UDC 517.581

N. X. Thao (Hanoi Univ. Technol., Vietnam),
N. O. Virchenko (Nat. Techn. Univ. Ukraine “KPI”, Kyiv)

ON THE POLYCONVOLUTION FOR THE FOURIER COSINE, FOURIER SINE, AND THE KONTOROVICH – LEBEDEV INTEGRAL TRANSFORMS*

ПРО ПОЛІЗГОРТКУ ДЛЯ КОСИНУС-ФУР’Є, СИНУС-ФУР’Є ТА КОНТОРОВИЧА – ЛЕБЕДЄВА ІНТЕГРАЛЬНИХ ПЕРЕТВОРЕНЬ

The polyconvolution $\underset{1}{*}(f,g,h)(x)$ of three functions f,g,h is constructed for the Fourier cosine (F_c) integral transform, the Fourier sine (F_s) integral transform, and the Kontorovich – Lebedev (K_{iy}) integral transform, whose factorization equality is of the form

$$F_c(\underset{1}{*}(f,g,h))(y) = (F_s f)(y).(F_s g)(y).(K_{iy}h) \quad \forall y > 0.$$

The relations of this polyconvolution to the Fourier convolution and the Fourier cosine convolution are obtained. In addition, the relations between the new polyconvolution product and other known convolution products are established. As application, we consider a class of integral equations with the Toeplitz kernel and the Hankel kernel, whose solutions in closed form can be obtained with the help of the new polyconvolution. Application in solving systems of integral equations is also presented.

Побудовано полізгортку $\underset{1}{*}(f,g,h)(x)$ трьох функцій f,g,h для косинус-Фур’є (F_c), синус-Фур’є (F_s) і Конторовича – Лебедєва (K_{iy}) інтегральних перетворень з рівністю факторизації у формі

$$F_c(\underset{1}{*}(f,g,h))(y) = (F_s f)(y).(F_s g)(y).(K_{iy}h) \quad \forall y > 0.$$

Одержано співвідношення цієї полізгортки із згорткою Фур’є і косинус-Фур’є згорткою. Також встановлено співвідношення між добутком нової полізгортки та добутками інших відомих згорток. Як застосування, розглянуто клас інтегральних рівнянь з ядрами Тепліца і Ганкеля, розв’язки цих рівнянь за допомогою нової полізгортки можна одержати у замкненій формі. Наведено також застосування до розв’язання систем інтегральних рівнянь.

Introduction. The convolution of two functions f and g for the Fourier transform is well-known [1]:

$$(f \underset{F}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}. \tag{0.1}$$

This convolution has the factorization equality as belows

$$F(f \underset{F}{*} g)(y) = (Ff)(y)(Fg)(y) \quad \forall y \in \mathbb{R},$$

here F denotes the Fourier transform [1]

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixy} f(x)dx.$$

The convolution of f and g for the Kontorovich – Lebedev integral transform has been

*This research is supported partially by NAFOSTED of Vietnam, grant 101.01.21.09.

studied in [2]

$$(f \underset{K-L}{*} g)(x) = \frac{1}{2x} \int_0^\infty \int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{xu}{v} + \frac{xv}{u} + \frac{uv}{x}\right)\right] f(u)g(v)dudv, \ x > 0, \quad (0.2)$$

for which the factorization identity holds

$$K_{iy}(f \underset{K-L}{*} g) = (K_{iy}f).(K_{iy}g) \quad \forall y > 0.$$

Here K_{iy} is the Kontorovich – Lebedev transform [2]

$$K_{ix}[f] = \int_0^\infty K_{ix}(t)f(t)dt,$$

and $K_{ix}(t)$ is the Macdonald function [3].

The convolution of two function f and g for the Fourier cosine is of the form [1]

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)[g(|x-y|) + g(x+y)]dy, \quad x > 0, \quad (0.3)$$

which satisfied the factorization equality

$$F_c(f \underset{1}{*} g)(y) = (F_c f)(y)(F_c g)(y) \quad \forall y > 0.$$

Here the Fourier cosine transform is of the form [1]

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos yx.f(x)dx, \quad y > 0.$$

The convolution with a weight function $\gamma(x) = \sin x$ of two functions f and g for the Fourier sine transform has introduced in [4]

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(y)[\mathrm{sign}(x+y-1)g(|x+y-1|) + \mathrm{sign}(x-y+1)g(|x-y+1|) -$$

$$- g(x+y+1) - \mathrm{sign}(x-y-1)g(|x-y-1|)]dy, \quad x > 0, \quad (0.4)$$

and the factorization identity holds

$$F_s(f \overset{\gamma}{*} g)(y) = \sin y (F_s f)(y)(F_s g)(y) \quad \forall y > 0.$$

