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K. Eftekharinasab (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

SARD'S THEOREM FOR MAPPINGS BETWEEN FRÉCHET MANIFOLDS

ТЕОРЕМА САРДА ДЛЯ ВІДОБРАЖЕНЬ МІЖ МНОГОВИДАМИ ФРЕШЕ

We prove an infinite-dimensional version of Sard's theorem for Fréchet manifolds. Let M and respectively N be bounded Fréchet manifolds with compatible metrics d_M (respectively d_N) modelled on Fréchet spaces E (respectively F) with standard metrics. Let $f\colon M \to N$ be an MC^k-Lipschitz–Fredholm map with $k > \max\{\operatorname{Ind} f, 0\}$. Then the set of regular values of f is residual in N.

Доведено нескінченновимірну версію теореми Сарда для многовидів Фреше. Припустимо, що M і відповідно N – обмежені многовиди із сумісними метриками d_M (відповідно d_N), які змодельовані на просторах Фреше E (відповідно F) зі стандартними метриками. Нехай $f\colon M \to N$ буде MC^k-відображенням Ліпшиця–Фредгольма з $k > \max\{\operatorname{Ind} f, 0\}$. Тоді множина регулярних значень f є залишковою в N.

1. Introduction. Sard's theorem in infinite-dimensional spaces may fail as showed in [1], by giving a counterexample of real smooth map on a Hilbert space with critical values containing open set. However, in [2] Smale proved that if $f\colon M \to N$ is a C^k-Fredholm map between Banach manifolds with $k > \max\{\operatorname{Ind} f, 0\}$, then the set of regular values of f is residual in N. The condition that f be Fredholm is necessary from the counterexample in [1]. In this paper we generalize the Smale's theorem for Fréchet manifolds. To carry out this at first, we need to establish the stability of Fredholm operators under small perturbation which requires to define an appropriate topology on the space of linear continuous maps. But it seems that it is formidable to work with the candidate topologies (cf. [3]), due to the fact that if E, F and G are Fréchet spaces, the evalution map is not continious for any vector space topology on the space $\mathcal{L}(E,F)$ of linear maps, and if E is not normable or F is not empty then the composition map $\pi\colon \mathcal{L}(F,G) \times \mathcal{L}(E,F) \to \mathcal{L}(E,G)$ defined by $\pi(\iota,\kappa) = \iota \circ \kappa$, is not bilinear continuous. An idea to remedy these problems could be to replace the space $C\mathcal{L}(E,F)$ of all linear continuous maps between two Fréchet spaces (E,d) and (F,g) by the space $\mathcal{L}_{d,g}(E,F)$ of all linear globally Lipschitz continuous maps. Then $\mathcal{L}_{d,g}(E,F)$ will have desired properties (see Remark 2.1, Proposition 2.1, and Proposition 2.2). In addition, we have to restrict the class of maps that we use to those ones which for them an inverse function theorem is avilable, we need this theorem to prove a local representation theorem for Lipschitz–Fredholm maps which plays an essential role in proving the main theorem. The class of maps that we will consider is the class of the so-called MC^k-maps introduced in [4, 5]. Because as mentioned there exists a suitable topology on the space of linear globaly Lipshitz continuous maps and in this category an inverse function theorem was obtained (see [4, 5]).

2. Preliminaries and notations. In this section we set up notations and conventions which will be used. Most of the terminologies are taken from [5] but, we avoid differing metric Fréchet space with Fréchet space.

2.1. Lipschitz maps and the space $\mathcal{L}_{d,g}(E,F)$. Suppose (X,d) is a metric space. We let $B^d_r(x)$ to be an open ball centered at x with radius r, and $\overline{B}^d_r(x)$ its closure. If

X is a metrizable topological vector space, we set $\|x\|_d = d(x, o)$, for $x \in X$. Given a linear map $L\colon V \to W$ between vector spaces, we write $L.x$ instead of $L(x)$.

