Per}(x_i)$ for some $n_i \in \mathbb{Z} \setminus \{0\}$.

We claim that $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = 0$. Indeed, take any $\varepsilon > 0$. Then there is a neighbourhood W of z such that $|\xi(y) - \xi(x)| < \varepsilon$ for all $x, y \in W$. Let $N > 0$ be such that $x_i, y_i \in W$ for $i > N$,

$$\mathrm{Per}(x_i) \leq n_i \mathrm{Per}(x_i) = \xi(y_i) - \xi(x_i) < \varepsilon, \quad i > N.$$

This implies $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = 0$, whence, by Lemma 2.1, $z \in \Sigma$. But in this case there exists a neighbourhood W_1 of z and $\varepsilon > 0$ such that $\mathbf{F}(W_1 \times [0, \varepsilon]) \subset V$. Take $x_i \in W_1$ such that $\mathrm{Per}(x_i) < \varepsilon$, then

$$o_{x_i} = \mathbf{F}(x_i \times [0, \mathrm{Per}(x_i)) \subset \mathbf{F}(W_1 \times [0, \varepsilon]) \subset V.$$

In other words $o_{x_i} \cap V = o_{x_i}$ is connected, whence by Lemma 1.1 ξ is constant on o_{x_i}. Therefore $\xi(x_i) = \xi(y_i)$ which contradicts to the assumption.

Lemma 2.4 (Continuity of extensions of regular P-functions). *Let $V \subset M$ be an open subset and $\xi \in RP(V)$ be a regular P-function on V. Put $U = \mathbf{F}(V \times \mathbb{R})$. Then ξ extends to a P-function $\widetilde{\xi}$ on all of U.*

If M is a $\mathcal{C}^r$ manifold, $\mathbf{F}$ is $\mathcal{C}^r$ on $V \times \mathbb{R}$, and ξ is $\mathcal{C}^r$ on V, then $\widetilde{\xi}$ is $\mathcal{C}^r$ on U.

Proof. The definition of $\widetilde{\xi}$ is evident: if $y \in U$, so $y = \mathbf{F}(x, \tau)$ for some $(x, \tau) \in V \times \mathbb{R}$, then we put $\tilde{\xi}(y) = \xi(x)$. Since ξ is regular, this definition does not depend on a particular choice of such (x, τ).

It remains to prove continuity of $\widetilde{\xi}$ on U. Let $y = \mathbf{F}(x, t) \in U$ for some $(x, t) \in V \times \mathbb{R}$. Since V is open, there exists a neighbourhood W of y in U such that $\mathbf{F}_{-t}(W) \subset V$. Then $\widetilde{\xi}$ can be defined on W by $\widetilde{\xi}(z) = \xi \circ \mathbf{F}_{-t}(z)$ for all $z \in W$. This shows continuity of $\widetilde{\xi}$ on W.

Moreover, if M is a $\mathcal{C}^r$ manifold, ξ and $\mathbf{F}$ are $\mathcal{C}^r$, then so is $\widetilde{\xi}$.

In order to formulate the last preliminary result we recall the following well-known theorem of M. Newman:

Theorem 2.1 (M. Newman [7], see also [8–10]). *If a compact Lie group effectively acts on a connected manifold M, then the set Σ of fixed points of this action is nowhere dense in M and, by* [9], *it does not separate M.*

Lemma 2.5 (Condition of divisibility by integers)**.** *Let $V \subset M$ be a connected open subset and $\xi\colon V \to \mathbb{R}$ be a regular P-function. Suppose that there exist an integer $p \geq 2$ and a non-empty open subset $W \subset V$ such that $\mathbf{F}(x, \xi(x)/p) = x$ for all $x \in W$, so the restriction of ξ/p to W is a P-function. Then ξ/p is also a P-function on all of V.*

Proof. By Lemma 2.4 we can assume that V is $\mathbf{F}$-invariant. Moreover, it suffices to consider the case when p is a prime. Define the following map $h\colon V \to V$ by $h(x) = \mathbf{F}(x, \xi(x)/p)$. Since ξ is constant along orbits of $\mathbf{F}$, it follows that $\xi(h(x)) = \xi(x)$, whence

$$h \circ h(x) = \mathbf{F}\big(h(x),\ \xi(h(x))/p\big) = \mathbf{F}(\mathbf{F}\big(x, \xi(x)/p, \xi(x)/p\big) = \mathbf{F}\big(x, 2\xi(x)/p\big).$$

Similarly,

$$h^k(x) = \mathbf{F}\big(x, k\xi(x)/p\big), \quad k \in \mathbb{N}.$$

In particular, we obtain that $h^p = id_V$, and thus h yields a $\mathbb{Z}_p$-action on V. But by assumption this action is trivial on the non-empty open set W. Then by M. Newman's Theorem 2.1 the action is trivial on all of V, so ξ/p is a P-function on V.