Here the Fourier sine is of the form [1]

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin yx.f(x)dx, \quad y > 0.$$

The generalized convolution of two functions f, g for the Fourier sine and Fourier cosine transforms has studied in [1]

$$(f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty f(u)[g(|x-u|) - g(x+u)]du, \ x > 0, \tag{0.5}$$

and the respectively factorization identity is [1]

$$F_s(f \underset{2}{*} g)(y) = (F_s f)(y).(F_c g)(y) \quad \forall y > 0.$$

The generalized convolution of two functions f and g for the Fourier cosine and the Fourier sine transforms is defined by [5]

$$(f \underset{3}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty f(u)[\text{sign}(u-x) g(|u-x|) + g(u+x)]du, \quad x > 0. \tag{0.6}$$

For this generalized convolution the factorization equality holds [5]

$$F_c(f \underset{3}{*} g)(y) = (F_s f)(y)(F_s g)(y) \quad \forall y > 0.$$

The generalized convolution with the weight function $\gamma(x) = \sin x$ for the Fourier cosine and the Fourier sine transforms of f and g has introduced in [6]

$$(f \overset{\gamma}{\underset{1}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty f(u)[g(|x+u-1|) + g(|x-u+1|) - g(x+u+1) -$$

$$-g(|x-u-1|)]du, \quad x > 0. \tag{0.7}$$

It satisfies the factorization property [6]

$$F_c(f \overset{\gamma}{\underset{1}{*}} g)(y) = \sin y \, (F_s f)(y)(F_c g)(y) \quad \forall y > 0.$$

The generalized convolution with the weight function $\gamma(x) = \sin x$ of f and g for the Fourier sine and Fourier cosine has studied in [7]

$$(f \overset{\gamma}{\underset{2}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty f(u)[g(|x+u-1|) + g(|x-u-1|) - g(x+u+1) -$$

$$-g(|x-u+1|)]du, \quad x > 0, \tag{0.8}$$

and satisfy the factorization identity

$$F_s(f \overset{\gamma}{\underset{2}{*}} g)(y) = \sin y \, (F_c f)(y)(F_c g)(y) \quad \forall y > 0.$$

In 1997, Kakichev V. A. introduced a constructive method for defining a polyconvolution $\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)(x)$ of functions $f_1, f_2, \ldots, f_n$ with a weight function γ for the integral transforms $K, K_1, K_2, \ldots, K_n$, for which the factorization property holds [8]

$$K[\overset{\gamma}{*}(f_1, f_2, \ldots, f_n)](y) = \gamma(y) \prod_{i=1}^n (K_i f_i)(y), \quad n \geq 3.$$

Polyconvolutions for the Hilbert, Stieltjes, Fourier cosine and Fourier sine integral transforms has been studied in.

The polyconvolution of f, g and h for the Fourier cosine and the Fourier sine transforms has the form [9]

$$*(f,g,h)(x) = \frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty f(u)g(v)[h(|x+u-v|)+h(x-u+v|)-$$

$$-h(|x-u-v|)-h(x+u+v)]dudv, \quad x>0, \tag{0.9}$$

which satisfies the following factorization property:

$$F_c(*(f,g,h))(y) = (F_s f)(y).(F_s g)(y).(F_c h)(y) \quad \forall y>0.$$

Recent years, many sciences interested in the theory of convolution for the integral transforms and gave several interesting application (see [10]). Specially, the integral equations with the Toeplitz plus Hankel kernel

$$f(x) + \int\limits_0^\infty [k_1(x+y)+k_2(x-y)]f(y)dy = g(x), \quad x>0, \tag{0.10}$$

where k_1, k_2, g are known functions, and f is unknow function. Many partial cases of this equation can be solved in closed form with the help of the convolutions and generalized convolutions. In this paper, we construct and investigate the polyconvolution for the Fourier cosine, Fourier sine and the Kontorovich – Lebedev transforms. Several properties of this new polyconvolution and its application on solving integral equation with Toeplitz plus Hankel equation and systems of integral equations are obtained.