Fréchet space is a locally convex topological vector space F whose topology can be defined by a complete translational-invariant metric d on F. Recall that a neighbourhood $U \subset F$ of zero is **absolutely convex** if it is convex and balanced. In the case of real Fréchet space, it makes balls to be invariant under reflections $x \mapsto -x$. In the complex case, they have to be invariant under multiplication with a complex unit. Every Fréchet space F admits a translational-invariant metric d defining the topology of F with absolutely convex balls. If α_n is an arbitrary sequence of positive real numbers converging to zero and ρ_n is any sequence of continuous semi-norms defining the topology of F. Then

$$d_{\alpha,\,\rho}\colon F \times F \longrightarrow [0, \infty),$$

$$d_{\alpha,\,\rho}(e, f) = \sup \alpha_n \frac{\rho_n(e-f)}{1+\rho_n(e-f)}$$

is a metric with absolutely convex balls. The metrics of the form $d_{\alpha,\,\rho}$ will be called standard metrics.

Definition 2.1. *Suppose (E, d) and (F, g) are two Fréchet spaces. Define $\mathcal{L}_{d,g}(E, F)$ to be the set of all globally Lipschitz linear maps, i. e., maps $L\colon E \to F$ such that for them:*

$$\|L\|_{d,g} = \sup_{x \in E \setminus \{0\}} \frac{\|L.x\|_g}{\|x\|_d} < \infty.$$

We abbreviate $\mathcal{L}_d(E) = \mathcal{L}_{d,d}(E, E)$; $\|L\|_d = \|L\|_{d,d}$ for $L \in \mathcal{L}_d(E)$.

***Remark* 2.1** ([5], Remark 1.9). *$\mathcal{L}_{d,g}(E, F)$ and the functions $\|.\|_{d,g}$ have the following useful properties:*

(i) *$\|L.x\|_g \leq \|L\|_{d,g} \|x\|_d$ for all $x \in E$. Moreover, $0 \in \mathcal{L}_{d,g}(E, F)$ with $\| 0 \|_{d,g} = 0$. If L is not identically zero, then $\|L_{d,g}\| > 0$.*

(ii) *If (G, h) is another Fréchet space, then*

$$\|H \circ L\|_{d,h} \leq \|H\|_{g,h} \|L\|_{d,g},$$

for $L \in \mathcal{L}_{d,g}(E, F)$, and $H \in \mathcal{L}_{g,h}(F, G)$.

(iii) *If $L, H \in \mathcal{L}_{d,g}(E, F)$, then*

$$\|L + H\|_{d,g} \leq \|L\|_{d,g} + \|H\|_{d,g} < \infty.$$

(iv) *If g is a standard metric, then*

$$D_{d,g}\colon \mathcal{L}_{d,g}(E, F) \times \mathcal{L}_{d,g}(E, F) \longrightarrow [0, \infty), \qquad (L, H) \longrightarrow \|L - H\|_{d,g} \tag{1}$$

is a translational-invariant metric on $\mathcal{L}_{d,g}(E, F)$ making it into an abelian topological group.

Proposition 2.1 ([5], Proposition 2.1). *Let (E, d) and (F, g) be Fréchet spaces, and g a standard metric. Then the following hold:*

(i) *$\mathcal{L}_{d,g}(E, F)$ is a vector subspace of the space of all maps from E to F.*

(ii) *The evaluation map*

$$\mathcal{L}_{d,g}(E, F) \times E \longrightarrow F,$$

$$(L, x) \longrightarrow L.x$$

is bilinear continuous.

(iii) *If (G, h) is another Fréchet space with standard metric, then the composition map*

$$\mathcal{L}_{d,g}(F, G) \times \mathcal{L}_{g,h}(E, F) \longrightarrow \mathcal{L}_{d,h}(E, G),$$

$$(L, H) \longrightarrow L \circ H$$

is bilinear continuous.

(iv) *The metric* (1) *is complete, and has absolutely convex balls.*

(v) *The group of automorphisms,* $\operatorname{Aut}(E)$ *is open in $\mathcal{L}_d(E)$ with respect to the topology induced by the metric* (1). *And the inversion map* $\operatorname{Aut}(E) \to \operatorname{Aut}(E)$, $A \to$ $\to A^{-1}$ *is continuous.*

Proposition 2.2. *Let (E, d) and (F, g) be Fréchet spaces, and g a standard metric. The set of isomorphisms of E to F,* $\operatorname{Iso}(E, F)$ *is open in $\mathcal{L}_{d,g}(E, F)$ with respect to the topology induced by the metric* (1).

