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WELL-POSED REDUCTION FORMULAE FOR q-KAMPÉ DE FÉRIET FUNCTION

КОРЕКТНІ ФОРМУЛИ РЕДУКЦІЇ ДЛЯ q-ФУНКЦІЇ КАМПЕ ДЕ ФЕР'Є

By means of the limiting case $n \to \infty$ of Watson's q-Whipple transformation, we investigate transformations on the nonterminating q-Kampé de Fériet series. Further new transformation and well-posed reduction formulae are established for the basic Clausen hypergeometric series. Several remarkable formulae are found also for new function classes beyond q-Kampé de Fériet function.

За допомогою граничного випадку $n \to \infty$ для ватсонівського q-перетворення Віппла досліджено перетворення нескінченного q-ряду Кампе де Фер'є. Крім того, встановлено нові формули перетворень та коректні формули редукції для базового гіпергеометричного ряду Клаузена. Декілька важливих формул знайдено також для нових класів функцій, до яких не належить q-функція Кампе де Фер'є.

1. Introduction and motivation. For the two indeterminates x and q, the shifted factorial of x with base q reads as

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-qx)\dots(1-q^{n-1}x) \quad \text{for} \quad n \in \mathbb{N}.$$

When $|q| < 1$, there are two well-defined infinite product expressions

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = (x;q)_\infty \,/\, (q^n x;q)_\infty .$$

The product and fraction of shifted factorials are abbreviated respectively to

$$[\,\alpha,\ \beta,\ \dots,\ \gamma; q\,]_n = (\alpha;q)_n\,(\beta;q)_n \dots (\gamma;q)_n\,,$$

$$\left[\begin{matrix}\alpha, & \beta, & \dots, & \gamma\\ A, & B, & \dots, & C\end{matrix}\,\middle|\, q\right]_n = \frac{(\alpha;q)_n\,(\beta;q)_n \dots (\gamma;q)_n}{(A;q)_n\,(B;q)_n \dots (C;q)_n}.$$

Following Bailey [1] and Gasper, Rahman [5], the basic hypergeometric series is defined by

$${}_{1+r}\phi_s\left[\begin{matrix}a_0, & a_1, & \dots, & a_r\\ & b_1, & \dots, & b_s\end{matrix}\,\middle|\, q; z\right] =$$

$$= \sum_{n=0}^{\infty}\left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r}\left[\begin{matrix}a_0, & a_1, & \dots, & a_r\\ q, & b_1, & \dots, & b_s\end{matrix}\,\middle|\, q\right]_n z^n,$$

where the base q will be restricted to $|q| < 1$ for nonterminating q-series.

Among the q-series transformations, Watson's one is fundamental (cf. [5], III-18)

$${}_8\phi_7\left[\begin{matrix}a,\ q\sqrt{a},\ -q\sqrt{a},\ b,\ c,\ d,\ e,\ q^{-n}\\ \sqrt{a},\ -\sqrt{a},\ qa/b,\ qa/c,\ qa/d,\ qa/e,\ q^{n+1}a\end{matrix}\,\middle|\, q; \frac{q^{n+2}a^2}{bcde}\right] = \qquad (1a)$$

$$= \left[\begin{matrix} qa,\ qa/bc \\ qa/b, qa/c \end{matrix} \middle| q\right]_n {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b, & c, & qa/de \\ & qa/d, & qa/e, & q^{-n}bc/a \end{matrix} \middle| q;\ q\right]. \tag{1b}$$

Its limiting case $n \to \infty$ reads equivalently as the nonterminating transformation

$$ {}_3\phi_2 \left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix} \middle| q; \frac{bd}{ace}\right] = \left[\begin{matrix} bd/ae, bd/ce \\ bd/e, bd/ace \end{matrix} \middle| q\right]_\infty \times \tag{2a}$$

$$\times \sum_{i=0}^{\infty} (-1)^i \frac{1 - q^{2i-1}bd/e}{1 - bd/qe} q^{\binom{i}{2}} \left[\begin{matrix} bd/qe, & a, & c, & b/e, & d/e \\ q, & bd/ae, & bd/ce, & d, & b \end{matrix} \middle| q\right]_i \left(\frac{bd}{ac}\right)^i. \tag{2b}$$

As the q-analogue of Kampé de Fériet function, Srivastava and Karlsson [10, p. 349] define the generalized bivariate basic hypergeometric function by

$$\Phi^{\lambda:r;s}_{\mu:u;v} \left[\begin{matrix} \alpha_1, \ldots, \alpha_\lambda : & a_1, \ldots, a_r; & c_1, \ldots, c_s; & q : x, y \\ \beta_1, \ldots, \beta_\mu : & b_1, \ldots, b_u; & d_1, \ldots, d_v; & i, j, k \end{matrix}\right] =$$

$$= \sum_{m,n=0}^{\infty} \frac{[\alpha_1, \ldots, \alpha_\lambda; q]_{m+n}}{[\beta_1, \ldots, \beta_\mu; q]_{m+n}} \frac{[a_1, \ldots, a_r; q]_m [c_1, \ldots, c_s; q]_n}{[b_1, \ldots, b_u; q]_m [d_1, \ldots, d_v; q]_n} \frac{x^m y^n q^{i\binom{m}{2} + j\binom{n}{2} + kmn}}{(q;q)_m (q;q)_n}.$$

When $i, j, k \in \mathbb{N}_0$, this double series $\Phi^{\lambda:r;s}_{\mu:u;v}$ is convergent for $|x| < 1$, $|y| < 1$ and $|q| < 1$. There has not been much attention to this series in the literature. By means of the q-analogue of Kummer – Thomae – Whipple and the Hall transformation on ${}_3\phi_2$-series (cf. [5], III-9 and III-10), Chu, Jia [3] and Jia, Wang [7] investigated systematically summation and reduction formulae respectively for the $\Phi^{0:3;\lambda}_{1:1;\mu}$ and $\Phi^{1:2;\lambda}_{1:1;\mu}$ series. Chu, Jia [2] and Chu, Srivastava [4] derived several transformation and reduction formulae respectively by inversion techniques and formal power series method. Jeugt [6] determined invariant transformation group for the double Clausen series with $\lambda + r = 3$ and $\mu + u = 2$. Further works can be found in Jia, Wang [8] and Singh [9] as well as those cited by Chu, Jia [3].

The purpose of this paper is to investigate the above defined nonterminating q-Kampé de Fériet function exclusively by employing the limiting transformation (2a), (2b). The rest of the paper will be divided into five sections, devoted respectively to the five series labeled by

$$\Phi^{0:3;\lambda}_{1:1;\mu}, \quad \Phi^{2:1;\lambda}_{2:0;\mu}, \quad \Phi^{1:2;\lambda}_{2:0;\mu}, \quad \Phi^{1:2;\lambda}_{1:1;\mu}, \quad \Phi^{0:3;\lambda}_{2:0;\mu}.$$

Several transformation and well-posed reduction formulae on these five series will be established which can briefly be commented as follows:

Most of the reduction formulae displayed from Section 2 to Section 4 are well-posed, unlike the usual ones appeared in Chu, Jia [3] and Jia, Wang [7].

Even though the two double series $\Phi^{0:3;\lambda}_{1:1;\mu}$ and $\Phi^{1:2;\lambda}_{1:1;\mu}$ have intensively been studied by Chu, Jia [3] and Jia, Wang [7], the results shown in Section 2 and Section 5 are substantially different.

It is remarkable that there exist reduction formulae for new function classes beyond the q-Kampé de Fériet function, which are exemplified for the two transformations proven in Sections 5 and 6.

2. Nonterminating double series $\Phi^{0:3;\lambda}_{1:1;\mu}$. By means of the two transformations for ${}_3\phi_2$-series (cf. [5], III-9 and III-10), the double $\Phi^{0:3;\lambda}_{1:1;\mu}$ series has intensively been investigated by Chu, Jia [3] (§ 2), where numerous reduction and summation formulae have been obtained. Instead, we shall utilize (2a), (2b) to transform this $\Phi^{0:3;\lambda}_{1:1;\mu}$ series into $\Phi^{2:3;\lambda+1}_{3:1;\mu+1}$ series modified by a "well-posed" factor, which yields four well-posed reduction formulae for the former and two unusual ones for the latter.

Theorem 1. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$\left[\begin{matrix} bd/e, bd/ace \\ bd/ae, bd/ce \end{matrix}\middle| q\right]_\infty \sum_{i,j=0}^{\infty} \frac{\Omega(j)}{(b;q)_{i+j}} \left[\begin{matrix} a, c, e \\ q,\ d \end{matrix}\middle| q\right]_i \left(\frac{q^j bd}{ace}\right)^i = \tag{3a}$$

$$= \sum_{i,j=0}^{\infty} \frac{1-q^{2i+j-1}bd/e}{1-bd/qe} \left[\begin{matrix} b/e,\ bd/qe \\ b, bd/ae, bd/ce \end{matrix}\middle| q\right]_{i+j} \times \tag{3b}$$

$$\times q^{\binom{i}{2}+ij} \left[\begin{matrix} a, c, d/e \\ q,\ d \end{matrix}\middle| q\right]_i \left(-\frac{bd}{ac}\right)^i \left[\begin{matrix} bd/ace \\ b/e \end{matrix}\middle| q\right]_j \Omega(j) \tag{3c}$$

provided that both double series displayed above are absolutely convergent.