1. Polyconvolution.

Definition 1. *The polyconvolution of functions f, g and h for the Fourier cosine, Fourier sine and the Kontorovich – Lebedev integral transforms is defined as follows*

$$\underset{1}{*}(f,g,h)(x) = \int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty \theta(x,u,v,w)f(u)g(v)h(w)dudvdw, \quad x>0, \tag{1.1}$$

where

$$\theta(x,u,v,w) = \frac{1}{2\sqrt{2\pi}}[e^{-w\cosh(x+u-v)}+$$

$$+e^{-w\cosh(x-u+v)} - e^{-w\cosh(x+u+v)} - e^{-w\cosh(x-u-v)}].$$

Theorem 1. *Let f, g be functions in $L_1(\mathbb{R}_+)$, and let h be function in $L_1\left(\frac{1}{\sqrt{w}}, \mathbb{R}_+\right)$, then the polyconvolution* (1.1) *belongs to $L_1(\mathbb{R}_+)$ and satisfies the factorization property*

$$F_c(\underset{1}{*}(f,g,h))(y) = (F_s f)(y).(F_s g)(y).(K_{iy}h) \quad \forall y>0. \tag{1.2}$$

Proof. Since $|e^{-w\cosh(x+u-v)} - e^{-w\cosh(x+u+v)}| \leqslant \frac{1}{\sqrt{w}}$, for sufficient large $w>0$, we have

$$|\underset{1}{*}(f,g,h)(x)| \leqslant \frac{1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty \int_0^\infty |f(u)||g(v)||h(w)||\theta(x,u,v,w)|dudvdw \leqslant$$

$$\leqslant \sqrt{\frac{1}{2\pi}} \int_0^\infty |f(u)|du. \int_0^\infty |g(v)|dv. \int_0^\infty \frac{1}{\sqrt{w}}|h(w)|dw < +\infty.$$

On the other hand, note that $\cosh(x+u-v) \geqslant \dfrac{(x+u-v)^2}{2}$, we have

$$e^{-w\cosh(x+u-v)} \leqslant e^{-w\frac{(x+u-v)^2}{2}} \quad \forall w > 0.$$

Then we have

$$\int_0^\infty e^{-w\cosh(x+u-v)}dx \leqslant \sqrt{\frac{2}{w}} \int_0^\infty e^{-\left(\sqrt{\frac{w}{2}}(x+u-v)\right)^2} d\left(\sqrt{\frac{w}{2}}(x+u-v)\right) \leqslant$$

$$\leqslant 2\sqrt{\frac{2}{w}} \int_0^\infty e^{-s^2}ds = \sqrt{\frac{2\pi}{w}}.$$

Using this estimation we obtain

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-w\cosh(x+u-v))}|f(u)||g(v)||h(w)|dudvdwdx \leqslant$$

$$\leqslant \int_0^\infty \int_0^\infty \int_0^\infty \sqrt{\frac{2\pi}{w}}|h(w)||f(u)||g(v)|dudvdw =$$

$$= \sqrt{2\pi} \int_0^\infty |f(u)|du. \int_0^\infty |g(v)|dv. \int_0^\infty \frac{1}{\sqrt{w}}|h(w)|dw < +\infty. \tag{1.3}$$

The following estimation can be obtained by similar way

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-w\cosh(x-u+v))}|f(u)||g(v)||h(w)|dudvdwdx < +\infty, \tag{1.4}$$

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-w\cosh(x+u+v))}|f(u)||g(v)||h(w)|dudvdwdx < +\infty, \tag{1.5}$$

$$\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-w\cosh(x-u-v))}|f(u)||g(v)||h(w)|dudvdwdx < +\infty. \tag{1.6}$$

From formulas (1.1), (1.3) – (1.6) we have

$$\int_0^\infty |\underset{1}{*}(f,g,h)(x)|dx < +\infty.$$

It shows that the polyconvolution (1.1) belonging to $L_1(\mathbb{R}_+)$.

We now prove the factorization equality (1.2). We get

$$(F_s f)(y)(F_s g)(y)(K_{iy}h) =$$

$$= \frac{2}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty \sin(yu)\sin(yv)K_{iy}(w)f(u)g(v)h(w)dudvdw =$$

$$= \frac{1}{\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \cos(y\alpha)e^{-w\cosh\alpha}(\cos y(u-v)-$$

$$-\cos y(u+v))f(u)g(v)h(w)dudvdwd\alpha =$$

$$= \frac{1}{2\pi}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty [\cos y(\alpha+u-v)+\cos y(\alpha-u+v)-\cos y(\alpha+u+v)-$$

$$-\cos y(\alpha-u-v)]e^{-w\cosh\alpha}f(u)g(v)h(w)dudvdwd\alpha. \tag{1.7}$$

Changing variables we have

$$\int_0^\infty [\cos y(\alpha+u-v)-\cos y(\alpha+u+v)]e^{-w\cosh\alpha}d\alpha =$$

$$= \int_0^\infty \cos xy[e^{-w\cosh(x-u+v)}-e^{-w\cosh(x-u-v)}]dx. \tag{1.8}$$

Similar,

$$\int_0^\infty [\cos y(\alpha-u+v)-\cos y(\alpha-u-v)]e^{-w\cosh\alpha}d\alpha =$$

$$= \int_0^\infty \cos xy[e^{-w\cosh(x+u-v)}-e^{-w\cosh(x+u+v)}]dx. \tag{1.9}$$

From formulaes (1.7) – (1.9) we have

$$(F_s f)(y)(F_s g)(y)(K_{iy}h) = F_c(*(f,g,h))(y).$$

Theorem 1 is proved.

