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DEFORMATIONS OF CIRCLE-VALUED MORSE FUNCTIONS ON SURFACES*

ДЕФОРМАЦІЇ ВІДОБРАЖЕНЬ МОРСА ПОВЕРХОНЬ У КОЛО

Let M be a smooth connected orientable compact surface. Denote by $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ the space of all Morse functions $f : M \to S^1$ having no critical points on ∂M and such that for every connected component V of ∂M, the restriction $f : V \to S^1$ is either a constant map or a covering map. Endow $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ with C^∞-topology. In this paper the connected components of $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ are classified. This result extends the results of S. V. Matveev, V. V. Sharko, and the author for the case of Morse functions being locally constant on ∂M.

Нехай M — гладка зв'язна орієнтовна компактна поверхня. Позначимо через $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ простір усіх відображень Морса $f : M \to S^1$, які не мають критичних точок на ∂M, а для кожної компоненти зв'язності V межі ∂M обмеження $f : V \to S^1$ є або постійним або накриваючим відображенням. Наділимо $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ топологією C^∞. У статті наведено класифікацію компонент зв'язності простору $\mathcal{F}_{\mathrm{cov}}(M, S^1)$. Цей результат узагальнює результати С. В. Матвєєва, В. В. Шарка та автора про функції Морса, що є локально постійними на ∂M.

1. Introduction. Let M be a compact surface and P be either the real line $\mathbb{R}$ or the circle S^1. Denote by $\mathcal{F}'(M, P)$ the subset of $\mathcal{C}^\infty(M, S^1)$ consisting of maps $f\colon M \to P$ such that

(1) all critical points of f are non-degenerate and belongs to the interior of M, so f is a *Morse* function.

Let also $\mathcal{F}_{l.c.}(M, P)$ be the subset of $\mathcal{F}'(M, P)$ consisting of maps $f\colon M \to P$ such that

(2) $f|_{\partial M}$ is a *locally constant* map, that is for every connected component W of ∂M the restriction of f to W is a constant map.

Moreover, for the case $P = S^1$ let $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ be another subset of $\mathcal{F}'(M, S^1)$ consisting of maps $f\colon M \to S^1$ such that

(2′) for every connected component W of ∂M the restriction of f to W is either a *constant* map or a *covering* map.

Thus
$$\mathcal{F}_{l.c.}(M, S^1) \subset \mathcal{F}_{\mathrm{cov}}(M, S^1).$$

Endow all these spaces $\mathcal{F}'(M, P)$, $\mathcal{F}_{l.c.}(M, P)$, and $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ with the corresponding C^∞-topologies. The connected components of the spaces $\mathcal{F}_{l.c.}(M, P)$ were described in [1 – 4]. The aim of this note is to describe the connected components of the space $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ for the case when M is orientable.

To formulate the result fix an orientation of P and let $f \in \mathcal{F}'(M, P)$. Then for each (non-degenerate) critical point of f we can define its index with respect to a given orientation of S^1. Denote by $c_i = c_i(f)$, $i = 0, 1, 2$, the total number of critical points of f of index i.

Moreover, suppose W is a connected component of ∂M such that the restriction of f to W is a constant map. Then we associate to W the number $\varepsilon_W(f) := +1$ (resp.

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$\varepsilon_k(f) := -1$) whenever the value $f(W)$ is a local maximum (resp. minimum) with respect to the orientation of P. If $f|_W$ is non-constant, then we put $\varepsilon_W(f) = 0$.

The following theorem describes the connected components of $\mathcal{F}_{l.c.}(M, P)$.

Theorem 1 [1–4]. *Let $f, g \in \mathcal{F}_{l.c.}(M, P)$. Then they belong to the same path component of $\mathcal{F}_{l.c.}(M, P)$ iff the following three conditions hold true*:

(i) *f and g are homotopic as continuous maps* (*for the case $P = \mathbb{R}$ this condition is, of course, trivial*);

(ii) *$c_i(f) = c_i(g)$ for $i = 0, 1, 2$*;

(iii) *$\varepsilon_W(f) = \varepsilon_W(g)$ for every connected component W of ∂M.*

If $P = \mathbb{R}$ and $f = g$ on some neighbourhood of ∂M, then one can choose a homotopy between f and g fixed near ∂M.