Proof. The theorem follows directly by writing the double sum in (3a) as

$$\sum_{i,j=0}^{\infty} \frac{\Omega(j)}{(b;q)_{i+j}} \left[\begin{matrix} a, c, e \\ q,\ d \end{matrix}\middle| q\right]_i \left(\frac{q^j bd}{ace}\right)^i = \sum_{j=0}^{\infty} \frac{\Omega(j)}{(b;q)_j} {}_3\phi_2 \left[\begin{matrix} a, & c, & e \\ & q^j b, & d \end{matrix}\middle| q; \frac{q^j bd}{ace}\right] \tag{4}$$

and then transforming, via (2a), (2b), the ${}_3\phi_2$-series into

$${}_3\phi_2 \left[\begin{matrix} a, c, e \\ q^j b, d \end{matrix}\middle| q; \frac{q^j bd}{ace}\right] = \left[\begin{matrix} q^j bd/ae, q^j bd/ce \\ q^j bd/e, q^j bd/ace \end{matrix}\middle| q\right]_\infty \sum_{i=0}^{\infty} (-1)^i \frac{1-q^{2i+j-1}bd/e}{1-q^{j-1}bd/e} \times$$

$$\times q^{\binom{i}{2}+ij} \left[\begin{matrix} q^{j-1}bd/e, a, c, q^j b/e, d/e \\ q, q^j bd/ae, q^j bd/ce, d, q^j b \end{matrix}\middle| q\right]_i \left(\frac{bd}{ac}\right)^i =$$

$$= \left[\begin{matrix} bd/ae, bd/ce \\ bd/e, bd/ace \end{matrix}\middle| q\right]_\infty \left[\begin{matrix} b, bd/ace \\ b/e \end{matrix}\middle| q\right]_j \sum_{i=0}^{\infty} (-1)^i \left[\begin{matrix} a, c, d/e \\ q,\ d \end{matrix}\middle| q\right]_i \times$$

$$\times \frac{1-q^{2i+j-1}bd/e}{1-bd/qe} q^{\binom{i}{2}+ij} \left[\begin{matrix} b/e,\ bd/qe \\ b, bd/ae, bd/ce \end{matrix}\middle| q\right]_{i+j} \left(\frac{bd}{ac}\right)^i.$$

Theorem 1 is proved.

Instead, applying the q-analogue of Kummer – Thomae – Whipple transformation (cf. [5], III-9), we can reformulate the ${}_3\phi_2$-series displayed in (4) as

$${}_3\phi_2 \left[\begin{matrix} a, & c, & e \\ & q^j b, & d \end{matrix}\middle| q; \frac{q^j bd}{ace}\right] = \left[\begin{matrix} d/a, q^j bd/ce \\ d,\ q^j bd/ace \end{matrix}\middle| q\right]_\infty {}_3\phi_2 \left[\begin{matrix} a, q^j b/c, q^j b/e \\ q^j b, q^j bd/ce \end{matrix}\middle| q; \frac{d}{a}\right]$$

where the ${}_3\phi_2$-series on the right-hand side of the last equation can further be restated by means of (2a), (2b) as

$${}_3\phi_2 \left[\begin{matrix} a, q^j b/c, q^j b/e \\ q^j b, q^j bd/ce \end{matrix}\middle| q; \frac{d}{a}\right] = \left[\begin{matrix} q^j bd/ac, q^j bd/ae \\ d/a,\ q^{2j} b^2 d/ace \end{matrix}\middle| q\right]_\infty \sum_{i=0}^{\infty} \frac{1-q^{2i+2j-1}b^2 d/ace}{1-q^{2j-1}b^2 d/ace} \times$$

$$\times(-1)^i q^{\binom{i}{2}} \left[\begin{matrix} q^{2j-1}b^2d/ace, & q^jb/a, & q^jb/c, & q^jb/e, & q^jbd/ace \\ q, & q^jb, & q^jbd/ac, & q^jbd/ae, & q^jbd/ce \end{matrix} \middle| q\right]_i d^i.$$

Substituting them successively into (4), we find the following alternative expression:

$$\left[\begin{matrix} d,\ bd/ace,\ b^2d/ace \\ bd/ac, bd/ae, bd/ce \end{matrix} \middle| q\right]_\infty \sum_{i,j=0}^{\infty} \frac{\Omega(j)}{(b;q)_{i+j}} \left[\begin{matrix} a, c, e \\ q,\ d \end{matrix} \middle| q\right]_i \left(\frac{q^jbd}{ace}\right)^i =$$

$$= \sum_{j=0}^{\infty} \frac{\Omega(j)}{[b/a, b/c, b/e; q]_j} \sum_{i=0}^{\infty} (-1)^i \frac{1-q^{2i+2j-1}b^2d/ace}{1-b^2d/qace} q^{\binom{i}{2}} \times$$

$$\times \left[\begin{matrix} b/a, & b/c, & b/e, & bd/ace \\ b, & bd/ac, & bd/ae, & bd/ce \end{matrix} \middle| q\right]_{i+j} \frac{(b^2d/qace;q)_{i+2j}}{(q;q)_i} d^i.$$

Letting $n := i+j$ and then keeping in mind of the relation

$$\frac{(b^2d/qace;q)_{n+j}}{(q;q)_{n-j}} = (-1)^j q^{jn-\binom{j}{2}} \frac{(b^2d/qace;q)_n}{(q;q)_n} \left[q^{-n}, q^{n-1}b^2d/ace; q\right]_j$$

we can equivalently reformulate Theorem 1 as another transformation.

Theorem 2. *For an arbitrary sequence* $\{\Omega(j)\}$, *there holds the transformation*

$$\left[\begin{matrix} d,\ bd/ace,\ b^2d/ace \\ bd/ac, bd/ae, bd/ce \end{matrix} \middle| q\right]_\infty \sum_{i,j=0}^{\infty} \frac{\Omega(j)}{(b;q)_{i+j}} \left[\begin{matrix} a, c, e \\ q,\ d \end{matrix} \middle| q\right]_i \left(\frac{q^jbd}{ace}\right)^i = \tag{5a}$$

$$= \sum_{n=0}^{\infty} \frac{1-q^{2n-1}b^2d/ace}{1-b^2d/qace} \left[\begin{matrix} b/a, & b/c, & b/e, & bd/ace, & b^2d/qace \\ q, & b, & bd/ac, & bd/ae, & bd/ce \end{matrix} \middle| q\right]_n \times \tag{5b}$$

$$\times(-1)^n q^{\binom{n}{2}} d^n \sum_{j=0}^{n} \left(\frac{q}{d}\right)^j \left[\begin{matrix} q^{-n}, q^{n-1}b^2d/ace \\ b/a,\ b/c,\ b/e \end{matrix} \middle| q\right]_j \Omega(j) \tag{5c}$$

provided that both double series displayed above are absolutely convergent.

According to Theorems 1 and 2, we are now going to derive five reduction transformation formulae by concretely specifying $\Omega(j)$ in five different manners.

2.1. For the $\Omega(j)$ sequence given by

$$\Omega(j) = \left[\begin{matrix} \alpha,\ b/a,\ b/c,\ b/e \\ q, w, b^2d\alpha/wace \end{matrix} \middle| q\right]_j d^j$$

evaluating the inner sum with respect to j in (5) by means of the q-Pfaff–Saalschütz theorem (cf. [5], II-12)

$${}_3\phi_2 \left[\begin{matrix} q^{-n}, & a, & b \\ & c, & q^{1-n}ab/c \end{matrix} \middle| q;q\right] = \left[\begin{matrix} c/a, c/b \\ c,\ c/ab \end{matrix} \middle| q\right]_n \tag{6}$$

and then simplifying the corresponding equation displayed in Theorem 2, we obtain the following reduction formula.

Proposition 1 (well-posed reduction formula).

$$\sum_{i,j=0}^{\infty}\frac{q^{ij}d^j}{(b;q)_{i+j}}\left[\begin{matrix}a,c,e\\ q,\ d\end{matrix}\middle|\,q\right]_i\left[\begin{matrix}\alpha,\ b/a,\ b/c,\ b/e\\ q,w,b^2d\alpha/wace\end{matrix}\middle|\,q\right]_j\left(\frac{bd}{ace}\right)^i=$$

$$=\left[\begin{matrix}bd/ac,bd/ae,bd/ce\\ d,\ bd/ace,\ b^2d/ace\end{matrix}\middle|\,q\right]_\infty\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\frac{1-q^{2n-1}b^2d/ace}{1-b^2d/qace}\times$$

$$\times\left[\begin{matrix}b/a, & b/c, & b/e, & w/\alpha, & bd/ace, & b^2d/qace, & b^2d/wace\\ q, & b, & w, & bd/ac, & bd/ae, & bd/ce, & b^2d\alpha/wace\end{matrix}\middle|\,q\right]_n(d\alpha)^n.$$

The limiting case $\alpha\to\infty$ of this proposition yields an interesting transformation.