Proposition. *Let* $f, g \in L_1(\mathbb{R}_+)$, *and let* $h \in L_1\left(\frac{1}{\sqrt{w}}, \mathbb{R}_+\right)$, *then the identity holds*

$$\underset{1}{*}(f,g,h) = \sqrt{\frac{\pi}{2}}\int_0^\infty h(w)\big((g \underset{1}{*} e^{-w\cosh t}) \underset{F}{*} (f(|t|))\big)(x)dw. \tag{1.10}$$

Proof. From the definition (1.1) of the polyconvolution and the convolution (0.3) we have

$$\underset{1}{*}(f,g,h)(x) = \frac{1}{2}\int\limits_0^\infty\int\limits_0^\infty f(u)h(w)[(g\underset{1}{*}e^{-w\cosh t})(x+u)+(g\underset{1}{*}e^{-w\cosh t})(x-u)]dudw. \tag{1.11}$$

Therefore, in view of formula (0.1) we obtain

$$\underset{1}{*}(f,g,h) = \sqrt{\frac{\pi}{2}}\int\limits_0^\infty h(w)\big((g\underset{1}{*}e^{-w\cosh t})\underset{F}{*}(f(|t|))\big)(x)dw.$$

Theorem 2. *Let f, g, h be functions in $L_1(\mathbb{R}_+)$, and let k be functions in $L\left(\frac{1}{\sqrt{w}},\mathbb{R}_+\right)$, then the following properties hold*

a) $\underset{1}{*}(f\underset{2}{*}g,h,k) = \underset{1}{*}(f,h\underset{2}{*}g,k)$;

b) $\underset{1}{*}(f\overset{\gamma}{*}g,h,k) = \underset{1}{*}(f,g\overset{\gamma}{*}h,k)$.

Proof. We only need to prove the assertion a), since the second one can be obtained similarly. From Theorem 1 and (0.5) we have

$$F_c(\underset{1}{*}(f,h\underset{2}{*}g,k))(y) = F_s(f\underset{2}{*}g)(y)(F_sh)(y)(K_{iy}k) =$$

$$= (F_sf)(y)(F_cg)(y)(F_sh)(y)(K_{iy}k) =$$

$$= (F_sf)(y)F_s(h\underset{2}{*}g)(y)(K_{iy}k) = F_s(\underset{1}{*}(f\underset{2}{*}g,h,k))(y).$$

Then we obtain assertion a).

Definition 2. *Let f be a function in $L_1(\mathbb{R}_+)$ and g be a function in $L_1(\beta,\mathbb{R}_+)$, $\beta(v) = \frac{2}{\sqrt{v}}$. Then their norm are defined as follows*

$$\|f\|_{L_1(\mathbb{R}_+)} = \int\limits_0^\infty |f(x)|dx, \ \textit{and} \ \ \|g\|_{L_1(\beta,\mathbb{R}_+)} = \int\limits_0^\infty \beta(v)|f(v)|dv.$$

Theorem 3. *Let f,g be functions in $L_1(\mathbb{R}_+)$, and let h be function in $L_1(\beta,\mathbb{R}_+)$, then the estimation holds*

$$\|\underset{1}{*}(f,g,h)\|_{L_1(\mathbb{R}_+)} \leqslant \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\beta,\mathbb{R}_+)}.$$

Proof. From formulas (1.1), (1.3) – (1.6) we have

$$\int |\underset{1}{*}(f,g,h)(x)|dx \leqslant \int\limits_0^\infty \frac{2}{\sqrt{w}}|h(w)dw.\int\limits_0^\infty |f(u)|du.\int\limits_0^\infty |g(v)|dv.$$

Therefore,

$$\|\underset{1}{*}(f,g,h)\|_{L_1(\mathbb{R}_+)} \leqslant \|f\|_{L_1(\mathbb{R}_+)}\|g\|_{L_1(\mathbb{R}_+)}\|h\|_{L_1(\beta,\mathbb{R}_+)}.$$