The case $P = \mathbb{R}$ was independently established by V. V. Sharko [1] and S. V. Matveev. Matveev's proof was generalized to the case of height functions and published in the paper [2] by E. Kudryavtseva. The case $P = S^1$ was proven by the author in [3]. Moreover, in [4] Theorem 1 was reproved by another methods.

The present notes establishes the following result.

Theorem 2. *Suppose M is orientable. Let $f, g \in \mathcal{F}_{\text{cov}}(M, S^1)$. Then they belong to the same path component of $\mathcal{F}_{\text{cov}}(M, S^1)$ iff the following three conditions hold true*:

(i) *f and g are homotopic as continuous maps*;

(ii) *$c_i(f) = c_i(g)$ for $i = 0, 1, 2$*;

(iii) *$\varepsilon_W(f) = \varepsilon_W(g)$ for every connected component W of ∂M such that $f|_W$ is a constant map.*

Notice that the formulations of both Theorems 1 and 2 look the same. The difference is that in Theorem 1 every $f \in \mathcal{F}_{l.c.}(M, P)$ takes constant values of connected components of ∂W, while in Theorem 2 the restrictions of $f \in \mathcal{F}_{\text{cov}}(M, S^1)$ to boundary components W of M may also be covering maps and the degrees of such restrictions $f\colon W \to S^1$ are encoded by homotopy condition (i).

I would like to thank A. Pajitnov for posing me question about connected components of $\mathcal{F}_{\text{cov}}(M, S^1)$ and useful discussions.

The proof of Theorem 2 follows the line of [3, 4]. First we prove $\mathbb{R}$-variant of Theorem 2 similarly to [4], see Theorem 3 below, and then deduce Theorem 2 from Theorem 3 similarly to [3]. Therefore we mostly sketch the proofs indicating only the principal differences.

2. $\mathbb{R}$-variant of Theorem 2 for surfaces with corners. Let $f \in \mathcal{F}_{\text{cov}}(M, S^1)$. Say that $v \in S^1$ is an *exceptional* value of f, if v is either a critical value of f or there exists a connected component W of ∂M such that $f(W) = v$.

Let $v \in S^1$ be a non-exceptional value of f. Then its inverse image $f^{-1}(v)$ is a proper 1-submanifold of M which does not contain connected components of ∂M. Thus $f^{-1}(v)$ is a disjoint union of circles and arcs with ends on ∂M and transversal to ∂M at these points. Let $\widehat{M}$ be a surface obtained by cutting M along $f^{-1}(v)$.

Then $\widehat{M}$ can be regarded as a surface with corners and f induces a function $\widehat{f}\colon \widehat{M} \to$ $\to [0, 1]$ such that

(a) $\widehat{f}|_{\text{Int}\widehat{M}}$ is Morse and has no critical points on $\partial\widehat{M}$;

(b) let W be a connected component of $\partial\widehat{M}$; then either $\widehat{f}|_W$ is constant, or there are $4k_W$ points on W for some $k_W \geq 1$ dividing W into $4k_W$ arcs

$$A_1, B_1, C_1, D_1, \ldots, A_{k_W}, B_{k_W}, C_{k_W}, D_{k_W}$$

such that $\widehat{f}$ strictly decreases on A_i, strictly increases on C_i, $\widehat{f}(B_i) = 1$, and $\widehat{f}(D_i) = 0$ for each $i = 1, \ldots, k_W$.

We will now define the space of all such functions and describe its connected components.

2.1. Space $\mathcal{F}_\xi(M, I)$. Let M be a compact, possibly non-connected, surface. For every connected component W of ∂M fix an orientation and a number $k_W \geq 0$, and divide W into $4k_W$ consecutive arcs

$$A_1, B_1, C_1, D_1, \ldots, A_{k_W}, B_{k_W}, C_{k_W}, D_{k_W}$$

directed along the orientation of W. If $k_W = 0$ then we do not divide W at all.

Denote this subdivision of ∂M by ξ and the set of ends of these arcs by $K = K(\xi)$. We will regard K as „corners" of M.

Let also T_+ (resp. T_1, T_-, and T_0) be the union of all closed arcs A_i (resp. B_i, C_i, D_i) over all boundary components of M.

Let $\mathcal{F}_\xi(M, I)$ be the space of all continuous functions $f \colon M \to I = [0, 1]$ satisfying the following three conditions.