***Corollary* 1** (well-posed reduction formula).

$$\sum_{i,j=0}^{\infty}\frac{q^{ij}}{(b;q)_{i+j}}\left[\begin{matrix}a,c,e\\ q,\ d\end{matrix}\middle|\,q\right]_i\left[\begin{matrix}b/a,b/c,b/e\\ q,\ w\end{matrix}\middle|\,q\right]_j\left(\frac{bd}{ace}\right)^i\left(\frac{wace}{b^2}\right)^j=$$

$$=\left[\begin{matrix}bd/ac,bd/ae,bd/ce\\ d,\ bd/ace,\ b^2d/ace\end{matrix}\middle|\,q\right]_\infty\times$$

$$\times{}_8\phi_7\left[\begin{matrix}\frac{b^2d}{qace}, & \sqrt{\frac{qb^2d}{ace}}, & -\sqrt{\frac{qb^2d}{ace}}, & \frac{b}{a}, & \frac{b}{c}, & \frac{b}{e}, & \frac{bd}{ace}, & \frac{b^2d}{wace}\\ & \sqrt{\frac{b^2d}{qace}}, & -\sqrt{\frac{b^2d}{qace}}, & b, & w, & \frac{bd}{ac}, & \frac{bd}{ae}, & \frac{bd}{ce}\end{matrix}\middle|\,q;\ \frac{wace}{b^2}\right].$$

2.2. For the $\Omega(j)$ sequence defined by

$$\Omega(j)=q^{\binom{j}{2}}\frac{[b/a,b/c,b/e;q]_j}{(q;q)_j(b^2d/ace;q^2)_j}d^j$$

evaluating the inner sum with respect to j displayed in (5) through the q-analogue of Gauss' ${}_2F_1\left(\frac{1}{2}\right)$ sum (cf. [5], II-11)

$${}_2\phi_2\left[\begin{matrix}a, & b\\ \sqrt{qab}, & -\sqrt{qab}\end{matrix}\middle|\,q;-q\right]=\left[\begin{matrix}qa,qb\\ q,qab\end{matrix}\middle|\,q^2\right]_\infty \tag{7}$$

and then simplifying the corresponding equation in Theorem 2, we get the following reduction formula.

Proposition 2 (well-posed reduction formula).

$$\sum_{i,j=0}^{\infty}q^{\binom{j}{2}}\left[\begin{matrix}a,c,e\\ q,\ d\end{matrix}\middle|\,q\right]_i\frac{(q^id)^j[b/a,b/c,b/e;q]_j}{(b;q)_{i+j}(q;q)_j(b^2d/ace;q^2)_j}\left(\frac{bd}{ace}\right)^i=$$

$$=\left[\begin{matrix}bd/ac,bd/ae,bd/ce\\ d,\ bd/ace,\ b^2d/ace\end{matrix}\middle|\,q\right]_\infty\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\frac{1-q^{4n-1}b^2d/ace}{1-b^2d/qace}\times$$

$$\times\left[\begin{matrix}b/a, & b/c, & b/e, & bd/ace\\ b, & bd/ac, & bd/ae, & bd/ce\end{matrix}\middle|\,q\right]_{2n}\frac{(b^2d/qace;q^2)_n}{(q^2;q^2)_n}d^{2n}.$$

2.3. For the $\Omega(j)$ sequence specified by

$$\Omega(j) = \left[\begin{matrix} b/a, b/c, b/e \\ q, \quad w \end{matrix} \middle| q\right]_j \frac{(w;q^2)_j}{(b^2d/ace;q^2)_j} d^j$$

evaluating the inner sum with respect to j displayed in (5) via Andrews' terminating q-analogue of the Watson ${}_3F_2$-sum (cf. [5], II-17)

$${}_4\phi_3 \left[\begin{matrix} q^{-n}, & q^{n+1}a^2, & c, & -c \\ & c^2, & qa, & -qa \end{matrix} \middle| q;q\right] =$$

$$= \begin{cases} 0, & n - \text{odd}, \\ c^n \left[\begin{matrix} q, \; q^2a^2/c^2 \\ q^2a^2, \; qc^2 \end{matrix} \middle| q^2\right]_m, & n = 2m, \end{cases} \tag{8}$$

and then simplifying the corresponding equation in Theorem 2, we find the following reduction formula.

Proposition 3 (well-posed reduction formula)**.**

$$\sum_{i,j=0}^{\infty} \frac{(q^i d)^j}{(b;q)_{i+j}} \left[\begin{matrix} a, c, e \\ q, \; d \end{matrix} \middle| q\right]_i \left[\begin{matrix} b/a, b/c, b/e \\ q, \quad w \end{matrix} \middle| q\right]_j \frac{(w;q^2)_j}{(b^2d/ace;q^2)_j} \left(\frac{bd}{ace}\right)^i =$$

$$= \left[\begin{matrix} bd/ac, bd/ae, bd/ce \\ d, \; bd/ace, \; b^2d/ace \end{matrix} \middle| q\right]_\infty \sum_{n=0}^{\infty} q^{\binom{2n}{2}} \frac{1 - q^{4n-1}b^2d/ace}{1 - b^2d/qace} \times$$

$$\times \left[\begin{matrix} b/a, & b/c, & b/e, & bd/ace \\ b, & bd/ac, & bd/ae, & bd/ce \end{matrix} \middle| q\right]_{2n} \left[\begin{matrix} b^2d/qace, & b^2d/wace \\ q^2, & qw \end{matrix} \middle| q^2\right]_n (d^2w)^n.$$

2.4. In Theorem 1, rewrite the double sum (3a) as

$$\sum_{i=0}^{\infty} \left[\begin{matrix} a, & c, & e \\ q, & b, & d \end{matrix} \middle| q\right]_i \left(\frac{bd}{ace}\right)^i \sum_{j=0}^{\infty} \frac{\Omega(j)}{(q^i b;q)_j} q^{ij}. \tag{9}$$

Specializing the $\Omega(j)$ sequence explicitly by

$$\Omega(j) = \left[\begin{matrix} \beta, \gamma \\ q \end{matrix} \middle| q\right]_j \left(\frac{b}{\beta\gamma}\right)^j$$

and then evaluating the sum with respect to j displayed in (9) by means of the q-Gauss summation theorem (cf. [5], II-8)

$${}_2\phi_1 \left[\begin{matrix} a, & b \\ & c \end{matrix} \middle| q; \frac{c}{ab}\right] = \left[\begin{matrix} c/a, c/b \\ c, \; c/ab \end{matrix} \middle| q\right]_\infty \tag{10}$$

we have from Theorem 1 the following interesting reduction formula.

Proposition 4 (reduction formula)**.**

$$\sum_{i,j=0}^{\infty} \frac{1 - q^{2i+j-1}bd/e}{1 - bd/qe} \left[\begin{matrix} b/e, \; bd/qe \\ b, bd/ae, bd/ce \end{matrix} \middle| q\right]_{i+j} \times$$

$$\times q^{\binom{i}{2}} \begin{bmatrix} a, c, d/e \\ q,\ d \end{bmatrix} q \Bigg]_i \left(-\frac{bd}{ac}\right)^i \begin{bmatrix} \beta, \gamma, bd/ace \\ q,\ b/e \end{bmatrix} q \Bigg]_j \left(\frac{q^i b}{\beta\gamma}\right)^j =$$

$$= \left[\begin{matrix} b/\beta, b/\gamma, bd/e, bd/ace \\ b, b/\beta\gamma, bd/ae, bd/ce \end{matrix} \middle| q \right]_\infty {}_4\phi_3 \left[\begin{matrix} a, & c, & e, & b/\beta\gamma \\ & d, & b/\beta, & b/\gamma \end{matrix} \middle| q; \frac{bd}{ace}\right].$$

2.5. Alternatively, specializing the $\Omega(j)$ sequence explicitly by

$$\Omega(j) = q^{\binom{j}{2}} \frac{[\beta, q/\beta; q]_j}{(q^2; q^2)_j} b^j$$

and then evaluating the sum with respect to j displayed in (9) through the q-analogue of Bailey's ${}_2F_1\left(\frac{1}{2}\right)$ sum (cf. [5], II-10)

$$ {}_2\phi_2 \left[\begin{matrix} a, & q/a \\ -q, & b \end{matrix} \middle| q; -b\right] = \frac{[ab, qb/a; q^2]_\infty}{(b; q)_\infty} \tag{11}$$

we derive from Theorem 1 another strange reduction formula.