2. Applications. Consider the integral equation

$$f(x) + \int\limits_0^\infty \theta_1(x,u)f(u)du + \int\limits_0^\infty \theta_2(x,u)f(u)du +$$

$$+\int_0^\infty\int_0^\infty\int_0^\infty \theta(x,u,v,w)f(u)l(v)k(w)dudvdw = p(x), \quad x>0. \tag{2.1}$$

Here $\theta(x,u,v,w)$ is given by the Definition 1, and $\theta_1(x,u)$ and $\theta_2(x,u)$ are defined by

$$\theta_1(x,u) = \frac{1}{\sqrt{2\pi}}[g(|x-u|) - g(x+u)],$$

$$\theta_2(x,u) = \frac{1}{2\sqrt{2\pi}}[h(|x+u-1|)\operatorname{sign}(x+u-1) + h(|x-u+1|)\operatorname{sign}(x-u+1)-$$

$$-h(x+u+1) - h(|x-u-1|)\operatorname{sign}(x-u-1)].$$

Beside, g, h, l, k, p are known functions, f is unknow function.

Theorem 4. *Let $g, h_1, h_2, l, p_1, p_2 \in L_1(\mathbb{R}_+)$, $p = p_1 + p_2$, and let $k \in \in L_1\left(\frac{1}{\sqrt{w}}, \mathbb{R}_+\right)$, $h = h_1 \underset{2}{*} h_2$ such that*

$$1 + (F_c g)(y) + \sin(F_s h)(y) \neq 0 \quad \forall y > 0,$$

and

$$p_2(x) = \underset{1}{*}(p_1, l, k) - l \underset{1}{*} (\underset{1}{*}(p_1, l, k))(x),$$

where $l \in L_1(\mathbb{R}_+)$ is defined uniquely by

$$(F_c l)(y) = \frac{(F_c g)(y) + \sin(F_s h)(y)}{1 + (F_c g)(y) + \sin(F_s h)(y)}.$$

Then the equation (2.1) *has a unique solution in $L_1(\mathbb{R}_+)$ whose closed form is*

$$f(x) = p_1(x) - (p_1 \underset{2}{*} l)(x).$$

Proof. First, similarly to the proof of the Theorem 1, we obtain the following lemma.

Lemma 1. *Let $f, g \in L_1(\mathbb{R}_+)$, then $(f \overset{\gamma}{\underset{3}{*}} g)(x)$ belongs to $L_1(\mathbb{R})$ the identity holds*

$$F(f \overset{\gamma}{\underset{3}{*}} g)(y) = -i \sin y (F_s f)(y)(F_s g)(y),$$

where

$$(f \overset{\gamma}{\underset{3}{*}} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x+u-1|)\operatorname{sign}(x+u-1)+$$

$$+g(|x-u+1|)\operatorname{sign}(x-u+1) - g(|x+u+1|)\operatorname{sign}(x+u+1)-$$

$$-g(|x-u-1|)\operatorname{sign}(x-u-1)]du.$$

Lemma 2. *Let $f, g \in L_1(\mathbb{R}_+)$, then $(f \overset{\gamma}{\underset{4}{*}} g)(x)$ belongs to $L_1(\mathbb{R})$ the identity holds*

$$F(f \overset{\gamma}{\underset{4}{*}} g)(y) = -i(F_s f)(y)(F_c g)(y),$$

where

$$(f \overset{\gamma}{\underset{4}{*}} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x-u|) - g(|x+u|)]du.$$

We now prove the Theorem 4 with the help of the Fourier transform, Lemmas 1 and 2, Theorem 1, the generalized convolution (0.5) and the convolution (0.4). Extend f, p_1 oddly, and p_2 evenly over whole real-line, we have

$$-i(F_sf)(y) - i(F_sf)(y).(F_cg)(y) - i(F_sf)(y).(F_sh)(y)\sin y +$$
$$+(F_sf)(y).(F_sl)(y)(K_{iy}k) = -i(F_sp_1)(y) + (F_cp_2)(y). \tag{2.2}$$

Note that the equation (2.2) is equivalent to the following system:

$$(F_sf)(y)(1 + (F_cg)(y) + \sin y(F_sh)(y)) = (F_sp_1)(y), \tag{2.3}$$

$$(F_sf)(y).(F_sl)(y)(K_{iy}k) = (F_cp_2)(y). \tag{2.4}$$

From (2.3) and the given condition we have

$$(F_sf)(y) = (F_sp_1)(y)\left(1 - \frac{(F_cg)(y) + \sin y(F_sh)(y)}{1 + (F_cg)(y) + \sin y(F_sh)(y)}\right). \tag{2.5}$$