(a) The restriction of f to $M \setminus K$ is C^∞, and all partial derivatives of f of all orders continuously extend to all of M.

(b) All critical points of f are non-degenerate and belong to $\mathrm{Int}M$,

$$f(\mathrm{Int}M) \subset (0, 1), \quad f^{-1}(0) = T_0, \quad f^{-1}(1) = T_1,$$

and $f|_{T_+}$ (resp. $f|_{T_-}$) has strictly positive (resp. negative) derivative.

(c) Let W be a connected component of ∂M such that $k_W = 0$. Then $f|_W$ is constant and $\widehat{f}(W) \in (0, 1)$.

Notice that condition (a) means that f is a C^∞-function on a surface with corners and condition (b) implies that f strictly increases (decreases) on each arc A_i (C_i),

Again we associate to every $f \in \mathcal{F}_\xi(M, I)$ the total number $c_i(f)$ of critical points at each index $i = 0, 1, 2$. Moreover, to every connected component W of ∂M with $k_W = 0$ we associate the number $\varepsilon_W(f) = \pm 1$ as above.

The following theorem extends $\mathbb{R}$-case of Theorem 1 to orientable surfaces with corners.

Theorem 3. *Suppose M is orientable and connected. Then $f, g \in \mathcal{F}_\xi(M, I)$ belongs to the same path component of $\mathcal{F}_\xi(M, I)$ iff*

(i) $c_i(f) = c_i(g)$ *for* $i = 0, 1, 2$;

(ii) $\varepsilon_W(f) = \varepsilon_W(g)$ *for every connected component W of ∂M with $k_W = 0$.*

Moreover, if $f = g$ on some neighbourhood of $T_0 \cup T_1$, then there exists a homotopy relatively $T_0 \cup T_1$ between these functions in $\mathcal{F}_\xi(M, I)$.

The proof will be given in Section 4. Now we will deduce from this result Theorem 2.

3. Proof of Theorem 2. Necessity is obvious, therefore we will prove only sufficiency.

Let $f, g \in \mathcal{F}_{\mathrm{cov}}(M, S^1)$. Consider the following conditions (P_n), $n \geq 0$, (Q), and (R) for f and g.

(P_n) f (resp. g) is homotopic in $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ to a map $\tilde{f}$ (resp. $\tilde{g}$) such that for some common non-exceptional value $v \in S^1$ of $\tilde{f}$ and $\tilde{g}$ the intersection $\tilde{f}^{-1}(v) \cap \tilde{g}^{-1}(v)$ is transversal and consists of at most n points.

(Q) f (resp. g) is homotopic in $\mathcal{F}_{\mathrm{cov}}(M, S^1)$ to a map $\tilde{f}$ (resp. $\tilde{g}$) such that for some common non-exceptional value $v \in S^1$ of $\tilde{f}$ and $\tilde{g}$,

(i) $\tilde{f}^{-1}(v) = \tilde{g}^{-1}(v)$,

(ii) $\tilde{f} = \tilde{g}$ on some neighbourhood of $\tilde{f}^{-1}(v)$,

(iii) and for every connected component M_1 of $M \setminus \tilde{f}^{-1}(v)$ the restrictions $\tilde{f}$ and $\tilde{g}$ onto M_1 have the same numbers of critical points at each index.

(R) f is homotopic to g in $\mathcal{F}_{\mathrm{cov}}(M, S^1)$.

Notice that f and g always satisfy (P_n) for some $n \geq 0$. We have to prove for them condition (R). This is given by the following lemma, which completes the proof of Theorem 2.

Lemma 1. *Let $f, g \in \mathcal{F}_{\mathrm{cov}}(M, S^1)$. Suppose that $f, g \in \mathcal{F}_{\mathrm{cov}}(M, S^1)$ satisfy conditions* (i) – (iii) *of Theorem* 2. *Then the following implications hold*:

$$(P_n) \Rightarrow (P_{n-1}) \Rightarrow \ldots \Rightarrow (P_0) \Rightarrow (Q) \Rightarrow (R).$$

Proof. Implications $(P_n) \Rightarrow (P_{n-1})$ and $(P_0) \Rightarrow (Q)$ can be deduced from Theorem 3 almost by the same arguments as [3] (Theorems 3, 4) were deduced from the $\mathbb{R}$-case of Theorem 1. The principal difference here is that one should work with 1-submanifolds with boundary rather than with closed 1-submanifolds. The proof is left for the reader.