Proof. Fix an isomorphism $i\colon E \to F$. Define a map

$$i^*\colon \mathcal{L}_d(E) \longrightarrow \mathcal{L}_{d,g}(E, F),$$

$$i^* = i \circ k, \ k \in \mathcal{L}_d(E)$$

i^* is bijective because i is isomorphism. The composition $i \circ k$ is bilinear continuous by Proposition 2.1 (iii). Thus, i^* is homeomorphism by virtue of the open mapping theorem. And since the group of automorphisms of E, $\operatorname{Aut}(E)$, is open in $\mathcal{L}_d(E)$ by Proposition 2.2 (v), it follows that its image under i^* which is the group of isomorphisms $\operatorname{Iso}(E, F)$, is open in $\mathcal{L}_{d,g}(E, F)$.

Proposition 2.2 is proved.

2.2. *Differentiation and MC^k-maps.*

Definition 2.2. *Let E, F be two Fréchet spaces, U an open subset of E, and $P\colon U \to F$ a continuous map. P is called differentiable at the point p iff there exists a linear map $\mathrm{d}_p P\colon E \to F$ with*

$$\mathrm{d}_p P(h) = \lim_{t \to 0} \frac{P(p + th) - P(p)}{t}$$

for all $h \in F$. If P is differentiable at all points of U, and if $h \to \mathrm{d}_p P(h)$ for all $p \in U$ is a continuous mapping from E into F and finally if $P'\colon U \times E \to F$, $(u, h) \to \mathrm{d}_u P(h)$ is continuous in product topology, then P is called ***Keller-differentiable****. By induction we define $P^{(k+1)}\colon U \times E^{k+1} \to F$ by*

$$P^{(k+1)}(u, f_1, \ldots, f_{k+1}) = \lim_{t \to 0} \frac{P^{(k)}(u + tf_{k+1})(f_1, \ldots, f_k) - P^{(k)}(u)(f_1, \ldots, f_k)}{t}.$$

In this sense P is called smooth, if $P^{(k)}$ exists for all $k \in \mathbb{N}_0$.

Definition 2.3. *Let (E, d) and (F, g) be two Fréchet spaces with standard metrics, U an open subset of E. A map $P\colon U \to F$ is called* **b-differentiable** *if it is Keller-differentiable, $\mathrm{d}_p P \in \mathcal{L}_{d,g}(E, F)$ for all $p \in U$ and the induced map $\mathrm{d}_p P\colon U \to \mathcal{L}_{d,g}(E, F)$ is continuous. We say P is MC^0 if it is continuous. It is called MC^1, if it is b-differentiable. In this case we write $P^0 = P$ and $P^{(1)} = P'$. Let $\mathcal{L}_{d,g}(E, F)_0$ be the connected component of $\mathcal{L}_{d,g}(E, F)$ containing the zero map. If P is b-differentiable, and if there exists a connected open neighbourhood V of x_0 in U for each such that $P' |_V - P'(x_0)\colon V \to \mathcal{L}_{d,g}(E, F)_0$ is MC^{k-1}, then P is called an MC^k-map. Define $P^{(k)}$ at x_0 by $P^{(k)} |_V = (P' |_V - P'(x_0))^{(k-1)}$. The map P is MC^∞ if it is MC^k for all $k \in \mathbb{N}_0$.*

We should mention that an appropriate version of the Chain rule is available and the composition of composable MC^k-maps are again MC^k. **Fréchet manifold** is a Hausdorff Second countable topological space with an atlas of coordinate charts taking their values in Fréchet spaces, such that the coordinate transition functions are all smooth maps between Fréchet spaces. If these Fréchet spaces are endowed with Fréchet metrics we require those metrics to be standard and the transition functions to be globally Lipschitz and MC^∞. In this case, the manifold is called **bounded Fréchet manifold**.

Definition 2.4. *A compatible metric on a Fréchet manifold M is a metric d on M such that there is a Fréchet subatlas of M such that in each chart U, d is equivalent to the Fréchet metric d_U.*

If a Fréchet manifold carries a compatible metric, then it is bounded ([4], Theorem 3.33). And since we need manifolds to carry a compatible metric, we only deal with bounded Fréchet manifolds.

Suppose M is a bounded Fréchet manifold. Naturally, we define a bounded Fréchet vector bundle over M, this is a Fréchet vector bundle whose total space is a bounded Fréchet manifold. The tangent bundle TM is a bounded Fréchet vector bundle over M whose coordinate transition functions are just the tangents TP of the coordinate transition functions P for M.