***Corollary* 2.1.** *Let ξ be a regular P-function on a connected open subset $V \subset M$.*

(i) *If $V \cap \mathrm{Int}(\Sigma) \neq \varnothing$, then $\xi = 0$ on $V \setminus \mathrm{Int}(\Sigma)$.*

(ii) *If $V \cap \mathrm{Int}(\Sigma) = \varnothing$ and $\xi = 0$ on some open non-empty subset $W \subset V$, then $\xi = 0$ on all of V.*

Proof. Evidently, it suffices to show that in both cases $\xi = 0$ on $V \setminus \Sigma$.

In the case (i) put $W = V \cap \mathrm{Int}(\Sigma)$.

Let p be any prime. Then in both cases $\mathbf{F}(y, \xi(y)/p) = y$ for all $y \in W$, where W is a non-empty open set. Hence by Lemma 2.5 $\mathbf{F}(y, \xi(y)/p) = y$ for all $y \in V$, that is ξ/p is a P-function on V. Thus if $\xi(x) = n\mathrm{Per}(x) \neq 0$ for some $x \in V \setminus \Sigma$ and $n \in \mathbb{Z}$, then n is divided by p. Since p is arbitrary, we get $n = 0$.

3. Proof of Theorem 1.1. (A). Suppose $\mathrm{Int}(\Sigma) \cap V \neq \varnothing$. We should prove that the following set

$$P' = \big\{\xi \in \mathcal{C}(V, \mathbb{R})\colon\ \xi|_{V \setminus \mathrm{Int}(\Xi)} = 0\big\}$$

coincides with $P(V)$. Evidently, $P' \subset P(V)$.

Conversely, let $\xi \in P(V)$. We claim that *for every connected component T of $V \setminus \overline{\mathrm{Int}(\Sigma)}$ there exists $z \in T$ such that $\xi(z) = 0$*. By Lemma 2.2 this will imply that $\xi|_T = 0$. Since T is arbitrary we will get that $\xi = 0$ on all of $V \setminus \mathrm{Int}(\Sigma)$ and, in particular, that ξ is a regular P-function.

As V is connected, the following set is non-empty, see Fig. 2.1:

$$B := \overline{T} \cap V \cap \left(\overline{\mathrm{Int}(\Sigma)} \setminus \mathrm{Int}(\Sigma)\right) \neq \varnothing.$$

Let $x \in B \subset V = \mathrm{Int}(V)$. Then by Lemma 2.3 there exists an open connected neighbourhood W such that $\xi|_W$ is a regular P-function. Then we have that $W \cap \mathrm{Int}(\Sigma) \neq \varnothing$ and $W \cap T \neq \varnothing$ as well. Since ξ is regular on W, it follows from (i) of Corollary 2.1 that $\xi = 0$ on $W \setminus \mathrm{Int}(\Sigma)$ and, in particular, on $W \cap T$.

(B). Suppose that $\mathrm{Int}(\Sigma) \cap V = \varnothing$ and $P(V) \neq \{0\}$, so there exists $\xi \in P(V)$ which is not identically zero on V. We have to show that $P(V) = \{n\theta\}_{n \in \mathbb{Z}}$ for some P-function $\theta\colon V \to \mathbb{R}$ satisfying (1)–(4).

Denote by Y the subset of V consisting of all points x having one of the following two properties:

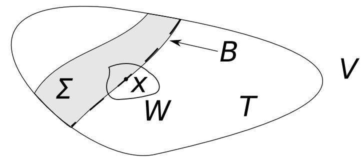

Fig. 2.1

(L$_1$) $x \in V \setminus \Sigma$ and $\xi(x) = 0$;

(L$_2$) $x \in V \cap \Sigma$ and there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$ converging to x and such that $\xi(x_i) = 0$ for all $i \in \mathbb{N}$.

Evidently, $\xi = 0$ on Y.