$(Q) \Rightarrow (R)$. Cut M along $f^{-1}(v)$ and denote the obtained surface with corners by $\widehat{M}$. Then f (resp. g) induces on $\widehat{M}$ a function $\widehat{f}$ (resp. $\widehat{g}$) belonging to $\mathcal{F}_{\xi}(M', I)$. Moreover, it follows from conditions (i) – (iii) of Theorem 1 for f and g and assumption (iii) of (Q) that for every connected component M_1 of $\widehat{M}$ the restrictions of $\widehat{f}$ and $\widehat{g}$ to M_1 satisfy conditions (i) and (ii) of Theorem 3. Hence they are homotopic in $\mathcal{F}_{\xi}(M', I)$ relatively some neighbourhood of the set $T_0 \cup T_1$ corresponding to $f^{-1}(v)$. This homotopy yields a desired homotopy between f and g in $\mathcal{F}_{\mathrm{cov}}(M, S^1)$.

Lemma 1 is proved.

4. Proof of Theorem 3. We will follow the line of the proof of Theorem 1, see [2, 4]. Suppose $f, g \in \mathcal{F}_{\xi}(M, I)$ satisfy assumptions (i) and (ii) of Theorem 3. The idea is to reduce the situation to the case when $g = f \circ h$ for some diffeomorphism h of M fixed near ∂M, and then show that $f \circ h$ is homotopic in $\mathcal{F}_{\xi}(M, I)$ to f, see Lemmas 4 – 6.

4.1. KR-graph. For $f \in \mathcal{F}_{\xi}(M, I)$ define the *Kronrod – Reeb* graph (or simply *KR-graph*) Γ_f of f as a topological space obtained by shrinking to a point every connected component of $f^{-1}(v)$ for each $v \in I$. It easily follows from the assumptions on f that Γ_f has a natural structure of a 1-dimensional CW-complex. The vertices of f corresponds to the connected components of level sets $f^{-1}(v)$ containing critical points of f.

Notice that f can be represented as the following composite of maps:

$$f = f_{KR} \circ p_f \colon M \xrightarrow{p_f} \Gamma_f \xrightarrow{f_{KR}} I,$$

where p_f is a factor map and f_{KR} is the induced function on Γ_f which we will call the *KR-function* of f.

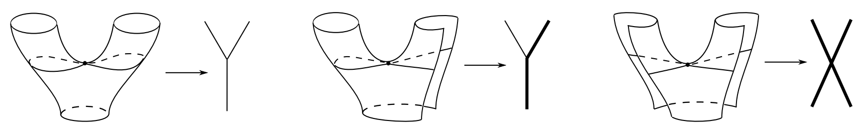
Say that f is *generic* if it takes distinct values at distinct critical points and connected components W of ∂M with $k_W = 0$. It is easy to show that every $f \in \mathcal{F}_{\xi}(M, I)$ is homotopic in $\mathcal{F}_{\xi}(M, I)$ to a generic function.

Notice that for each non-exceptional value v of f every connected component P of $f^{-1}(v)$ is either an arc or a circle. We will distinguish the corresponding points on Γ_f

as follows: if P is an arc, then we denote the corresponding point on Γ_f in bold. Thus on the KR-graph of f we will have two types of edges *bold* and *thin*.

Moreover, every vertex w of degree 1 of Γ_f corresponds either to a local extreme of f or to a boundary component W of ∂M with $k_W = 0$. In the first case w will be called an *e-vertex*, and a *∂-vertex* otherwise. ∂-vertexes will be denoted in bold.

Possible types of vertexes of Γ_f corresponding to saddle critical points together with the corresponding critical level sets are shown in Fig. 1.



a b c

Fig. 1. Structure of f near saddle critical points.

Definition 1. *Let $f, g \in \mathcal{F}_\xi(M, I)$. Say that KR-functions of f and g are* ***equivalent*** *if there exist a homeomorphism $H\colon \Gamma_f \to \Gamma_g$ between their KR-graphs and a homeomorphism $\Phi\colon I \to I$ which preserves orientation such that $g_{KR} = \Phi^{-1} \circ f_{KR} \circ H$ and H maps bold edges* (*resp. thin edges, ∂-vertexes*) *of Γ_g to bold edges* (*resp. thin edges, ∂-vertexes*) *of Γ_f.*

We will always draw a KR-graph so that the corresponding KR-function will be the projection to the vertical line. This determines KR-function up to equivalence in the sense of Definition 1.

