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ON A SPHERICAL CODE IN THE SPACE OF SPHERICAL HARMONICS

ПРО СФЕРИЧНИЙ КОД У ПРОСТОРІ СФЕРИЧНИХ ГАРМОНІК

We propose a new method for the construction of new “nice” configurations of vectors on the unit sphere S^d with the use of spaces of spherical harmonics.

Запропоновано новий метод для побудови нових „гарних” конфігурацій векторів на одиничній сфері S^d з використанням просторів сферичних гармонік.

1. Introduction. This paper is inspired by classical book J. H. Conway and N. J. A. Sloane [1] and recent paper of H. Cohn and A. Kumar [2]. The exceptional arrangement of points on the spheres are discussed there. Especially interesting are constructions coming from well known E_8 lattice and Leech lattice Λ_{24}. The main idea of the paper is to use these arrangements for construction new good arrangements in the spaces of spherical harmonics $\mathcal{H}_k^d$. Recently we have use dramatically the calculations in these spaces to obtain new asymptotic existence bounds for spherical designs, see [3]. Below we need a few facts on spherical harmonics. Let Δ be the Laplace operator in $\mathbb{R}^{d+1}$

$$\Delta = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2}.$$

We say that a polynomial P in $\mathbb{R}^{d+1}$ is harmonic if $\Delta P = 0$. For integer $k \geq 1$ the restriction to S^d of a homogeneous harmonic polynomial of degree k is called a spherical harmonic of degree k. The vector space of all spherical harmonics of degree k will be denoted by $\mathcal{H}_k^d$ (see [4] for details). The dimension of $\mathcal{H}_k^d$ is given by

$$\dim \mathcal{H}_k^d = \frac{2k+d-1}{k+d-1}\binom{d+k-1}{k}.$$

Consider usual inner product in $\mathcal{H}_k^d$

$$\langle P, Q \rangle := \int_{S^d} P(x)\, Q(x)\, d\mu_d(x),$$

where $\mu_d(x)$ is normalized Lebesgue measure on the unit sphere S^d. Now, for each point $x \in S^d$ there exists a unique polynomial $P_x \in \mathcal{H}_k^d$ such that

$$\langle P_x, Q \rangle = Q(x) \quad \text{for all} \quad Q \in \mathcal{H}_k^d.$$

It is well known that $P_x(y) = g((x, y))$, where g is a corresponding Gegenbauer polynomial. Let G_x be normalized polynomial P_x, that is $G_x = P_x / g(1)^{1/2}$. Note that $\langle G_{x_1}, G_{x_2} \rangle = g((x_1, x_2)) / g(1)$. So, if we have some arrangement $X = \{x_1, \ldots, x_N\}$ on S^d with known distribution of inner products (x_i, x_j), then, for each k, we have corresponding set $G_X = \{G_{x_1}, \ldots, G_{x_N}\}$ in $\mathcal{H}_k^d$, also with known distribution of inner products. Using this construction we will obtain in the next section the optimal antipodal spherical $(35, 240, 1/7)$ code from minimal vectors of E_8 lattice. Here is the definition.

Definition 1. *An antipodal set* $X = \{x_1, \ldots, x_N\}$ *on* S^d *is called antipodal spherical* $(d+1, N, a)$ *code, if* $|(x_i, x_j)| \leq a$, *for some* $a > 0$ *and for all* x_i, $x_j \in X$, $i \neq j$, *which are not antipodal. Such code is called optimal if for any antipodal set* $Y = \{y_1, \ldots, y_N\}$ *on* S^d *there exists* y_i, $y_j \in Y$, $i \neq j$, *which are not antipodal and* $|(y_i, y_j)| \geq a$.

In the other words, antipodal spherical $(d+1, N, a)$ code is optimal if a is a minimal possible number for fixed N, d.

Definition 2. *An antipodal set* $X = \{x_1, \ldots, x_N\}$ *on* S^d *forms spherical* 3-*design if and only if*

$$\frac{1}{N^2} \sum_{i,j=1}^{N} (x_i, x_j)^2 = \frac{1}{d+1}.$$

Note, that for all $x_1, \ldots, x_N \in S^d$ the following inequality hold

$$\frac{1}{N^2} \sum_{i,j=1}^{N} (x_i, x_j)^2 \geq \frac{1}{d+1}.$$

Another equivalent definition is the following:

The set of points $x_1, \ldots, x_N \in S^d$ is called a spherical 3-design if

$$\int_{S^d} P(x)\, d\mu_d(x) = \frac{1}{N} \sum_{i=1}^{N} P(x_i)$$

for all algebraic polynomials in $d+1$ variables and of total degree at most 3, where μ_d is normalized Lebesgue measure on S^d.

Thus we will prove the following theorem.

Theorem 1. *There exists an optimal antipodal spherical* $(35, 240, 1/7)$ *code, those vectors form spherical* 3-*design.*

2. Construction and the proof of optimality. ***Proof of Theorem* 1.** Let $X = \{x_1, \ldots, x_{120}\}$ be any subset of 240 normalized minimal vectors of E_8 lattice, such that no pair of antipodal vectors presents in X. Take in the space $\mathcal{H}_2^7$ the polynomials

$$G_{x_i}(y) = g_2((x_i, y)), \qquad i = \overline{1, \ldots, 120},$$

where $g_2(t) = \frac{8}{7}t^2 - \frac{1}{7}$ is a corresponding normalized Gegenbauer polynomial. Since $(x_i, x_j) = 0$ or $\pm 1/2$, for $i \neq j$, then $\langle G_{x_i}, G_{x_j} \rangle = g_2((x_i, y_j)) = \pm 1/7$! It looks really like a mystery the fact that $\left| g_2((x_i, x_j)) \right| = \text{const}$, for any different x_i, $x_j \in X$. But exactly this is essential for the proof of optimality of our code. Since, $\dim \mathcal{H}_2^7 = 35$, then the points $G_{x_1}, \ldots, G_{x_{120}}, -G_{x_1}, \ldots, -G_{x_{120}}$ provide antipodal spherical $(35, 240, 1/7)$ code. Here is a proof of optimality. Take arbitrary antipodal set of points $Y = \{y_1, \ldots, y_{240}\}$ in $\mathbb{R}^{35}$. Then, the inequality

$$\frac{1}{240^2} \sum_{i,j=1}^{240} (y_i, y_j)^2 \geq 1/35,$$

implies that $(y_i, y_j)^2 \geq 1/49$, for some y_i, $y_j \in Y$, $i \neq j$, which are not antipodal. This immediately gives us an optimality of our construction. The other reason why it works, that is our set is also spherical 3-design in $\mathbb{R}^{35}$. We are still not able generalize this construction even for Leech lattice Λ_{24}. We also don't know whether the construction described above is an optimal spherical $(35, 240, 1/7)$ code.

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