Lemma 3.1. *Y is open and closed in V. Hence if V is connected and $\xi(x) = 0$ for some $x \in V \setminus \Sigma$, then $\xi = 0$ on all of V.*

Proof. *Y is open.* Let $x \in Y$. We will show that there exists an open neighbourhood W of x such that $W \subset Y$.

If $x \in V \setminus \Sigma$, then, by Lemma 2.2, $\xi = 0$ on some neighbourhood $W \subset V \setminus \Sigma$ of x. Hence, by (L$_1$), $W \subset Y$.

Suppose $x \in \Sigma \cap V \subset V = \mathrm{Int}(V)$. Then by Lemma 2.3 there exists an open connected neighbourhood W_x of x such that $\xi|_{W_x}$ is regular. We claim that $W_x \subset Y$.

First we show that $\xi = 0$ on W_x. Indeed, by (L$_2$) there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset \subset V \setminus \Sigma$ converging to x and such that $\xi(x_i) = 0$ for all $i \in \mathbb{N}$. In particular, $x_i \in W_x$ for some $i \in \mathbb{N}$. Let C be the connected component of $W_x \setminus \Sigma$ containing x_i. Then $\xi = 0$ on an open set $C \subset W_x$, whence, by (ii) of Corollary 2.1, $\xi = 0$ on W_x.

Therefore $W_x \setminus \Sigma \subset Y$. Let $y \in W_x \cap \Sigma$. Since $W_x \cap \Sigma$ is nowhere dense in W_x, there exists a sequence $\{y_i\}_{i\in\mathbb{N}} \subset W_x \setminus \Sigma$ converging to y. But then $\xi(y_i) = 0$, whence, by (L$_2$), $y \in Y$ as well.

Y is closed. Let $\{x_i\}_{i\in\mathbb{N}} \subset Y$ be a sequence converging to some $x \in V$. We have to show that $x \in Y$. Since $\xi(x_i) = 0$, we have $\xi(x) = 0$ as well.

If $x \in V \setminus \Sigma$, then by (L$_1$) $x \in Y$.

Suppose $x \in V \cap \Sigma$. Then we can assume that either $\{x_i\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$ or $\{x_i\}_{i\in\mathbb{N}} \subset \subset V \cap \Sigma$. In the first case $x \in Y$ by (L$_2$).

Suppose $\{x_i\}_{i\in\mathbb{N}} \subset V \cap \Sigma$. Since $x_i \in Y$, it follows from (L$_2$) for x_i that there exists a sequence $\{y_i^j\}_{j\in\mathbb{N}} \subset V \setminus \Sigma$ converging to x_i and such that $\xi(y_i^j) = 0$. Then for each $i \in \mathbb{N}$ we can find $n(i) \in \mathbb{N}$ such that the *diagonal* sequence $\{y_i^{n(i)}\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$ converges to x, and satisfies $\xi(y_i^{n(i)}) = 0$. Hence, by (L$_2$), $x \in Y$.

The lemma is proved.

Thus we can assume that $\xi \neq 0$ on $V \setminus \Sigma$. In particular, all points in $V \setminus \Sigma$ are periodic.

Take any $x \in V \setminus \Sigma$ and consider the following *homomorphism*

$$e_x\colon\ P(V) \to \mathbb{Z}, \quad e_x(\nu) = \nu(x)/\mathrm{Per}(x),$$

for $\nu \in P(V)$. If $\nu(x) = 0$, then, as noted above, $\nu = 0$ on all of V, whence e_x *is a monomorphism*. Moreover, $e_x(\xi) = \xi(x) \neq 0$, whence e_x yields an isomorphism of $P(V)$ onto a non-zero subgroup $k\mathbb{Z}$ of $\mathbb{Z}$ for some $k \in \mathbb{N}$. Put $\theta = e_x^{-1}(k)$. Then $P(V) = \{n\theta\}_{n\in\mathbb{Z}}$.

It remains to verify properties of θ.

(2) $\Rightarrow$ (1). We have that $\theta(x) = \mathrm{Per}(x) > 0$ on an open and everywhere dense subset $Q \subset V$, whence $\theta \geq 0$ on V. On the other hand, by Lemma 3.1, $\theta \neq 0$ on $V \setminus \Sigma$, whence $\theta > 0$ on $V \setminus \Sigma$.

(2) $\Rightarrow$ (3). We have to show that θ is regular, that is

$$\theta(x) = \theta(\mathbf{F}_\tau(x))$$

for any $x \in V \setminus \Sigma$ and $\tau \in \mathbb{R}$ such that $\mathbf{F}_\tau(x) \in V$.