Since $h = h_1 \underset{2}{*} h_2$ we have

$$\sin y(F_sh)(y) = F_c(h_1 \overset{\gamma}{\underset{1}{*}} h_2)(y).$$

In virtue of the Wiener–Levy theorem [10], and the given condition, there exists a function $l \in L_1(\mathbb{R}_+)$ such that

$$(F_cl)(y) = \frac{(F_cg)(y) + F_c(h_1 \overset{\gamma}{\underset{1}{*}} h_2)(y)}{1 + (F_cg)(y) + F_c(h_1 \overset{\gamma}{\underset{1}{*}} h_2)(y)}. \tag{2.6}$$

From (2.4) – (2.6) we have

$$(F_sf)(y) = (1 - (F_cl)(y))(F_sp_1)(y).$$

Therefore,

$$f(x) = p_1(x) - (p_1 \underset{2}{*} l)(x). \tag{2.7}$$

Substitute (2.7) into (2.4) we obtain

$$(F_cp_2)(y) = (1 - (F_cl)(y))(F_sp_1)(y)(F_sl)(y)(K_{iy}k).$$

Hence, using formula (0.5) and Theorem 1 we have

$$p_2(x) = \underset{1}{*}(p_1, l, k)(x) - (l \underset{1}{*} (\underset{1}{*}(p_1, l, k)))(x), \quad x > 0. \tag{2.8}$$

From (2.3), (2.4), (2.7), (2.8), the solution of equation (2.1) has a closed form in $L_1(\mathbb{R}_+)$ as

$$f(x) = p_1(x) - (p_1 \underset{2}{*} l)(x).$$

Remark. The integral equation (2.1) is a special case of the integral equation with the Toeplitz plus Hankel kernel (0.10) with

$$k_1(t) = -\frac{1}{\sqrt{2\pi}}g(t) - \frac{1}{2\sqrt{2\pi}}[h(t+1) - h(|t-1|)\,\mathrm{sign}(t-1)] -$$

$$-\frac{1}{2\sqrt{2\pi}}\int\limits_0^\infty\int\limits_0^\infty l(v)k(w)[e^{-w\cosh(t+v)}-e^{-w\cosh(t-v)}]dvdw,$$

$$k_2(t)=\frac{1}{\sqrt{2\pi}}g(|t|)+\frac{1}{2\sqrt{2\pi}}[h(|t+1|)\operatorname{sign}(t+1)-h(|t-1|)\operatorname{sign}(t-1)]+$$

$$+\frac{1}{2\sqrt{2\pi}}\int\limits_0^\infty\int\limits_0^\infty l(v)k(w)[e^{-w\cosh(t+v)}-e^{-w\cosh(t-v)}]dvdw.$$

Next, we consider the following system of two integral equations for $x>0$:

$$\begin{gathered}f(x)+\int\limits_0^\infty\theta_3(x,u)g(u)du+\int\limits_0^\infty\theta_4(x,u)g(u)du+\\+\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty\theta(x,u,v,w)h(u)g(v)h(w)dudvdw=p(x),\\\int\limits_0^\infty\theta_5(x,u)f(u)du+\int\limits_0^\infty\theta_6(x,u)f(u)du+g(x)=q(x).\end{gathered}\tag{2.9}$$

Here $\theta(x,u,v,w)$ is defined by (1.1), and

$$\theta_3(x,u)=\frac{1}{\sqrt{2\pi}}[h(u+x)+h(|u-x|)\operatorname{sign}(u-x)],$$

$$\theta_4(x,u)=\frac{1}{2\sqrt{2\pi}}[k(|x+u-1|)+k(|x-u+1|)-k(x+u+1)-k(|x-u-1|)],$$

$$\theta_5(x,u)=\frac{1}{\sqrt{2\pi}}[\psi(|x-u|)\operatorname{sign}(x-u)+\psi(x+u)],$$

$$\theta_6(x,u)=\frac{1}{2\sqrt{2\pi}}[\xi(|x+u-1|)+\xi(|x-u-1|)-\xi(x+u+1)-\xi(|x-u+1|)],$$

$h,k,l,\varphi,\psi,\xi,p,q$ are known functions, f,g are unknown functions.