Proposition 5 (reduction formula)**.**

$$\sum_{i,j=0}^{\infty} \frac{1 - q^{2i+j-1} bd/e}{1 - bd/qe} \left[\begin{matrix} b/e,\ bd/qe \\ b, bd/ae, bd/ce \end{matrix} \middle| q\right]_{i+j} \left(-\frac{bd}{ac}\right)^i \times$$

$$\times q^{\binom{i+j}{2}} \left[\begin{matrix} a, c, d/e \\ q,\ d \end{matrix} \middle| q\right]_i \left[\begin{matrix} \beta, q/\beta, bd/ace \\ q,\ -q,\ b/e \end{matrix} \middle| q\right]_j b^j =$$

$$= \left[\begin{matrix} b/\beta, bd/e, bd/ace \\ b,\ bd/ae,\ bd/ce \end{matrix} \middle| q\right]_\infty \sum_{i=0}^{\infty} \left[\begin{matrix} a, & c, & e \\ q, & d, & b/\beta \end{matrix} \middle| q\right]_i \left[\begin{matrix} q^i b\beta \\ q^i b/\beta \end{matrix} \middle| q^2\right]_\infty \left(\frac{bd}{ace}\right)^i.$$

3. Nonterminating double series $\Phi^{2:1;\lambda}_{2:0;\mu}$. There is a theorem connecting the $\Phi^{0:3;\lambda}_{1:1;\mu}$ series in the last section to $\Phi^{2:1;\lambda}_{2:0;\mu+1}$ series due to Chu, Jia [3] (Theorem 2.2), where no reduction formulae was given for this last series. By means of (2a), (2b), this section will prove a transformation theorem for this $\Phi^{2:1;\lambda}_{2:0;\mu}$ series and then derive four well-posed reduction formulae.

Theorem 3. *For an arbitrary sequence* $\{\Omega(j)\}$, *there holds the transformation*

$$\left[\begin{matrix} bd/e, bd/ace \\ bd/ae, bd/ce \end{matrix} \middle| q\right]_\infty \sum_{i,j=0}^{\infty} \left(\frac{bd}{ace}\right)^i \left[\begin{matrix} a, & c \\ b, & d \end{matrix} \middle| q\right]_{i+j} \left[\begin{matrix} e \\ q \end{matrix} \middle| q\right]_i \Omega(j) = \tag{12a}$$

$$= \sum_{n=0}^{\infty} \frac{1 - q^{2n-1} bd/e}{1 - bd/qe} \left[\begin{matrix} a, & c, & b/e, & d/e, & bd/qe \\ q, & b, & d, & bd/ae, & bd/ce \end{matrix} \middle| q\right]_n \times \tag{12b}$$

$$\times (-1)^n q^{\binom{n}{2}} \left(\frac{bd}{ac}\right)^n \sum_{j=0}^{n} \left(\frac{qac}{bd}\right)^j \left[\begin{matrix} q^{-n}, & q^{n-1} bd/e \\ b/e, & d/e \end{matrix} \middle| q\right]_j \Omega(j) \tag{12c}$$

provided that both double series displayed above are absolutely convergent.

Proof. Expressing the double sum in (12a) as

$$\sum_{i,j=0}^{\infty}\left(\frac{bd}{ace}\right)^i\left[\begin{matrix}a, & c\\ b, & d\end{matrix}\,\middle|\, q\right]_{i+j}\left[\begin{matrix}e\\ q\end{matrix}\,\middle|\, q\right]_i\Omega(j)= \tag{13a}$$

$$=\sum_{j=0}^{\infty}\Omega(j)\left[\begin{matrix}a, & c\\ b, & d\end{matrix}\,\middle|\, q\right]_j {}_3\phi_2\left[\begin{matrix}q^ja, q^jc, e\\ q^jb, q^jd\end{matrix}\,\middle|\, q;\frac{bd}{ace}\right] \tag{13b}$$

and then transforming, by (2a), (2b), the above ${}_3\phi_2$-series into

$${}_3\phi_2\left[\begin{matrix}q^ja, q^jc, e\\ q^jb, q^jd\end{matrix}\,\middle|\, q;\frac{bd}{ace}\right]=\left[\begin{matrix}q^jbd/ae, q^jbd/ce\\ q^{2j}bd/e,\ bd/ace\end{matrix}\,\middle|\, q\right]_\infty\sum_{i=0}^{\infty}\frac{1-q^{2i+2j-1}bd/e}{1-q^{2j-1}bd/e}\times$$

$$\times(-1)^iq^{\binom{i}{2}}\left[\begin{matrix}q^{2j-1}bd/e, & q^ja, & q^jc, & q^jd/e, & q^jb/e\\ q, & q^jbd/ae, & q^jbd/ce, & q^jb, & q^jd\end{matrix}\,\middle|\, q\right]_i\left(\frac{bd}{ac}\right)^i=$$

$$=\left[\begin{matrix}bd/ae, bd/ce\\ bd/e, bd/ace\end{matrix}\,\middle|\, q\right]_\infty\left[\begin{matrix}b, \quad d\\ a, c, b/e, d/e\end{matrix}\,\middle|\, q\right]_j\sum_{i=0}^{\infty}\frac{1-q^{2i+2j-1}bd/e}{1-bd/qe}\times$$

$$\times(-1)^iq^{\binom{i}{2}}\left[\begin{matrix}a, & c, & b/e, & d/e\\ b, & d, & bd/ae, & bd/ce\end{matrix}\,\middle|\, q\right]_{i+j}\frac{(bd/qe;q)_{i+2j}}{(q;q)_i}\left(\frac{bd}{ac}\right)^i$$

we derive the following equality:

$$\text{Eq(12a)}=\sum_{j=0}^{\infty}\frac{\Omega(j)}{[b/e,d/e;q]_j}\sum_{i=0}^{\infty}(-1)^i\frac{1-q^{2i+2j-1}bd/e}{1-bd/qe}q^{\binom{i}{2}}\times$$

$$\times\left[\begin{matrix}a, & c, & b/e, & d/e\\ b, & d, & bd/ae, & bd/ce\end{matrix}\,\middle|\, q\right]_{i+j}\frac{(bd/qe;q)_{i+2j}}{(q;q)_i}\left(\frac{bd}{ac}\right)^i.$$

Relabeling the summation indices by $n:=i+j$ leads us to another expression

$$\text{Eq(12a)}=\sum_{n=0}^{\infty}\frac{1-q^{2n-1}bd/e}{1-bd/qe}\left[\begin{matrix}a, & c, & b/e, & d/e\\ b, & d, & bd/ae, & bd/ce\end{matrix}\,\middle|\, q\right]_n\times$$

$$\times\sum_{j=0}^{n}\frac{\Omega(j)}{[b/e,d/e;q]_j}\frac{(bd/qe;q)_{n+j}}{(q;q)_{n-j}}(-1)^{n-j}q^{\binom{n-j}{2}}\left(\frac{bd}{ac}\right)^{n-j}$$

which is equivalent to the transformation in Theorem 3 in view of the relation

$$\frac{(bd/qe;q)_{n+j}}{(q;q)_{n-j}}=(-1)^jq^{jn-\binom{j}{2}}\frac{(bd/qe;q)_n}{(q;q)_n}\left[q^{-n},q^{n-1}bd/e;q\right]_j.$$

Theorem 3 is proved.

By specifying the $\Omega(j)$ sequence in terms of shifted factorial fractions, we shall derive from Theorem 3 three reduction formulae.

3.1. In Theorem 3, specializing the $\Omega(j)$ sequence firstly by

$$\Omega(j)=\left[\begin{matrix}\alpha, & b/e, & d/e\\ q, & w, & bd\alpha/we\end{matrix}\middle|\, q\right]_j\left(\frac{bd}{ac}\right)^j$$

and then evaluating the inner sum with respect to j displayed in (12c) by means of (6) lead us to the following reduction formula.

Proposition 6 (well-posed reduction formula)**.**

$$\left[\begin{matrix}bd/e, bd/ace\\ bd/ae, bd/ce\end{matrix}\middle|\, q\right]_\infty\sum_{i,j=0}^{\infty}\left[\begin{matrix}a, & c\\ b, & d\end{matrix}\middle|\, q\right]_{i+j}\left[\begin{matrix}e\\ q\end{matrix}\middle|\, q\right]_i\times$$

$$\times\left[\begin{matrix}\alpha, & b/e, & d/e\\ q, & w, & bd\alpha/we\end{matrix}\middle|\, q\right]_j\left(\frac{bd}{ace}\right)^i\left(\frac{bd}{ac}\right)^j=$$

$$=\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{1-q^{2n-1}bd/e}{1-bd/qe}\times$$

$$\times\left[\begin{matrix}a, & c, & b/e, & d/e, & bd/qe, & w/\alpha, & bd/we\\ q, & b, & d, & w, & bd/ae, & bd/ce, & bd\alpha/we\end{matrix}\middle|\, q\right]_n\left(\frac{bd\alpha}{ac}\right)^n.$$

When $\alpha\to\infty$, this proposition results in an interesting transformation.