First notice that for any open subsets $A, B \subset M$ we have that

$$\overline{A \cap \overline{B}} = \overline{\overline{A} \cap B} = \overline{A \cap B}. \tag{3.1}$$

Since Q is open and everywhere dense in V, it follows that

$$\mathbf{F}_\tau(x) \in V \cap \mathbf{F}_\tau(V) \subset \overline{\overline{Q} \cap \mathbf{F}_\tau(V)} \overset{(3.1)}{=}$$

$$\overset{(3.1)}{=} \overline{Q \cap \overline{\mathbf{F}_\tau(V)}} = \overline{Q \cap \overline{\mathbf{F}_\tau(Q)}} \overset{(3.1)}{=} \overline{Q \cap \mathbf{F}_\tau(Q)}.$$

In other words, there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset Q$ converging to x and such that $\{\mathbf{F}_\tau(x_i)\}_{i\in\mathbb{N}} \subset Q$. Then $\theta(\mathbf{F}_\tau(x_i)) = \theta(x_i) = \mathrm{Per}(x_i)$. Whence

$$\theta(\mathbf{F}_\tau(x)) = \lim_{i\to\infty} \theta(\mathbf{F}_\tau(x_i)) = \lim_{i\to\infty} \theta(x_i) = \theta(x).$$

(3) $\Rightarrow$ (4). See Lemma 2.4.

(2) The proof consists of the following three statements.

Claim 3.1. *Let $x \in V \setminus \Sigma$. Then there exist an open connected neighbourhood W_x of x in V, a regular P-function $\theta_x \in P(W_x)$, a number $m_x \in \mathbb{Z} \setminus \{0\}$, and an open and everywhere dense subset $Q_x \subset W_x$ consisting of periodic points such that*

(a) $P(W_x) = \{m\theta_x\}_{m\in\mathbb{Z}}$,

(b) $\theta = m_x\theta_x$ *on* W_x,

(c) $\theta_x(y) = \mathrm{Per}(y)$ *for all* $y \in Q_x$.

Proof. By Lemma 2.3 there exists an open connected neighbourhood W_x of x such that $\overline{W_x} \subset V \setminus \Sigma$ and $\theta|_{W_x}$ is regular. Notice that if we decrease W_x, then the restriction of θ to W_x remains regular. Therefore we can additionally assume that there exists $\varepsilon \in (0, \mathrm{Per}(x))$ such that

(i) $\theta(y) < \theta(x) + \varepsilon$ for all $y \in \overline{W_x}$;

(ii) $\mathrm{Per}(x) < \mathrm{Per}(y) + \varepsilon$ for all $y \in \overline{W_x}$;

(iii) there is $N > 0$ such that $n_y := \theta(y)/\mathrm{Per}(y) < N$ for all $y \in \overline{W_x}$.

Indeed, (i) follows from continuity of θ, and (ii) from *lower semicontinuity* of Per, c.f. [11].

More precisely, suppose (ii) fails. Then there exists a sequence $\{x_i\}_{i\in\mathbb{N}} \subset V \setminus \Sigma$ converging to x and such that $\mathrm{Per}(x) \geq \mathrm{Per}(x_i) + \varepsilon$. In particular, periods of x_i are bounded above and we can assume that $\lim\limits_{i\to\infty} \mathrm{Per}(x_i) = \tau < \infty$ for some τ. Then

$$\mathrm{Per}(x) \geq \tau + \varepsilon > \tau. \tag{3.2}$$

But $\mathbf{F}(x, \tau) = \lim\limits_{i\to\infty} \mathbf{F}(x_i, \mathrm{Per}(x_i)) = x$, so $\tau = n\mathrm{Per}(x) \geq \mathrm{Per}(x)$ for some $n \in \mathbb{N}$, which contradicts to (3.2). This proves (ii).

To establish (iii) notice that it follows from (i) and (ii) that

$$n_y(\mathrm{Per}(x)-\varepsilon) < n_y\mathrm{Per}(y) = \theta(y) < \theta(x)+\varepsilon,$$

whence

$$N := \frac{\theta(x)+\varepsilon}{\mathrm{Per}(x)-\varepsilon} > n_y.$$

This proves (iii).