Definition 2.5. *A map $P\colon M \to N$ is an MC^k of bounded Fréchet manifolds if we can find charts around any point in M and its image in N such that the local representative of P in these charts is an MC^k-map of Fréchet spaces. It is called MC^∞ if the local representative in the charts is MC^∞.*

For $k \geq 1$, an MC^k-map $P\colon M \to N$ of bounded Fréchet manifolds induces a tangent map $TP\colon TM \to TN$ of their tangent bundles which takes the fibre over $f \in M$ into the fibre over $P(f) \in N$ and is linear on each fibre. The local representatives for the tangent map TP are just the tangents of the local representatives for P. The **derivative** of P at f is the linear map

$$\mathrm{D}\, P(f)\colon TM_f \longrightarrow TN_{P(f)}$$

induced by TP on the tangent spaces. When the manifolds are Fréchet spaces this agrees with Definition 2.3.

3. Lipschitz – Fredholm maps and stability.

Definition 3.1. *Let (E,d) and (F,g) be Fréchet spaces, and g a standard metric. A map $\varphi \in \mathcal{L}_{d,g}(E,F)$ is called **Lipschitz – Fredholm operator** if it satisfies the following conditions:*

1. *The image of φ is closed.*
2. *The dimension of the kernel of φ is finite.*
3. *The co-dimension of the image of φ is finite.*

We denote by $\mathcal{LF}(E,F)$ the set of all Lipschitz – Fredholm operators from E into F. For $\varphi \in \mathcal{LF}(E,F)$ we define the **index** of φ to be

$$\operatorname{Ind}\varphi = \dim\ker\varphi - \operatorname{codim}\operatorname{Img}\varphi.$$

A subset G of a Fréchet space F is called topologically complemented or split in F, if F is homeomorphic to the topological direct sum $G \oplus H$, where H is a subspace of F. We call H a topological complement of G in F.

Theorem 3.1 ([4], Theorem 3.14). *Let F be a Fréchet space. Then:*

(i) *every finite-dimensional subspace of F is closed;*

(ii) *every closed subspace $G \subset F$ with* $\operatorname{codim}(G) = \dim(F/G) < \infty$ *is topologically complemented in F;*

(iii) *every finite-dimensional subspace of F is topologically complemented;*

(iv) *every linear isomorphism between the direct sum of two closed subspaces and F, $G \oplus H \to F$, is a homeomorphism.*

Theorem 3.2. *$\mathcal{LF}(E,F)$ is open in $\mathcal{L}_{d,g}(E,F)$ with respect to the topology defined by the metric* (1). *Furthermore, the function $T \to \operatorname{Ind} T$ is continuous on $\mathcal{LF}(E,F)$, hence constant on connected components of $\mathcal{LF}(E,F)$.*

Proof. Suppose $\varphi\colon E \to F$ is a Lipschitz – Fredholm operator. We have to find a neighbourhood N of φ in $\mathcal{L}_{d,g}(E,F)$ such that $N \subset \mathcal{LF}(E,F)$. We can write $E = \ker\varphi \oplus G$, where G is a topological complement of $\ker\varphi$ by Theorem 3.1 (iii). φ induces linear isomorphism of G into its image $\varphi(G)$ by virtue of the open mapping theorem, thus we can write $F = \varphi(G) \oplus H$ for some finite dimensional subspace H in F. The map

$$\overline{\varphi}\colon G \oplus H \longrightarrow \varphi(G) \oplus H = F,$$

$$\overline{\varphi}(x,y) \mapsto \varphi.x + y$$

is a linear isomorphism. But the set of linear isomorphisms is open in the space of linear Lipschitz maps, see Proposition 2.2. Now assume $\psi \in \mathcal{L}_{d,g}(E,F)$ is in an open neighbourhood of φ, say N, it is constructed as follows, consider the space of linear maps $\mathcal{L}(G \oplus H, F)$. Define the map α