Theorem 5. *Given that* $h,k,l,\psi,\xi_1,\xi_2,p,q\in L_1(\mathbb{R}_+)$ *and* $\varphi\in L_1\left(\frac{1}{\sqrt{w}},\mathbb{R}_+\right)$, $\xi=\xi_1\underset{3}{*}\xi_2$ *such that* $1-(F_cr)(y)\neq 0\ \ \forall y>0$, *where*

$$\begin{gathered}r(x)=(h\underset{3}{*}\psi)(x)+(\psi\overset{\gamma}{\underset{1}{*}}k)(x)+\underset{1}{*}(\psi,l,\varphi)(x)+\\+(h\overset{\gamma}{\underset{1}{*}}\xi)(x)+(\xi_1\overset{\gamma}{\underset{1}{*}}(\xi_2\overset{\gamma}{\underset{1}{*}}k))(x)+\underset{1}{*}(\xi_1,\xi_2\overset{\gamma}{*}l,\varphi)(x).\end{gathered}$$

Then the system (2.9) *has a unique solution in* $L_1(\mathbb{R}_+)\times L_1(\mathbb{R}_+)$ *whose closed formed as follows*

$$\begin{gathered}f(x)=p(x)-(q\underset{3}{*}h)(x)-(q\overset{\gamma}{\underset{1}{*}}k)(x)-\underset{1}{*}(q,l,\varphi)(x)+\\+(\eta\underset{1}{*}p)(x)-(\eta\underset{1}{*}(q\underset{3}{*}h))(x)-(\eta\underset{1}{*}(q\overset{\gamma}{*}k))(x)-(\eta\underset{1}{*}(\underset{1}{*}(q,l,\varphi))(x),\end{gathered}$$

$$g(x) = q(x) - (\psi \underset{2}{*} p)(x) - (\xi \overset{\gamma}{\underset{2}{*}} p)(x) +$$

$$+ (q \underset{2}{*} \eta)(x) - ((\psi \underset{2}{*} p) \underset{2}{*} \eta)(x) - ((\xi \overset{\gamma}{\underset{2}{*}} p) \underset{2}{*} \eta)(x).$$

Here, $\eta \in L_1(\mathbb{R}_+)$ is defined by

$$F_c\eta = \frac{(F_c r)(y)}{1 - (F_c r)(y)}.$$

Proof. Using Theorem 1 and (0.5) – (0.8) we write the system (2.9) in the form

$$(F_c f)(y) + (F_s g)(y)[(F_s h)(y) + \sin y (F_s k)(y) + (F_s l)(y)(K_{iy}\varphi)] = (F_c p)(y),$$

$$(F_c f)(y)[(F_s \psi)(y) + \sin y (F_c \xi)(y) + (F_s g)(y) = (F_s q)(y). \tag{2.10}$$

We obtain a system of two linear equations for $(F_c f)(y)$ and $(F_s g)(y)$. We have

$$\Delta = \begin{vmatrix} 1 & (F_s h)(y) + \sin y (F_c k)(y) + (F_s l)(y)(K_{iy}\varphi) \\ (F_s \psi)(y) + \sin y (F_c \xi)(y) & 1 \end{vmatrix} =$$

$$= 1 - (F_c r)(y). \tag{2.11}$$

In view of the Wiener – Levy theorem [10], by the given condition, there is a unique function $\eta \in L_1(\mathbb{R}_+)$ such that

$$(F_c \eta)(y) = \frac{(F_c r)(y)}{1 - (F_c r)(y)}. \tag{2.12}$$

From (2.11) and (2.12) we have

$$\frac{1}{\Delta} = 1 + (F_c \eta)(y). \tag{2.13}$$

On the other hand,

$$\Delta_1 = \begin{vmatrix} (F_c p)(y) & (F_s h)(y) + \sin y (F_c k)(y) + (F_s l)(y)(K_{iy}\varphi) \\ (F_s q)(y) & 1 \end{vmatrix} =$$

$$= (F_c p)(y) - F_c(q \underset{3}{*} h)(y) - F_c(q \overset{\gamma}{\underset{1}{*}} k)(y) - F_c(*(q, l, \varphi))(y). \tag{2.14}$$

Hence, from (2.13), (2.14) we have

$$(F_c f)(y) = \frac{\Delta_1}{\Delta} =$$

$$= [1 + (F_c \eta)(y)][(F_c p)(y) - F_c(q \underset{3}{*} h)(y) - F_c(q \overset{\gamma}{\underset{1}{*}} k)(y) - F_c(\underset{1}{*}(q, l, \varphi))(y)] =$$

$$= (F_c p)(y) - F_c(q \underset{3}{*} h)(y) - F_c(q \overset{\gamma}{\underset{1}{*}} k)(y) - F_c(\underset{1}{*}(q, l, \varphi))(y) + F_c(\eta \underset{1}{*} p) -$$

$$- F_c(\eta \underset{1}{*} (q \underset{3}{*} h)(y) - F_c(\eta \underset{1}{*} (q \overset{\gamma}{\underset{1}{*}} k)(y) - F_c(\eta \underset{1}{*} (\underset{1}{*}(q, l, \varphi)))(y).$$