***Corollary* 2** (well-posed reduction formula)**.**

$$\left[\begin{matrix}bd/e, bd/ace\\ bd/ae, bd/ce\end{matrix}\middle|\, q\right]_\infty\sum_{i,j=0}^{\infty}\left[\begin{matrix}a, & c\\ b, & d\end{matrix}\middle|\, q\right]_{i+j}\left[\begin{matrix}e\\ q\end{matrix}\middle|\, q\right]_i\times$$

$$\times\left[\begin{matrix}b/e, & d/e\\ q, & w\end{matrix}\middle|\, q\right]_j\left(\frac{bd}{ace}\right)^i\left(\frac{we}{ac}\right)^j=$$

$$={}_8\phi_7\left[\begin{matrix}bd/qe, & q\sqrt{bd/qe}, & -q\sqrt{bd/qe}, & a, & c, & b/e, & d/e, & bd/we\\ & \sqrt{bd/qe}, & -\sqrt{bd/qe}, & b, & d, & w, & bd/ae, & bd/ce\end{matrix}\middle|\, q;\frac{we}{ac}\right].$$

3.2. In Theorem 3, specializing the $\Omega(j)$ sequence alternatively by

$$\Omega(j)=q^{\binom{j}{2}}\frac{[b/e,d/e;q]_j}{(q;q)_j(bd/e;q^2)_j}\left(\frac{bd}{ac}\right)^j$$

and then evaluating the inner sum with respect to j displayed in (12c) by means of (7) yield another reduction formula.

Proposition 7 (well-posed reduction formula)**.**

$$\left[\begin{matrix}bd/e, bd/ace\\ bd/ae, bd/ce\end{matrix}\middle|\, q\right]_\infty\sum_{i,j=0}^{\infty}q^{\binom{j}{2}}\left[\begin{matrix}a, & c\\ b, & d\end{matrix}\middle|\, q\right]_{i+j}\left[\begin{matrix}e\\ q\end{matrix}\middle|\, q\right]_i\times$$

$$\times\frac{[b/e,d/e;q]_j}{(q;q)_j(bd/e;q^2)_j}\left(\frac{bd}{ace}\right)^i\left(\frac{bd}{ac}\right)^j=$$

$$=\sum_{n=0}^{\infty}(-1)^n q^{\binom{n}{2}}\frac{1-q^{4n-1}bd/e}{1-bd/qe}\left[\begin{matrix}a, & c, & b/e, & d/e\\ b, & d, & bd/ae, & bd/ce\end{matrix}\middle|\, q\right]_{2n}\frac{(bd/qe;q^2)_n}{(q^2;q^2)_n}\left(\frac{bd}{ac}\right)^{2n}.$$

3.3. In Theorem 3, specializing the $\Omega(j)$ sequence finally by

$$\Omega(j)=\left[\begin{matrix}b/e, & d/e\\ q, & w\end{matrix}\middle|\,q\right]_j\frac{(w;q^2)_j}{(bd/e;q^2)_j}\left(\frac{bd}{ac}\right)^j$$

and then evaluating the inner sum with respect to j displayed in (12c) by means of (8) give rise to the following reduction formula.

Proposition 8 (well-posed reduction formula)**.**

$$\left[\begin{matrix}bd/e, bd/ace\\ bd/ae, bd/ce\end{matrix}\middle|\,q\right]_\infty\sum_{i,j=0}^{\infty}\left[\begin{matrix}a, & c\\ b, & d\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}e\\ q\end{matrix}\middle|\,q\right]_i\left[\begin{matrix}b/e, & d/e\\ q, & w\end{matrix}\middle|\,q\right]_j\times$$

$$\times\frac{(w;q^2)_j}{(bd/e;q^2)_j}\left(\frac{bd}{ace}\right)^i\left(\frac{bd}{ac}\right)^j=$$

$$=\sum_{n=0}^{\infty}q^{\binom{2n}{2}}\frac{1-q^{4n-1}bd/e}{1-bd/qe}\left[\begin{matrix}a, & c, & b/e, & d/e\\ b, & d, & bd/ae, & bd/ce\end{matrix}\middle|\,q\right]_{2n}\times$$

$$\times\left[\begin{matrix}bd/qe, & bd/we\\ q^2, & qw\end{matrix}\middle|\,q^2\right]_n\left(\frac{bd}{ac}\right)^{2n}w^n.$$

4. Nonterminating double series $\boldsymbol{\Phi^{1:2;\lambda}_{2:0;\mu}}$. This section is devoted to the transformation and well-posed reduction formulae for the $\Phi^{1:2;\lambda}_{2:0;\mu}$ series, which does not seem to have appeared previously in literature.

Theorem 4. *For an arbitrary sequence* $\{\Omega(j)\}$, *there holds the transformation*

$$\left[\begin{matrix}bd/e, bd/ace\\ bd/ae, bd/ce\end{matrix}\middle|\,q\right]_\infty\sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i\left[\begin{matrix}a\\ b,d\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle|\,q\right]_i\Omega(j)= \tag{14a}$$

$$=\sum_{n=0}^{\infty}(-1)^nq^{\binom{n}{2}}\frac{1-q^{2n-1}bd/e}{1-bd/qe}\left[\begin{matrix}a, & c, & b/e, & d/e, & bd/qe\\ q, & b, & d, & bd/ae, & bd/ce\end{matrix}\middle|\,q\right]_n\left(\frac{bd}{ac}\right)^n\times \tag{14b}$$

$$\times\sum_{j=0}^{n}(-1)^jq^{-\binom{j}{2}}\left(\frac{qa}{bd}\right)^j\left[\begin{matrix}q^{-n}, & q^{n-1}bd/e\\ q^{1-n}/c, & q^nbd/ce\end{matrix}\middle|\,q\right]_j\left[\begin{matrix}bd/ace\\ b/e, d/e\end{matrix}\middle|\,q\right]_j\Omega(j) \tag{14c}$$

provided that both double series displayed above are absolutely convergent.

Proof. Rewriting the double sum in (14a) as

$$\sum_{i,j=0}^{\infty}\left(\frac{q^jbd}{ace}\right)^i\left[\begin{matrix}a\\ b,d\end{matrix}\middle|\,q\right]_{i+j}\left[\begin{matrix}c,e\\ q\end{matrix}\middle|\,q\right]_i\Omega(j)= \tag{15a}$$

$$=\sum_{j=0}^{\infty}\Omega(j)\left[\begin{matrix}a\\ b,d\end{matrix}\middle|\,q\right]_j{}_3\phi_2\left[\begin{matrix}q^ja, & c, & e\\ & q^jb, & q^jd\end{matrix}\middle|\,q;\frac{q^jbd}{ace}\right] \tag{15b}$$

and then reformulating the last ${}_3\phi_2$-series via (2a), (2b) as

$${}_3\phi_2\left[\begin{matrix}q^ja, & c, & e\\ & q^jb, & q^jd\end{matrix}\middle|\,q;\frac{q^jbd}{ace}\right]=$$

$$= \begin{bmatrix} q^j bd/ae, q^{2j} bd/ce \\ q^{2j} bd/e, q^j bd/ace \end{bmatrix} q \Bigg]_\infty \sum_{i=0}^{\infty} \frac{1 - q^{2i+2j-1} bd/e}{1 - q^{2j-1} bd/e} \times$$

$$\times (-1)^i q^{\binom{i}{2}} \left[\begin{matrix} q^{2j-1} bd/e, & q^j a, & c, & q^j d/e, & q^j b/e \\ q, & q^j bd/ae, & q^{2j} bd/ce, & q^j b, & q^j d \end{matrix} \middle| q \right]_i \left(\frac{q^j bd}{ac} \right)^i =$$

$$= \left[\begin{matrix} bd/ae, bd/ce \\ bd/e, bd/ace \end{matrix} \middle| q \right]_\infty \left[\begin{matrix} b, d, bd/ace \\ a, b/e, d/e \end{matrix} \middle| q \right]_j \sum_{i=0}^{\infty} \frac{1 - q^{2i+2j-1} bd/e}{1 - bd/qe} \times$$

$$\times (-1)^i q^{\binom{i}{2}} \left[\begin{matrix} a, & b/e, & d/e \\ b, & d, & bd/ae \end{matrix} \middle| q \right]_{i+j} \left[\begin{matrix} c \\ q \end{matrix} \middle| q \right]_i \left[\begin{matrix} bd/qe \\ bd/ce \end{matrix} \middle| q \right]_{i+2j} \left(\frac{q^j bd}{ac} \right)^i$$

we get the following double sum expression

$$\text{Eq(14a)} = \sum_{i,j=0}^{\infty} (-1)^i \frac{1 - q^{2i+2j-1} bd/e}{1 - bd/qe} q^{\binom{i}{2}} \left[\begin{matrix} bd/ace \\ b/e, d/e \end{matrix} \middle| q \right]_j \Omega(j) \times$$

$$\times \left[\begin{matrix} c \\ q \end{matrix} \middle| q \right]_i \left[\begin{matrix} a, & b/e, & d/e \\ b, & d, & bd/ae \end{matrix} \middle| q \right]_{i+j} \left[\begin{matrix} bd/qe \\ bd/ce \end{matrix} \middle| q \right]_{i+2j} \left(\frac{q^j bd}{ac} \right)^i.$$

This leads to the transformation displayed in Theorem 4 after having changed the summation indices by $n := i + j$ and then applied the relation

$$\left[\begin{matrix} c \\ q \end{matrix} \middle| q \right]_{n-j} \left[\begin{matrix} bd/qe \\ bd/ce \end{matrix} \middle| q \right]_{n+j} = \left(\frac{q}{c} \right)^j \left[\begin{matrix} c, & bd/qe \\ q, & bd/ce \end{matrix} \middle| q \right]_n \left[\begin{matrix} q^{-n}, & q^{n-1} bd/e \\ q^n bd/ce, & q^{1-n}/c \end{matrix} \middle| q \right]_j.$$

Theorem 4 is proved.