$$\alpha\colon \mathcal{L}_{d,g}(E,F) \to \mathcal{L}(G \oplus H, F),$$

$$(\alpha \circ \psi)(x,y) = \psi.x + y.$$

Put $\mathrm{N} = \alpha^{-1}(\operatorname{Iso}(G \oplus H, F))$. Then the map

$$\overline{\psi}\colon G \oplus H \longrightarrow F,$$

$$\overline{\psi}(x,y) \mapsto \psi.x + y$$

is therefore, a linear isomorphism. It follows that $\dim\ker\psi \le \dim\ker\varphi$. Indeed, consider the projection map $\gamma\colon E = G \oplus \ker\varphi \to \ker\varphi$. Since $\overline{\psi}$ is isomorphism it follows that $G \cap \ker\psi = 0$. But $G = \ker\gamma$, whence $\ker(\gamma|_{\ker\psi}) = \ker\gamma \cap \ker\psi = G \cap \ker\psi = 0$. Thus $\gamma|_{\ker\psi}\colon \ker\psi \to \ker\psi$ is a monomorphism, whence $\dim\ker\psi < \dim\ker\varphi < < \infty$. Now let us show that $\operatorname{codim}\operatorname{Img}(\psi) = \dim\dfrac{F}{\psi(E)} < \infty$. Let K be the complement of $\gamma(\ker\psi)$ in $\ker\varphi$. Then $\dim K < \infty$, and there exists an isomorphism $E \cong G \oplus K \oplus \ker\psi$. Notice that $\psi|_G = \overline{\psi}|_{(G\oplus\{0\})}$ and we have natural identifications:

$$\frac{F}{\psi(G)} = \frac{F}{\overline{\psi}(G \oplus \{0\})} \overset{\iota}{\cong} H \quad \text{and} \quad \frac{F}{\psi(G \oplus K)} \overset{\kappa}{\cong} \frac{F}{\psi(E)}.$$

Then we have the following commutative diagram:

$$\begin{array}{ccc} \dfrac{F}{\psi(G)} & \xrightarrow{\ \iota\ } & H \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \zeta} \\ \dfrac{F}{\psi(G \oplus K)} & \xrightarrow{\ \kappa\ } & \dfrac{F}{\psi(E)} \end{array}$$

Since ζ is onto and H has a finite dimension, we see that $\dim\dfrac{F}{\psi(E)} < \infty$. So ψ is Lipschitz – Fredholm. Evidently,

$$\dim\ker\xi = \dim\ker\psi = \dim K \quad \text{and} \quad \operatorname{codim}\operatorname{Img}\psi = \dim H - \dim K.$$

Furthermore,

$$\dim\ker\varphi = \dim K + \dim\ker\psi \ \text{ and } \ \operatorname{codim}\operatorname{Img}\varphi = \dim H = \dim K + \dim\frac{F}{\psi(E)}.$$

So we have

$$\begin{aligned} \operatorname{Ind}\psi &= \dim\ker\psi - (\dim H - \dim K) = \\ &= \dim\ker\psi + \dim K - \dim H = \\ &= \dim\ker\varphi - \dim H = \\ &= \operatorname{Ind}\varphi. \end{aligned}$$

Theorem 3.2 is proved.

Definition 3.2. *Let M and respectively N be bounded Fréchet manifolds with compatible metrics d_M and respectively d_N modelled on Fréchet spaces E and respectively F with standard metrics. A Lipschitz – Fredholm map is an MC^1-map $f\colon M \to N$ such that for each $x \in M$, the derivative $\mathrm{D}\,P(x)\colon TM_f \longrightarrow TN_{P(x)}$ is a Lipschitz – Fredholm operator. The **index** of f denoted by $\operatorname{Ind} f$, is defined to be the index of $\mathrm{D}\,P(x)$ for some x. Since f is MC^1 and M is connected by Theorem 2.2, the definition does not depend on the choice of x.*

4. Sard's theorem. Let M and respectively N be bounded Fréchet manifolds with compatible metrics d_M and respectively d_N modelled on Fréchet spaces E and respectively F with standard metrics. Let $f\colon M \to N$ be any MC^1-map. A point $x \in M$ is called a **regular point** of f if $\mathrm{D}\,P(x)\colon TM_f \longrightarrow TN_{P(x)}$ is surjective, otherwise is called **critical**. The images of the critical points under f are called the **the critical values** and their complement the **regular values**. Note that if $y \in N$ is not in image of f it is a regular value. We denote the set of critical points of f by $\mathcal{C}_f$ and the set of regular values by $\mathcal{R}(f)$ or $\mathcal{R}_f$. In addition, for $A \subset M$ we define $\mathcal{R}_{f|_A}$ by $\mathcal{R}_f|A = N \setminus f(\mathcal{C}_f \cap A)$. In particular, if $U \subset M$ is open, $\mathcal{R}_f|U = \mathcal{R}(f|U)$.

