К О Р О Т К І П О В І Д О М Л Е Н Н Я

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A NOTE ON INVARIANT SUBMANIFOLDS OF (k, μ)-CONTACT MANIFOLDS

ПРО ДЕЯКІ ПІДМНОГОВИДИ (k, μ)-КОНТАКТНИХ МНОГОВИДІВ

The object of the present paper is to study invariant submanifolds of a (k, μ)-contact manifold and to find the necessary and sufficient conditions for an invariant submanifold of a (k, μ)-contact manifold to be totally geodesic.

Метою статті є вивчення інваріантних підмноговидів (k, μ)-контактного многовиду та встановлення необхідних і достатніх умов для того, щоб інваріантний підмноговид (k, μ)-контактного многовиду був цілком геодезичним.

1. Introduction. It is well known [1, 2] that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$, where R is the curvature tensor. On the other hand, on a manifold M equipped with a Sasakian structure (ϕ, ξ, η, g), one has

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad X, Y \in \Gamma(TM). \tag{1}$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case (1), Blair, Koufogiorgos and Papantoniou [3] introduced the case of contact metric manifolds with contact metric structure (ϕ, ξ, η, g) which satisfy

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{2}$$

for all $X, Y \in \Gamma(TM)$, where k and μ are real constants and $2h$ is the Lie derivative of ϕ in the direction ξ. A contact metric manifold belonging to this class is called a (k, μ)-contact manifold. In fact, there are many motivations for studying (k, μ)-contact manifolds: the first is that, in the non-Sasakian case (that is, for $k \neq 1$) the condition (2) determines the curvature completely; moreover, while the values of k and μ change, the form of (2) is invariant under D-homothetic deformations [3]; finally there is a complete classification of these manifolds, given in [4] by Boeckx, who proved also that any non-Sasakian (k, μ)-contact manifold is locally homogeneous and strongly locally ϕ-symmetric [5, 6]. There are also non-trivial examples of (k, μ)-contact manifolds, the most important being the unit tangent sphere bundle of a Riemannian manifold of constant sectional curvature with the usual contact metric structure.

An odd dimensional invariant submanifold of a (k, μ)-contact manifold is a submanifold for which the structure tensor field ϕ maps tangent vectors into tangent vectors. Such a submanifold inherits a contact metric stucture from the ambient space and it is in fact a (k, μ)-contact manifold [16].

In [11] Kon proved that an invariant submanifold of a Sasakian manifold is totally geodesic, provided the second fundamental form of the immersion is covariantly constant. Generalising this result of Kon the authors [16] proved that if the second fundamental form of an invariant submanifold in a (k,μ)-contact manifold is covariantly constant then either $k=0$ or the submanifold is totally geodesic.

Motivated by these works we have studied the possible necessary and sufficient conditions of an invariant submanifold of a (k,μ)-contact manifold to be totally geodesic. In this paper we have generalized the results of [16]. In the present paper we have proved that the recurrency, 2-recurrency and generalised 2-recurrency of the second fundamental form of an invariant submanifold of a (k,μ)-contact manifold are equivalent. And any one of these three conditions can be taken as a necessary and sufficient condition of the submanifold to be totally geodesic. Since $N(k)$-contact metric manifold is a special case of (k,μ)-contact manifold, therefore the above results also hold in any $N(k)$-contact metric manifold. Finally we have studied the semiparallelity of an invariant submanifold of a (k,μ)-contact manifold.

2. Preliminaries. An n-dimensional manifold $M^n (n$ is odd) is said to admit an almost contact structure [1, 15, 18] if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \tag{3}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0. \tag{4}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^n \times \mathbb{R}$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^n \times \mathbb{R}$. Let g be the compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Then M^n becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (3) it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X),$$

for any vector fields X, Y on the manifold. An almost contact metric structure becomes a contact metric structure if $g(X, \phi Y) = d\eta(X, Y)$, for all vector fields X, Y.

Let $f\colon (M, g) \longrightarrow (\bar{M}, \bar{g})$ be an isometric immersion of an n-dimensional Riemannian manifold (M, g) into $(n+d)$-dimensional Riemannian manifold $(\bar{M}, \bar{g})$, $n \geq 2$, $d \geq 1$. We denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections of M and $\bar{M}$ respectively, and by $T^{\perp}M$ its normal bundle. Then for vector fields X, $Y \in TM$, the second fundamental form σ is given by the formula $\sigma(X, Y) = \overline{\nabla}_X Y - \nabla_X Y$. Furthermore, for $N \in T^{\perp}(M)$, $A_N \colon TM \longrightarrow TM$ will denote the Weingarten operator in the direction of N, $A_N X = \nabla^{\perp}_X N - \overline{\nabla}_X N$, where $\nabla^{\perp}$ denotes the normal connection of M. The second fundamental form σ and A_N are related by $\bar{g}(\sigma(X, Y), N) = g(A_N X, Y)$, where