4.1. Specifying the $\Omega(j)$ sequence in Theorem 4 by

$$\Omega(j) = (-1)^j q^{\binom{j}{2}} \left[\begin{matrix} b/e, d/e, q/c^2 \\ q, \ bd/ace \end{matrix} \middle| q \right]_j \left(\frac{bd}{a} \right)^j$$

and then evaluating the inner sum with respect to j displayed in (14c) by means of the q-Pfaff – Saalschütz formula (6), we obtain the following reduction formula.

Proposition 9 (well-posed reduction formula)**.**

$$\sum_{i,j=0}^{\infty} (-1)^j q^{\binom{j}{2}} \left[\begin{matrix} a \\ b, d \end{matrix} \middle| q \right]_{i+j} \left[\begin{matrix} c, e \\ q \end{matrix} \middle| q \right]_i \left[\begin{matrix} b/e, d/e, q/c^2 \\ q, \ bd/ace \end{matrix} \middle| q \right]_j \left(\frac{q^j bd}{ace} \right)^i \left(\frac{bd}{a} \right)^j =$$

$$= \left[\begin{matrix} bd/ae, bd/ce \\ bd/e, bd/ace \end{matrix} \middle| q \right]_\infty \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{1 - q^{2n-1} bd/e}{1 - bd/qe} \times$$

$$\times \left[\begin{matrix} bd/qe, & a, & q/c, & b/e, & d/e \\ q, & b, & d, & bd/ae, & bcd/qe \end{matrix} \middle| q \right]_n \left[\begin{matrix} bcd/qe \\ bd/ce \end{matrix} \middle| q \right]_{2n} \left(\frac{bd}{ac} \right)^n.$$

4.2. Instead, specifying the $\Omega(j)$ sequence by

$$\Omega(j) =$$

$$= (-1)^j q^{\binom{j}{2}} \frac{1-q^{2j-1}bd/ce}{1-bd/qce} \left[\begin{matrix} \alpha/c, \beta/c, bd/qce, bd/\alpha\beta e \\ q,\ \alpha\beta/c,\ bd/\alpha e,\ bd/\beta e \end{matrix} \middle| q\right]_j \left[\begin{matrix} b/e, d/e \\ bd/ace \end{matrix} \middle| q\right]_j \left(\frac{bd}{a}\right)^j$$

and then evaluating the inner sum with respect to j displayed in (14c) by means of Jackson's q-analogue of Dougall's ${}_7F_6$-sum (cf. [5], II-22)

$${}_8\phi_7 \left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & q^{-n} \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e, & q^{n+1}a \end{matrix} \middle| q; q\right] =$$

$$= \left[\begin{matrix} qa, & qa/bc, & qa/bd, & qa/cd \\ qa/b, & qa/c, & qa/d, & qa/bcd \end{matrix} \middle| q\right]_n \quad \text{where } q^{n+1}a^2 = bcde$$

we get from Theorem 4 another reduction formula.

Proposition 10 (well-posed reduction formula)**.**

$$\sum_{i,j=0}^{\infty} \left[\begin{matrix} a \\ b, d \end{matrix} \middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q \end{matrix} \middle| q\right]_i \left[\begin{matrix} b/e, d/e \\ bd/ace \end{matrix} \middle| q\right]_j \left(\frac{q^j bd}{ace}\right)^i \left(\frac{bd}{a}\right)^j \times$$

$$\times (-1)^j q^{\binom{j}{2}} \frac{1-q^{2j-1}bd/ce}{1-bd/qce} \left[\begin{matrix} \alpha/c, \beta/c, bd/qce, bd/\alpha\beta e \\ q,\ \alpha\beta/c,\ bd/\alpha e,\ bd/\beta e \end{matrix} \middle| q\right]_j =$$

$$= \left[\begin{matrix} bd/ae, bd/ce \\ bd/e, bd/ace \end{matrix} \middle| q\right]_\infty \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{1-q^{2n-1}bd/e}{1-bd/qe} \times$$

$$\times \left[\begin{matrix} a, & \alpha, & \beta, & b/e, & d/e, & bd/qe, & bcd/\alpha\beta e \\ q, & b, & d, & \alpha\beta/c, & bd/ae, & bd/\alpha e, & bd/\beta e \end{matrix} \middle| q\right]_n \left(\frac{bd}{ac}\right)^n.$$

5. Nonterminating double series $\Phi^{1:2;\lambda}_{1:1;\mu}$. Applying the two transformations for ${}_3\phi_2$-series (cf. [5], III-9 and III-10), Jia, Wang [7] studied systematically this $\Phi^{1:2;\lambda}_{1:1;\mu}$ series and found several reduction and summation formulae. Alternatively, we shall employ (2a), (2b) to show a couple of new transformation theorems for this $\Phi^{1:2;\lambda}_{1:1;\mu}$ series and deduce from them four very strange reduction formulae, that differ substantially from those due to Jia, Wang [7].

Theorem 5. *For an arbitrary sequence* $\{\Omega(j)\}$, *there holds the transformation*

$$\left[\begin{matrix} bd/e, bd/ace \\ bd/ae, bd/ce \end{matrix} \middle| q\right]_\infty \sum_{i,j=0}^{\infty} \left(\frac{bd}{ace}\right)^i \left[\begin{matrix} a \\ b \end{matrix} \middle| q\right]_{i+j} \left[\begin{matrix} c, e \\ q, d \end{matrix} \middle| q\right]_i \Omega(j) = \tag{16a}$$

$$= \sum_{i,j=0}^{\infty} \frac{1-q^{2i+j-1}bd/e}{1-bd/qe} \left[\begin{matrix} a, b/e, bd/qe \\ b,\ bd/ce \end{matrix} \middle| q\right]_{i+j} \times \tag{16b}$$

$$\times q^{\binom{i}{2}} \left[\begin{matrix} c,\ d/e \\ q, d, bd/ae \end{matrix} \middle| q\right]_i \left(-\frac{bd}{ac}\right)^i \frac{\Omega(j)}{(b/e;q)_j} \tag{16c}$$

provided that both double series displayed above are absolutely convergent.

Proof. Rewrite the double sum in (16a) as

$$\sum_{i,j=0}^{\infty}\left(\frac{bd}{ace}\right)^i\left[\begin{matrix}a\\b\end{matrix}\middle|q\right]_{i+j}\left[\begin{matrix}c,e\\q,d\end{matrix}\middle|q\right]_i\Omega(j)= \tag{17a}$$

$$=\sum_{j=0}^{\infty}\Omega(j)\left[\begin{matrix}a\\b\end{matrix}\middle|q\right]_j {}_3\phi_2\left[\begin{matrix}q^ja, & c, & e\\ & q^jb, & d\end{matrix}\middle|q;\frac{bd}{ace}\right]. \tag{17b}$$

According to (2a), (2b), the last ${}_3\phi_2$-series can be reformulated as

$$ {}_3\phi_2\left[\begin{matrix}q^ja,c,e\\q^jb,d\end{matrix}\middle|q;\frac{bd}{ace}\right]=\left[\begin{matrix}bd/ae,q^jbd/ce\\q^jbd/e,bd/ace\end{matrix}\middle|q\right]_\infty\sum_{i=0}^{\infty}(-1)^i\frac{1-q^{2i+j-1}bd/e}{1-q^{j-1}bd/e}q^{\binom{i}{2}}\times$$

$$\times\left[\begin{matrix}q^{j-1}bd/e, & q^ja, & c, & q^jb/e, & d/e\\ q, & bd/ae, & q^jbd/ce, & d, & q^jb\end{matrix}\middle|q\right]_i\left(\frac{bd}{ac}\right)^i=$$

$$=\left[\begin{matrix}bd/ae,bd/ce\\bd/e,bd/ace\end{matrix}\middle|q\right]_\infty\left[\begin{matrix}b\\a,b/e\end{matrix}\middle|q\right]_j\sum_{i=0}^{\infty}\frac{1-q^{2i+j-1}bd/e}{1-bd/qe}\times$$

$$\times(-1)^iq^{\binom{i}{2}}\left[\begin{matrix}a,b/e,bd/qe\\b,\ bd/ce\end{matrix}\middle|q\right]_{i+j}\left[\begin{matrix}c,\ d/e\\q,d,bd/ae\end{matrix}\middle|q\right]_i\left(\frac{bd}{ac}\right)^i.$$

Substituting this expression into (17a), (17b) and then reordering the factors, we get the transformation displayed in Theorem 5.