Theorem 4.1 ([5], Proposition 7.1. Inverse function theorem for MC^k-maps). *Let (F, d) be a Fréchet space, with standard metric. Suppose $k \geq 1$, and $f\colon U \subseteq F \to F$ is an MC^k-map on open subset U. Let $f'(x_0) \in \mathrm{Aut}\,(F)$, for $x_0 \in U$. Then there exists an open neighbourhood $V \subseteq U$ of x_0 such that $f(V)$ is open in F and $f|_V\colon V \to f(V)$ is an MC^k-diffeomorphism.*

Theorem 4.2 (Local representation theorem). *Let $f\colon U \subseteq E \to F$ be an MC^k, $k \geq 1$, $u_0 \in U$ and suppose that $\mathrm{D}\,f(u_0)$ has closed split image F_1 with closed topological complement F_2 and split kernel E_2 with closed topological complement E_1. Then there are two open sets $U' \subset U \subset E = E_1 \oplus E_2$ and $V \subset F_1 \oplus E_2$ and an MC^k-diffeomorphism $\Psi\colon V \to U'$, such that $(f \circ \Psi)(u, v) = (u, \eta(u, v))$ for all $(u, v) \in V$, where $\eta\colon V \to E_2$ is an MC^k-map.*

Proof. Let $f = f_1 \times f_2$, where $f_i\colon U \to F_i$, $i = 1, 2$. By virtue of the open mapping theorem we have $\mathrm{D}_1\, f_1(u_0) = \mathrm{f}_1(u_0)|_{E_1} \in \mathrm{Iso}(E_1, F_1)$. Define the map

$$g\colon U \subset E_1 \oplus E_2 \to F_1 \oplus E_2,$$

$$g(u_1, u_2) = (f_1(u_1, u_2), u_2)$$

therefore,

$$\mathrm{D}\,g(u).(e_1, e_2) = \begin{pmatrix} \mathrm{D}_1\, f_1(u) & \mathrm{D}_2\, f_1(u) \\ 0 & I_{E_2} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

for all $u = (u_1, u_2) \in U$, $e_1 \in E_1$, $e_2 \in E_2$. By hypothesis $E_2 = \ker \mathrm{D}\,f(u_0) = \ker \mathrm{D}\,f_1(u_0)$ and hence $\mathrm{D}_2\, f_1(u_0) = \mathrm{D}_2\, f_1(u_0)|_{E_2} = 0$. Therefore, $\mathrm{D}\,g(u_0) \in \mathrm{Iso}\,(E, F_1 \oplus E_2)$.

By the inverse function theorem, there are open sets U' and V and an MC^k-diffeomorphism $\Psi\colon V \to U'$ such that $u_0 \in U' \subset U \subset E$, $g(u_0) \in V \subset F_1 \oplus E_2$, and $\Psi^{-1} = g|_{U'}$. Hence if $(u, v) \in V$, then $(u, v) = (g \circ \Psi)(u, v) = g(\Psi_1(u, v), \Psi_2(u, v)) = (f_1 \circ \Psi_1(u, v), \Psi_2(u, v))$, where $\Psi = \Psi_1 \times \Psi_2$. This shows that $\Psi_2(v, v) = v$ and $(f_1 \circ \Psi)(u, v) = u$. Define $\eta = f_2 \circ \Psi$, then $(f \circ \Psi)(u, v) = (f_1 \circ \Psi(u, v), f_2 \circ \Psi(u, v)) = (u, \eta(u, v))$.

Theorem 4.2 is proved.

A map f between topological spaces is called **locally closed** if for any point x in the domain of f there exists an open neighbourhood U such that $f|_{\overline{U}}$ is a closed map.

Lemma 4.1. *Let $f\colon E \to F$ be a Lipschitz–Fredholm map between Fréchet spaces with standard metrics. Then f is locally closed.*

Proof. Since f is Fredholm it has split image F_1 with topological complement F_2 and split kernel E_2 with topological complement E_1. By the local representation theorem there are two open sets $U \subset E_1 \oplus E_2$ and $V \subset F_1 \oplus E_2$ and an MC^k-diffeomorphism $\Psi\colon V \to U$ such that $(f \circ \Psi)(u, v) = (u, \eta(u, v))$ for all $(u, v) \in V$, where $\eta\colon V \to E_2$ is an MC^k-map. Suppose $U_1 \subset F_1$ and $U_2 \subset E_2$ are open subsets and $\overline{U_2}$ is compact.