Consider the group $P(W_x)$. As W_x is open and connected, we have that $P(W_x) = \{m\theta_x\}_{m\in\mathbb{Z}}$ for some $\theta_x \in \mathcal{C}(W,\mathbb{R})$. By assumption, θ is a P-function on W_x, whence $\theta|_{W_x} = m_x\theta_x$ for some $m_x \in \mathbb{Z}\setminus\{0\}$.

To construct Q_x notice that for each $y \in W_x\setminus\Sigma$ there exists a unique $n_y \in \mathbb{Z}$ such that $\theta_x(y) = n_y\mathrm{Per}(y)$. For every $n \in \mathbb{N}$ denote by T_n the subset of W_x consisting of all y such that n_y is divided by n. Since the values n_y are bounded above, it follows that T_n is non-empty only for finitely many n. Also notice that

$$W_x\setminus\Sigma = \bigcup_{n=1}^{N} T_n.$$

We claim that $\overline{T_n}$ *is nowhere dense for* $n \geq 2$. Indeed, suppose $\mathrm{Int}(\overline{T_n}) \neq \varnothing$. Then θ_x/n is a regular P-function on $\mathrm{Int}(\overline{T_n})$ and therefore, by Lemma 2.5, on all of W_x. However this is possible only for $n = 1$ as θ_x generates $P(W_x)$. Thus the subset $Q_x := \mathrm{Int}(\overline{T_1})\cap W_x$ is open and everywhere dense in W and $\theta(y) = \mathrm{Per}(y)$ for all $y \in Q_x$.

Claim 3.1 is proved.

Claim 3.2. *Let $x, y \in V\setminus\Sigma$. Then $\theta_x = \theta_y$ on $W_x\cap W_y$ and $m_x = m_y$.*

Proof. Indeed, since Q_x (Q_y) is open and everywhere dense in W_x (W_y), it follows that $Q_x\cap Q_y$ is open and everywhere dense in $W_x\cap W_y$. Moreover, for each $z \in Q_x\cap Q_y$ we have that $\theta_x(z) = \theta_y(z) = \mathrm{Per}(z)$. Then by continuity $\theta_x = \theta_y$ on $W_x\cap W_y$.

In particular, if $z \in Q_x\cap Q_y$, then $\theta(z) = m_x\mathrm{Per}(z) = m_y\mathrm{Per}(z)$, whence $m_x = m_y$.

Claim 3.2 is proved.

Let T be a connected component of $V\setminus\Sigma$. Then by Claim 3.2 m_x is the same for all $x \in T$ and we denote their common value by m_T. It also follows that the functions $\{\theta_x\}_{x\in T}$ define a continuous function $\theta_T\colon T \to \mathbb{R}$ such that $\theta|_T = m_T\,\theta_T$. Thus if we put $Q_T = \bigcup_{x\in T} Q_x$, then Q_T is open and everywhere dense in T and $\theta_T(y) = \mathrm{Per}(y)$ for all $y \in Q_T$.

Claim 3.3. *Let S and T be any connected components of $V\setminus\Sigma$ such that $\overline{S}\cap\overline{T} \neq \varnothing$. Then $m_S = m_T$.*

Proof. We can assume that $T \neq S$. Let $x \in \overline{S}\cap\overline{T} \subset V\cap\Sigma$ and W_x be an open, connected neighbourhood of x in V such that $\theta|_{W_x}$ is a regular P-function on W_x. Notice that $\theta_S = \theta/m_S$ is a regular P-function on the non-empty open set $W_x\cap S$, whence, by Lemma 2.5, θ/m_S is a P-function on all of W_x.

If $x \in Q_T\cap W_x$, then $\theta(x) = m_T\theta_T(x) = m_T\mathrm{Per}(x)$, therefore m_T is divided by m_S. By symmetry m_S is divided by m_T as well, whence $m_S = m_T$.

Claim 3.3 is proved.

Since V is connected, it follows from Claim 3.3 that the number m_T is the same for all connected components T of $V \setminus \Sigma$. Denote the common value of these numbers by m. Then θ/m is continuous on V and $\mathbf{F}(x, \theta(x)/m) = x$ for all $x \in V$. Since θ generates $P(V)$, we obtain that $m = 1$.

Let Q be the union of all Q_T, where T runs over the set of all connected components of $V \setminus \Sigma$. Since for every such component T we have that $\theta = m\theta_T = \theta_T$ on T, it follows that $\theta(x) = \mathrm{Per}(x)$ for all $x \in Q_T$.

Theorem 1.1 is proved.

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Received 22.02.10