Alternatively, permuting the parameters of ${}_3\phi_2$-series gives

$$ {}_3\phi_2\left[\begin{matrix}q^ja, & c, & e\\ & q^jb, & d\end{matrix}\middle|q;\frac{bd}{ace}\right]={}_3\phi_2\left[\begin{matrix}c, & e, & q^ja\\ & q^jb, & d\end{matrix}\middle|q;\frac{bd}{ace}\right].$$

Applying the formula (2a), (2b) to the ${}_3\phi_2$-series on the right-hand side yields

$$ {}_3\phi_2\left[\begin{matrix}c, & e, & q^ja\\ & q^jb, & d\end{matrix}\middle|q;\frac{bd}{ace}\right]=\left[\begin{matrix}bd/ac,bd/ae\\bd/a,bd/ace\end{matrix}\middle|q\right]_\infty\sum_{i=0}^{\infty}(-1)^i\frac{1-q^{2i-1}bd/a}{1-bd/qa}q^{\binom{i}{2}}\times$$

$$\times\left[\begin{matrix}bd/qa, & c, & e, & b/a, & q^{-j}d/a\\ q, & bd/ac, & bd/ae, & d, & q^jb\end{matrix}\middle|q\right]_i\left(\frac{q^jbd}{ce}\right)^i=$$

$$=\left[\begin{matrix}bd/ac,bd/ae\\bd/a,bd/ace\end{matrix}\middle|q\right]_\infty\sum_{i=0}^{\infty}\frac{1-q^{2i-1}bd/a}{1-bd/qa}q^{2\binom{i}{2}}\times$$

$$\times\left[\begin{matrix}bd/qa,\ c,\ e,\ b/a\\q,bd/ac,bd/ae,d\end{matrix}\middle|q\right]_i\frac{[b,qa/d;q]_j}{(b;q)_{i+j}(qa/d;q)_{j-i}}\left(\frac{bd^2}{ace}\right)^i.$$

Substituting this expression into (17a), (17b) and then reordering the factors, we obtain another transformation formula.

Theorem 6. *For an arbitrary sequence* $\{\Omega(j)\}$, *there holds the transformation*

$$\left[\begin{matrix}bd/a,bd/ace\\bd/ac,bd/ae\end{matrix}\middle|q\right]_\infty\sum_{i,j=0}^{\infty}\left(\frac{bd}{ace}\right)^i\left[\begin{matrix}a\\b\end{matrix}\middle|q\right]_{i+j}\left[\begin{matrix}c,e\\q,d\end{matrix}\middle|q\right]_i\Omega(j)= \tag{18a}$$

$$= \sum_{i,j=0}^{\infty} \frac{1-q^{2i-1}bd/a}{1-bd/qa} \left[\begin{matrix} c,\ e,\ b/a,\ bd/qa \\ q, d, bd/ac, bd/ae \end{matrix} \middle| q\right]_i \times \tag{18b}$$

$$\times q^{2\binom{i}{2}} \left(\frac{bd^2}{ace}\right)^i \frac{[a, qa/d; q]_j}{(b;q)_{i+j}(qa/d;q)_{j-i}} \Omega(j) \tag{18c}$$

provided that both double series displayed above are absolutely convergent.

For the two equations displayed in Theorems 5 and 6, our efforts have failed to reduce the double sums on the right-hand side. However, we do succeed in figuring out two instances in which their corresponding left double sums can be expressed in single ones, that lead us to four remarkable reduction formulae.

5.1. Rewrite the double sum in (16a) or the same (18a) as

$$\sum_{i=0}^{\infty} \left[\begin{matrix} a, & c, & e \\ q, & b, & d \end{matrix} \middle| q\right]_i \left(\frac{bd}{ace}\right)^i \sum_{j=0}^{\infty} \left[\begin{matrix} q^i a \\ q^i b \end{matrix} \middle| q\right]_j \Omega(j). \tag{19}$$

For the $\Omega(j)$ sequence specified by

$$\Omega(j) = \left[\begin{matrix} \beta \\ q \end{matrix} \middle| q\right]_j \left(\frac{b}{a\beta}\right)^j$$

the sum with respect to j displayed in (19) can be evaluated via (10) as

$${}_2\phi_1 \left[\begin{matrix} \beta, & q^i a \\ & q^i b \end{matrix} \middle| q; \frac{b}{a\beta}\right] = \left[\begin{matrix} b/a, q^i b/\beta \\ q^i b, b/a\beta \end{matrix} \middle| q\right]_\infty.$$

Then Theorems 5 and 6 under the last specification for the $\Omega(j)$ sequence give rise respectively to the following two reduction formulae.

Proposition 11 (reduction formula**).**

$$\sum_{i,j=0}^{\infty} \frac{1-q^{2i+j-1}bd/e}{1-bd/qe} \left[\begin{matrix} a, b/e, bd/qe \\ b,\ bd/ce \end{matrix} \middle| q\right]_{i+j} \times$$

$$\times q^{\binom{i}{2}} \left[\begin{matrix} c,\ d/e \\ q, d, bd/ae \end{matrix} \middle| q\right]_i \left(-\frac{bd}{ac}\right)^i \left[\begin{matrix} \beta \\ q, b/e \end{matrix} \middle| q\right]_j \left(\frac{b}{a\beta}\right)^j =$$

$$= \left[\begin{matrix} b/a, b/\beta, bd/e, bd/ace \\ b, b/a\beta, bd/ae, bd/ce \end{matrix} \middle| q\right]_\infty {}_3\phi_2 \left[\begin{matrix} a, & c, & e \\ & d, & b/\beta \end{matrix} \middle| q; \frac{bd}{ace}\right].$$

Proposition 12 (reduction formula**).**

$$\sum_{i,j=0}^{\infty} \frac{1-q^{2i-1}bd/a}{1-bd/qa} \left[\begin{matrix} c,\ e,\ b/a,\ bd/qa \\ q, d, bd/ac, bd/ae \end{matrix} \middle| q\right]_i \times$$

$$\times q^{2\binom{i}{2}} \left(\frac{bd^2}{ace}\right)^i \frac{[\beta, a, qa/d; q]_j}{(q;q)_j (b;q)_{i+j}(qa/d;q)_{j-i}} \left(\frac{b}{a\beta}\right)^j =$$

$$= \left[\begin{matrix} b/a, b/\beta, bd/a, bd/ace \\ b, b/a\beta, bd/ac, bd/ae \end{matrix} \middle| q\right]_\infty {}_3\phi_2 \left[\begin{matrix} a, & c, & e \\ & d, & b/\beta \end{matrix} \middle| q; \frac{bd}{ace}\right].$$

5.2. Alternatively, letting $\Omega(j)$ be the sequence

$$\Omega(j) = \frac{(qa/b;q)_j}{(q;q)_j}\left(-\frac{b}{a}\right)^j$$

and then evaluating the corresponding sum with respect to j displayed in (19) by means of the q-Kummer sum (cf. [5], II-9)

$${}_2\phi_1\left[\begin{matrix} q^i a, & qa/b \\ & q^i b \end{matrix}\middle| q; -b/a\right] = (-q;q)_\infty \frac{[q^{1+i}a, q^i b^2/a; q^2]_\infty}{[q^i b, -b/a; q]_\infty}$$

we derive from Theorems 5 and 6 the following respective reduction formulae.

Proposition 13 (reduction formula)**.**

$$\sum_{i,j=0}^{\infty} \frac{1-q^{2i+j-1}bd/e}{1-bd/qe}\left[\begin{matrix} a, b/e, bd/qe \\ b,\ bd/ce \end{matrix}\middle| q\right]_{i+j}\left(-\frac{bd}{ac}\right)^i \times$$

$$\times q^{\binom{i}{2}}\left[\begin{matrix} c,\ d/e \\ q, d, bd/ae \end{matrix}\middle| q\right]_i \left[\begin{matrix} qa/b \\ q, b/e \end{matrix}\middle| q\right]_j \left(-\frac{b}{a}\right)^j =$$

$$= \left[\begin{matrix} -q,\ a,\ bd/e,\ bd/ace \\ -b/a, b, bd/ae, bd/ce \end{matrix}\middle| q\right]_\infty \sum_{i=0}^{\infty}\left[\begin{matrix} c, & e \\ q, & d \end{matrix}\middle| q\right]_i \left[\begin{matrix} q^i b^2/a \\ q^i a \end{matrix}\middle| q^2\right]_\infty \left(\frac{bd}{ace}\right)^i.$$

Proposition 14 (reduction formula)**.**

$$\sum_{i,j=0}^{\infty} \frac{1-q^{2i-1}bd/a}{1-bd/qa}\left[\begin{matrix} c,\ e,\ b/a,\ bd/qa \\ q, d, bd/ac, bd/ae \end{matrix}\middle| q\right]_i \times$$

$$\times q^{2\binom{i}{2}}\left(\frac{bd^2}{ace}\right)^i \frac{[a, qa/b, qa/d; q]_j}{(q;q)_j (b;q)_{i+j} (qa/d;q)_{j-i}}\left(-\frac{b}{a}\right)^j =$$

$$= \left[\begin{matrix} -q,\ a,\ bd/a,\ bd/ace \\ -b/a, b, bd/ac, bd/ae \end{matrix}\middle| q\right]_\infty \sum_{i=0}^{\infty}\left[\begin{matrix} c, & e \\ q, & d \end{matrix}\middle| q\right]_i \left[\begin{matrix} q^i b^2/a \\ q^i a \end{matrix}\middle| q^2\right]_\infty \left(\frac{bd}{ace}\right)^i.$$

6. Nonterminating double series $\Phi^{0:3;\lambda}_{2:0;\mu}$. Finally in this section, we are going to investigate the $\Phi^{0:3;\lambda}_{2:0;\mu}$ series and prove one transformation theorem plus two interesting reduction formulae.