g is the induced metric of $\bar{g}$ for any vector fields X, Y tangent to M. The covariant derivative $\overline{\nabla}\sigma$ and second covariant derivative $\overline{\nabla}^2\sigma$ of σ are defined by

$$\big(\overline{\nabla}_X\sigma\big)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{5}$$

$$\begin{aligned}(\overline{\nabla}^2\sigma)(Z,W;X,Y) &= (\overline{\nabla}_X\overline{\nabla}_Y\sigma)(Z,W) = \\ &= \nabla^{\perp}_X((\overline{\nabla}_Y\sigma)(Z,W)) - (\overline{\nabla}_Y\sigma)(\nabla_X Z, W) = \\ &\quad -(\overline{\nabla}_X\sigma)(Z,\nabla_Y W) - (\overline{\nabla}_{\nabla_X Y}\sigma)(Z,W),\end{aligned} \tag{6}$$

respectively, where $\overline{\nabla}\sigma$ is a normal bundle valued tensor of type $(0,3)$ and $\overline{\nabla}$ is called the *van der Waerden – Bortolotti connection* of M.

The basic equation of Gauss is given by [7]

$$\begin{aligned}&\bar{R}(X,Y,Z,W) = \\ &= R(X,Y,Z,W) - g(\sigma(X,W),\sigma(Y,Z)) + g(\sigma(X,Z),\sigma(Y,W)).\end{aligned}$$

However, for a (k,μ)-contact metric manifold M^n of dimension n, we have [2]

$$(\overline{\nabla}_X\phi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX),$$

where $h = \dfrac{1}{2}\pounds_\xi\phi$. From the above equation we also have

$$\overline{\nabla}_X\xi = -\phi X - \phi h(X).$$

Now from the Gauss formula we have

$$\overline{\nabla}_X\xi = \nabla_X\xi + \sigma(X,\xi).$$

Since the submanifold M is invariant, we have from the above two equations,

$$\nabla_X\xi = -\phi X - \phi h(X) \quad \text{and} \quad \sigma(X,\xi) = 0. \tag{7}$$

3. Immersions of recurrent type. We denote by $\nabla^p T$ the covariant differential of the pth order, $p \geq 1$, of a $(0,k)$-tensor field T, $k \geq 1$, defined on a Riemannian manifold (M,g) with the Levi-Civita connection ∇. According to [14], the tensor T is said to be *recurrent* and 2-*recurrent*, if the following conditions hold on M

$$(\nabla T)(X_1,\ldots,X_k;X)T(Y_1,\ldots,Y_k) = (\nabla T)(Y_1,\ldots,Y_k;X)T(X_1,\ldots,X_k), \tag{8}$$

$$(\nabla^2 T)(X_1,\ldots,X_k;X,Y)T(Y_1,\ldots,Y_k) = (\nabla^2 T)(Y_1,\ldots,Y_k;X,Y)T(X_1,\ldots,X_k), \tag{9}$$

respectively, where X, Y, X_1, $Y_1,\ldots,X_k$, $Y_k \in TM$. From (8) it follows that at a point $x \in M$ if the tensor T is non-zero, then there exists a unique 1-form θ, respectively, a $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that

$$\nabla T = T \otimes \theta, \qquad \theta = d(\log\|T\|), \tag{10}$$

respectively

$$\nabla^2 T = T \otimes \psi,$$

holds on U, where $\|T\|$ denotes the norm of T.

The tensor T is said to be *generalized* 2*-recurrent* if

$$(\nabla^2 T)(X_1, \ldots, X_k; X, Y) - (\nabla T \otimes \theta)(X_1, \ldots, X_k; X, Y)T(Y_1, \ldots, Y_k) =$$
$$= (\nabla^2 T)(Y_1, \ldots, Y_k; X, Y) - (\nabla T \otimes \theta)(Y_1, \ldots, Y_k; X, Y)T(X_1, \ldots, X_k)$$

holds on M, where θ is a 1-form on M. From this it follows that at a point $x \in M$ if the tensor T is non-zero then there exists a unique $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that

$$\nabla^2 T = \nabla T \otimes \theta + T \otimes \psi,$$

holds on U.

The notion of generalized 2-recurrent tensors in Riemannian spaces is introduced by Ray [13].