It follows

$$f(x) = p(x) - (q \underset{3}{*} h)(x) - (q \overset{\gamma}{\underset{1}{*}} k)(x) - \underset{1}{*}(q, l, \varphi)(x) + (\eta \underset{1}{*} p)(x) -$$

$$-(\eta \underset{1}{*} (q \underset{3}{*} h))(x) - (\eta \underset{1}{*} (q \overset{\gamma}{*} k))(x) - (\eta \underset{1}{*} (\underset{1}{*}(q, l, \varphi))(x). \tag{2.15}$$

Similarly,

$$\Delta_2 = \begin{vmatrix} 1 & (F_c p)(y) \\ (F_s \psi)(y) + \sin y (F_c \xi)(y) & (F_s q)(y) \end{vmatrix} =$$

$$= (F_s q)(y) - F_s(\psi \underset{2}{*} p)(y) - F_s(\xi \overset{\gamma}{\underset{2}{*}} p)(y). \tag{2.16}$$

Using formula (2.13) and (2.16) we have

$$(F_s g)(y) = \frac{\Delta_2}{\Delta} =$$

$$= [1 + (F_c \eta)(y)][(F_s q)(y) - F_s(\psi \underset{2}{*} p)(y) - F_s(\xi \overset{\gamma}{\underset{2}{*}} p)(y)] =$$

$$= (F_s q)(y) - F_s(\psi \underset{2}{*} p)(y) - F_s(\xi \overset{\gamma}{\underset{2}{*}} p)(y) +$$

$$+ F_s(q \underset{2}{*} \eta)(y) - F_s((\psi \underset{2}{*} p) \underset{2}{*} \eta)(y) - F_s((\xi \overset{\gamma}{\underset{2}{*}} p) \underset{2}{*} \eta)(y).$$

It shows that

$$g(x) = q(x) - (\psi \underset{2}{*} p)(x) - (\xi \overset{\gamma}{\underset{2}{*}} p)(x) + (q \underset{2}{*} \eta)(x) - ((\psi \underset{2}{*} p) \underset{2}{*} \eta)(x) - ((\xi \overset{\gamma}{\underset{2}{*}} p) \underset{2}{*} \eta)(x). \tag{2.17}$$

From (2.10), (2.15), (2.17), system (2.9) has a solution (f, g) in $L_1(\mathbb{R}_+) \times L_1(\mathbb{R}_+)$.

Theorem 5 is proved.

1. *Sneddon I. N.* The use of integral transforms. – New York: McGray-Hill, 1951. – 668 p.
2. *Yakubovich S. B.* On the convolution for Kontorovich – Lebedev integral transform and its application to integral transform // Dokl. Akad. Nauk BSSR. – 1987. – **31**. – P. 101 – 103.
3. *Erdely A. et al.* Higher transcendental functions. – New York: McGraw-Hill, 1953. – **1**. – 299 p.
4. *Kakichev V. A.* On the convolution for integral transforms // Izv. Vysh. Uchebn. Zaved. Mat. – 1967. – № 2. – S. 53 – 62.
5. *Nguyen Xuan Thao, Kakichev V. A., Vu Kim Tuan.* On the generalized convolution for Fourier cosine and sine transforms // East-West J. Math. – 1998. – **1**. – P. 85 – 90.
6. *Nguyen Xuan Thao, Vu Kim Tuan, Nguyen Minh Khoa.* On the generalized convolution with a weight-function for the Fourier cosine and sine transforms // Frac. Cal. and Appl. Anal. – 2004. – **7**, № 3. – P. 323 – 337.
7. *Nguyen Xuan Thao, Nguyen Minh Khoa.* On the generalized convolution with a weight- function for the Fourier sine and cosine transforms // Integral Transforms and Special Functions. – 2006. – **17**, № 9. – P. 673 – 685.
8. *Kakichev V. A.* Polyconvolution. – Taganrog: TPTU, 1997. – 54 p.
9. *Nguyen Xuan Thao.* On the polyconvolution for integral transforms // Vestn. NovGU. Ser. Estestv. and Tehn. Nauki. – 1999. – **10**. – S. 101 – 110.
10. *Betancor J. J., Claudio J., Molina, Sandra M., Lourdes R.-M.* Distributional convolutions for Fourier transforms // J. Math. Anal. and Appl. – 2007. – **325**, № 1. – P. 459 – 468.

Received 10.12.09