Theorem 7. *For an arbitrary sequence $\{\Omega(j)\}$, there holds the transformation*

$$\left[\begin{matrix} bd/e, bd/ace \\ bd/ae, bd/ce \end{matrix}\middle| q\right]_\infty \sum_{i,j=0}^{\infty} \frac{\Omega(j)}{[b,d;q]_{i+j}}\left[\begin{matrix} a, c, e \\ q \end{matrix}\middle| q\right]_i \left(\frac{q^{2j}bd}{ace}\right)^i = \tag{20a}$$

$$= \sum_{i,j=0}^{\infty} \frac{1-q^{2i+2j-1}bd/e}{1-bd/qe}\left[\begin{matrix} bd/qe \\ bd/ae, bd/ce \end{matrix}\middle| q\right]_{i+2j}\left[\begin{matrix} a, c \\ q \end{matrix}\middle| q\right]_i \times \tag{20b}$$

$$\times q^{\binom{i}{2}+2ij}\left[\begin{matrix} b/e, & d/e \\ b, & d \end{matrix}\middle| q\right]_{i+j}\left(-\frac{bd}{ac}\right)^i \frac{(bd/ace;q)_{2j}}{[b/e, d/e; q]_j}\Omega(j) \tag{20c}$$

provided that both double series displayed above are absolutely convergent.

Proof. Rewrite the double sum in (20a) as

$$\sum_{i,j=0}^{\infty}\frac{\Omega(j)}{[b,d;q]_{i+j}}\left[\begin{matrix}a,c,e\\ q\end{matrix}\,\middle|\,q\right]_i\left(\frac{q^{2j}bd}{ace}\right)^i=\sum_{j=0}^{\infty}\frac{\Omega(j)}{[b,d;q]_j}{}_3\phi_2\left[\begin{matrix}a,c,e\\ q^jb,q^jd\end{matrix}\,\middle|\,q;\frac{q^{2j}bd}{ace}\right]. \tag{21}$$

According to (2a), (2b), we can reformulate the above ${}_3\phi_2$-series as

$$ {}_3\phi_2\left[\begin{matrix}a, & c, & e\\ & q^jb, & q^jd\end{matrix}\,\middle|\,q;\frac{q^{2j}bd}{ace}\right]=\left[\begin{matrix}q^{2j}bd/ae,q^{2j}bd/ce\\ q^{2j}bd/e,q^{2j}bd/ace\end{matrix}\,\middle|\,q\right]_\infty\sum_{i=0}^{\infty}\frac{1-q^{2i+2j-1}bd/e}{1-q^{2j-1}bd/e}\times$$

$$\times(-1)^iq^{\binom{i}{2}}\left[\begin{matrix}q^{2j-1}bd/e, & a, & c, & q^jd/e, & q^jb/e\\ q, & q^{2j}bd/ae, & q^{2j}bd/ce, & q^jb, & q^jd\end{matrix}\,\middle|\,q\right]_i\left(\frac{q^{2j}bd}{ac}\right)^i=$$

$$=\left[\begin{matrix}bd/ae,bd/ce\\ bd/e,bd/ace\end{matrix}\,\middle|\,q\right]_\infty\frac{[b,d;q]_j(bd/ace;q)_{2j}}{[b/e,d/e;q]_j}\sum_{i=0}^{\infty}\frac{1-q^{2i+2j-1}bd/e}{1-bd/qe}\times$$

$$\times(-1)^iq^{\binom{i}{2}+2ij}\left[\begin{matrix}b/e, & d/e\\ b, & d\end{matrix}\,\middle|\,q\right]_{i+j}\left[\begin{matrix}a,c\\ q\end{matrix}\,\middle|\,q\right]_i\left[\begin{matrix}bd/qe\\ bd/ae,bd/ce\end{matrix}\,\middle|\,q\right]_{i+2j}\left(\frac{bd}{ac}\right)^i.$$

Substituting this last expression into (21) and then simplifying the resulting equation, we get the transformation displayed in Theorem 7.

By specifying the $\Omega(j)$ sequence and then expressing the double sum (20a) as single series, we can prove two quite interesting reduction formulae.

6.1. Letting $b=-d=\sqrt{\alpha}$ and replacing e by $-e$, we can reformulate the double sum in (20a) as

$$\sum_{i=0}^{\infty}\left[\begin{matrix}a, & c, & -e\\ q, & \sqrt{\alpha}, & -\sqrt{\alpha}\end{matrix}\,\middle|\,q\right]_i\left(\frac{\alpha}{ace}\right)^i\sum_{j=0}^{\infty}\frac{\Omega(j)}{(q^{2i}\alpha;q^2)_j}q^{2ij}. \tag{22}$$

Specifying the $\Omega(j)$ sequence in Theorem 7 by

$$\Omega(j)=\left[\begin{matrix}\beta,\gamma\\ q^2\end{matrix}\,\middle|\,q^2\right]_j\left(\frac{\alpha}{\beta\gamma}\right)^j$$

and then evaluating the inner sum with respect to j displayed in (22) by means of the q-Gauss sum (10), we get the following reduction formula.

Proposition 15 (reduction formula)**.**

$$\sum_{i,j=0}^{\infty}\frac{1-q^{2i+2j-1}\alpha/e}{1-\alpha/qe}\left[\begin{matrix}\alpha/qe\\ \alpha/ae,\alpha/ce\end{matrix}\,\middle|\,q\right]_{i+2j}\left[\begin{matrix}\alpha/e^2\\ \alpha\end{matrix}\,\middle|\,q^2\right]_{i+j}\times$$

$$\times q^{\binom{i}{2}}\left[\begin{matrix}a,c\\ q\end{matrix}\,\middle|\,q\right]_i\left(\frac{q^{2j}\alpha}{ac}\right)^i(\alpha/ace;q)_{2j}\left[\begin{matrix}\beta,\ \gamma\\ q^2,\alpha/e^2\end{matrix}\,\middle|\,q^2\right]_j\left(\frac{\alpha}{\beta\gamma}\right)^j=$$

$$=\left[\begin{matrix}\alpha/\beta,\alpha/\gamma\\ \alpha,\alpha/\beta\gamma\end{matrix}\,\middle|\,q^2\right]_\infty\left[\begin{matrix}\alpha/e,\alpha/ace\\ \alpha/ae,\alpha/ce\end{matrix}\,\middle|\,q\right]_\infty\sum_{i=0}^{\infty}\frac{[a,c,-e;q]_i(\alpha/\beta\gamma;q^2)_i}{(q;q)_i[\alpha/\beta,\alpha/\gamma;q^2]_i}\left(\frac{\alpha}{ace}\right)^i.$$

6.2. Instead, specifying the $\Omega(j)$ sequence in Theorem 7 by

$$\Omega(j) = \begin{bmatrix} \beta, & q^2/\beta \\ q^2, & -q^2 \end{bmatrix} q^2 \Bigg]_j (q^{j-1}\alpha)^j$$

and then evaluating the inner sum with respect to j displayed in (22) by means of the q-analogue of Bailey's ${}_2F_1(\frac{1}{2})$ sum (11), we obtain another reduction formula.

Proposition 16 (reduction formula)**.**

$$\sum_{i,j=0}^{\infty} \frac{1 - q^{2i+2j-1}\alpha/e}{1 - \alpha/qe} \left[\begin{matrix} \alpha/qe \\ \alpha/ae, \alpha/ce \end{matrix} \middle| q \right]_{i+2j} \left[\begin{matrix} \alpha/e^2 \\ \alpha \end{matrix} \middle| q^2 \right]_{i+j} \left[\begin{matrix} a, c \\ q \end{matrix} \middle| q \right]_i q^{\binom{i}{2}} \times$$

$$\times \left(\frac{q^{2j}\alpha}{ac} \right)^i \left[\begin{matrix} \beta, q^2/\beta \\ \alpha/e^2 \end{matrix} \middle| q^2 \right]_j \frac{(\alpha/ace; q)_{2j}}{(q^4; q^4)_j} (q^{j-1}\alpha)^j =$$

$$= \left[\begin{matrix} \alpha/\beta \\ \alpha \end{matrix} \middle| q^2 \right]_\infty \left[\begin{matrix} \alpha/e, \alpha/ace \\ \alpha/ae, \alpha/ce \end{matrix} \middle| q \right]_\infty \sum_{i=0}^{\infty} \frac{[a, c, -e; q]_i}{(q; q)_i (\alpha/\beta; q^2)_i} \left[\begin{matrix} q^{2i}\alpha\beta \\ q^{2i}\alpha/\beta \end{matrix} \middle| q^4 \right]_\infty \left(\frac{\alpha}{ace} \right)^i.$$

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