The following statement can be proved similarly to [5, 6].

Lemma 2. *Suppose M is orientable, and let $f, g \in \mathcal{F}_\xi(M, I)$ be two generic functions such that their KR-functions are KR-equivalent. Then there exist a diffeomorphism $h\colon M \to M$ and a preserving orientation diffeomorphism $\phi\colon I \to I$ such that $g = \phi^{-1} \circ f \circ h$.*

Since ϕ is isotopic to id_I, it follows that g is homotopic in $\mathcal{F}_\xi(M, I)$ to $f \circ h$.

4.2. Canonical KR-graph. Consider the graphs shown in Fig. 2.

The graph $X^0(k)$, $k \geq 1$, consists of a bold line "intersected" by another $k-1$ bold lines, the graph $X^{\pm}(k)$ is obtained from $X^0(k)$ by adding a thin edge directed either up or down. The vertex of degree 1 on that thin edge can be either e- or ∂-one.

The graph Y is determined by five numbers: z, b_-, b_+, e_-, e_+, where z is the total number of cycles in Y, b_- (resp. e_-) is the total number of ∂-vertexes (resp. e-vertexes) being local minimums for the KR-function, and b_+ and e_+ correspond to local maximums.

We will assume that KR-function surjectively maps $X^*(k)$ onto $[0,1]$, while Y is mapped into interval $(0,1)$.

Definition 2. *Let $f \in \mathcal{F}_\xi(M, I)$. Say that f is* ***canonical*** *if it is generic and its KR-graph Γ_f has one of the following forms*:

(1) *coincides either with one of $X^*(k)$ for some $k \geq 1$, or with Y for some $e_\pm$, $b_\pm$, and z*;

(2) *is a union of $X^-(k)$ with $X^+(l)$ with common thin edge for some $k, l \geq 1$, see* Fig. 3, *a*;

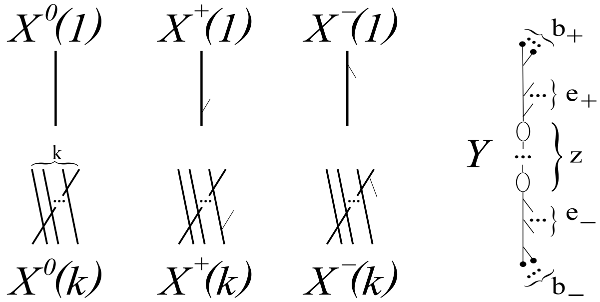


Fig. 2. Elementary blocks of canonical KR-graphs.

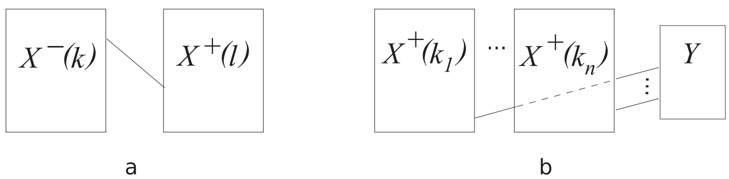


Fig. 3. Canonical KR-graph Γ_f.

(3) *is a union of some* $X^+(k_i)$, $i = 1, \ldots, n$, *connected along their thin edges with* Y, *see* Fig. 3, *b*.

Every maximal bold connected subgraph of Γ_f *will be called an* X***-block.*** *Evidently, such a block is isomorphic with* $X^0(k)$ *for some* k.

Lemma 3. *Let* $f \in \mathcal{F}_\xi(M, I)$ *be a canonical function. Then the numbers* $c_i(f)$, k_W, *and* $\varepsilon_W(f)$ *are completely determined by its KR-graph* Γ_f *and wise verse. Moreover, every* X*-block of* Γ_f *corresponds to a unique boundary component of* M. *In particular, the collection of* X*-blocks in* Γ_f *is determined (up to order) by the partition* ξ *of* ∂M, *and therefore does not depend on a canonical function* f.