J. Deprez defined the immersion to be $semiparallel$ if

$$\bar{R}(X,Y).\sigma = (\overline{\nabla}_X \overline{\nabla}_Y - \overline{\nabla}_Y \overline{\nabla}_X - \overline{\nabla}_{[X,Y]})\sigma = 0,$$

holds for all vector fields X, Y tangent to M. J. Deprez mainly paid attention to the case of semiparallel immersions in real space forms [8, 9]. Later, Lumiste showed that a semiparallel submanifold is the second order envelope of the family of parallel submanifolds [12]. In [10] H. Endo studied semiparallelity condition for a contact metric manifold. He showed that a semiparallel contact metric manifold is totally geodesic under certain conditions.

4. Recurrent submanifolds of (k, μ)-contact manifolds. To prove the main theorem we first state two lemmas.

Lemma 1 [19]**.** *Let M be a submanifold of a contact metric manifold $\overline{M}$. If ξ is orthogonal to M, then M is anti-invariant.*

Lemma 2 [17]**.** *We know that if (M, ϕ, ξ, η, g) be a contact Riemannian manifold and ξ belong to the (k, μ)-nullity distribution, then $k \leq 1$. If $k < 1$, then (M, ϕ, ξ, η, g) admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$, $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-k}$.*

Now, if $X \in D(\lambda)$, then $hX = \lambda X$ and if $X \in D(-\lambda)$, then $hX = -\lambda X$.

Theorem 1. *Let M be an invariant submanifold of a (k, μ)-contact manifold, with $k \neq 0$. Then the following conditions are equivalent*:

(i) *M is totally geodesic*;

(ii) *the second fundamental form σ is recurrent*;

(iii) *the second fundamental form σ is* 2*-recurrent*;

(iv) *the second fundamental form σ is generalized* 2*-recurrent*.

Proof. Suppose M is totally geodesic, then (ii), (iii) and (iv) are trivially true.

Now suppose σ is recurrent, then from (10), we get

$$(\bar{\nabla}_X \sigma)(Y, Z) = \theta(X)\sigma(Y, Z),$$

where θ is a 1-form on M. Then in view of (5), we obtain

$$\nabla_X^{\perp}(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = \theta(X)\sigma(Y, Z). \tag{11}$$

By Lemma 1, $\xi \in TM$. So, taking $Z = \xi$ in (11), we have

$$\nabla^{\perp}_X(\sigma(Y,\xi)) - \sigma(\nabla_X Y,\xi) - \sigma(Y,\nabla_X\xi) = \theta(X)\sigma(Y,\xi).$$

Then using (7), we obtain

$$\sigma(Y,\nabla_X\xi) = 0.$$

Using (7) we get

$$\sigma(Y,X) - \sigma(Y,hX) = 0.$$

Therefore, Lemma 2 yields $(1 \pm \lambda)\sigma(Y,X) = 0$, which implies $\sigma(Y,X) = 0$, provided $\lambda \neq \pm 1$, or $k \neq 0$.

Thus M is totally geodesic, provided $k \neq 0$.

Proceeding in a similar manner we can prove that if σ is 2-recurrent or generalized 2-recurrent, then also M is totally geodesic.

Theorem 1 is proved.

Theorem 2. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\bar{M}$. Then M is totally geodesic if and only if M is semiparallel, provided $k \neq \pm\mu\sqrt{1-k}$.*

Proof. We have

$$(\bar{R}(X,Y).\sigma)(V,W) =$$
$$= R^{\perp}(X,Y)(\sigma(V,W)) - \sigma(R(X,Y)V,W) - \sigma(V,R(X,Y)W).$$

Suppose M is semiparallel. Then $\bar{R}(X,Y).\sigma = 0$, that is, $\bar{R}(X,\xi).\sigma = 0$. Therefore, we have

$$R^{\perp}(X,\xi)(\sigma(V,W)) = \sigma(R(X,\xi)V,W) + \sigma(V,R(X,\xi)W).$$

Putting $V = \xi$, and using (7) we obtain

$$\sigma(R(X,\xi)\xi,W) = 0. \qquad (12)$$

Using (2) in (12) we obtain

$$(k \pm \mu\sqrt{1-k})\,\sigma(X,W) = 0.$$

Therefore, $\sigma(X,W) = 0$, provided $k \neq \pm\mu\sqrt{1-k}$. Hence M is totally geodesic. The converse statement is trivial. This completes the proof of the theorem.

The corollary follows immediately:

Corollary. *Let M be an invariant submanifold of a (k,μ)-contact manifold $\bar{M}$. Then the following conditions are equivalent*:

(i) *M is totally geodesic*;

(ii) $\bar{R}(X,\xi).\sigma = 0$;

(iii) $\bar{R}(X,Y).\sigma = 0$, *where X and Y are arbitrary vector fields on M.*

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