Proof. Since f is generic, $c_0(f)$ (resp. $c_2(f)$) is equal to the total number of vertexes of degree 1 being local minimums (resp. local maximums) of the restriction of f_{KR} to Y, while $c_1(f)$ is equal to the total number of vertexes of Γ_f of degrees 3 and 4.

Furthermore, it easily follows from Fig. 1, *c*, that every X-block N of Γ_f corresponds to a collar of some boundary component W of M such that k_W is equal to the total number of local minimums (= local maximums) of the restriction of f_{KR} to N.

Finally, every connected component W of ∂M with $k_W = 0$ corresponds to a ∂-vertex w on Y. Moreover, $\varepsilon_W = -1$ (resp. $\varepsilon_W = +1$) iff w is a local minimum (resp. local maximum) of the restriction of f_{KR} to Y.

Lemma 3 is proved.

Lemma 4. *Let* $f \in \mathcal{F}_\xi(M, I)$. *Then* f *is homotopic in* $\mathcal{F}_\xi(M, I)$ *to some canonical function.*

Proof. Consider the following elementary surgeries of a KR-graph shown in Fig. 4. It is easy to see that each of them can be realized by a deformation of f in $\mathcal{F}_\xi(M, I)$. Then similarly to [2] (Lemma 11) one can reduce any KR-graph of $f \in \mathcal{F}_\xi(M, I)$ to a canonical form using these surgeries. We leave the details for the reader.

Lemma 4 is proved.

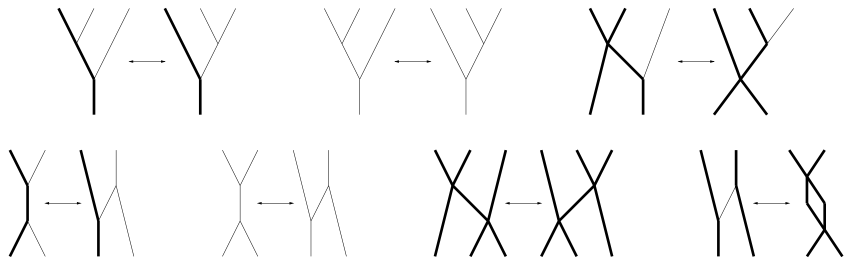


Fig. 4. Elementary surgeries of KR-graph.

Lemma 5. *Let $f, g \in \mathcal{F}_\xi(M, I)$ be two canonical functions satisfying assumptions* (i) *and* (ii) *of Theorem* 3. *Then f (resp. g) is homotopic in $\mathcal{F}_\xi(M, I)$ to another canonical function $\tilde{f}$ (resp. $\tilde{g}$) such that $\tilde{g} = \tilde{f} \circ h$ for some diffeomorphism $h\colon M \to M$ fixed near ∂M.*

Proof. It follows from Lemma 3 and assumptions on f and g that their KR-graphs have the same Y-blocks and the same (up to order) $X^{\pm}(k)$-blocks. Then, using surgeries of Figure 4 applied to Γ_g, we can reduce the situation to the case when KR-functions of f of g are KR-equivalent. Whence by Lemma 2 we can also assume that there exists a diffeomorphism $h\colon M \to M$ such that $g = f \circ h$. Moreover, changing g similarly to [2] or [4] one can choose h so that it preserves orientation of M, maps every connected component W of ∂M onto itself, and preserves subdivision ξ on W. Then using the assumptions on f and g near ∂M, one can show that h is isotopic to the identity near ∂M.

Lemma 5 is proved.

Lemma 6. *Let $h\colon M \to M$ be a diffeomorphism fixed near ∂M and $f \in \mathcal{F}_\xi(M, I)$ be a canonical function. Then $f \circ h$ is homotopic in $\mathcal{F}_\xi(M, I)$ to f relatively some neighbourhood of ∂M.*

Proof. Since every X-block of Γ_f corresponds to a collar $N(W)$ of some boundary component W of ∂M, we can assume that h is fixed on some neighbourhood of $N(W)$. Therefore we may cut off $N(W)$ from M and assume that f takes constant values at each boundary component of ∂M. Then f is homotopic to $f \circ h$ relatively some neighbourhood of ∂M by the arguments similar to the proof of Theorem 1, see [4].

Lemma 6 is proved.

Theorem 3 now follows from Lemmas 4 – 6.

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