Let $U' = U_1 \times U_2 \subset U$ so that $\overline{U'} = \overline{U}_1 \times \overline{U}_2$ and $\overline{U'} \subset \overline{U}$. Suppose $A \subset \overline{U'}$ is closed, and a sequence $\{(y_i, z_i) = (y_i, \eta(y_i, x_i))\} \subset f(A)$ converges to (y, z), where $\{(y_i, x_i)\}$ is a sequence in A, we need to show that $(y, z) \in f(A)$. By assumption we have $\{x_i\} \subset \overline{U}_2$, and since $\overline{U}_2$ is a compact subset of a finite dimensional topological vector space, we may assume $x_i \to x \in \overline{U}_2$. Then $(y_i, x_i) \to (y, x)$. Since A is closed, $(y, x) \in A$. By continuity of f we see that $\{f(y_i, x_i) = (y_i, z_i)\}$ converges to $f(y, x)$, but $f(y, x) \in f(A)$ thus, $f(A)$ is closed.

Lemma 4.1 is proved.

Theorem 4.3. *Let M and respectively N be bounded Fréchet manifolds with compatible metrics d_M and respectively d_N modelled on Fréchet spaces E and respectively F with standard metrics. Let $f\colon M \to N$ be an MC^k-Lipschitz – Fredholm map with $k > \max\{\operatorname{Ind} f, 0\}$. Then the set of regular values of f is residual in N.*

Proof. It is enough to verify that every $m \in M$ has an open neighbourhood Z such that $\mathcal{R}(f|\bar{Z})$ is open and dense in N. Since M is second countable we can find a countable open cover $\{Z_i\}$ of M with $\mathcal{R}_f|\bar{Z}_i$ open and dense. Since $\mathcal{R}_f = \bigcap_i \mathcal{R}_f|\bar{Z}_i$, it will follows $\mathcal{R}_f$ is residual.

Choose $m \in M$, we will construct a neighborhood Z of m so that $\mathcal{R}(f|\bar{Z})$ is open and dense. By the local representation theorem we may find charts (U, ϕ) at m and (V, ψ) at $f(m)$ such that $\phi(U) \subset E \times \Re^n$, $\psi(V) \subset E \times \Re^p$, and local representative $f_{\phi\psi} = \psi \circ f \circ \psi^{-1}$ of f has the form $f_{\phi\psi}(e, x) = (e, \eta(e, x))$ for $(e, x) \in \phi(U)$, where E is a Fréchet space, $x \in \Re^n, e \in E$, and $\eta\colon \phi(U) \to \Re^p$. The index of $T_m f$ is $n - p$ and so $k > \max\{n - p, 0\}$. To show that $\mathcal{R}(f|U)$ is dense in N, it is enough to show that $\mathcal{R}(f_{\phi\psi})$ is dense in $E \times \Re^p$. For $e \in E$, $(e, x) \in \psi(U)$, define $\eta_e(x) = \eta(e, x)$. For each e, η_e is a C^k-map defined on open set of $\Re^n$ then by Sard's theorem, $\mathcal{R}(\eta_e)$ is dense in $\Re^p$ for each $e \in E$. But for $(e, x) \in \psi(U)$, we have

$$\mathrm{D}\, f_{\phi\psi}(e, x) = \begin{pmatrix} I & 0 \\ * & \mathrm{D}\, \eta_e(x) \end{pmatrix}.$$

So $\mathrm{D}\, f_{\phi\psi}(e, x)$ is surjective if and only if $\mathrm{D}\, \eta_e(x)$ is surjective, hence for $e \in E$

$$\{e\} \times \mathcal{R}(\eta_e) = \mathcal{R}(f_{\phi\psi}) \cap (\{e\} \times \Re^p).$$

And so $\mathcal{R}(f_{\phi\psi})$ intersects every plane $\{e\} \times \Re^p$ in a dense set and is, therefore, dense in $E \times \Re^p$. So $\mathcal{R}(f|U)$ is dense.

Since f is locally closed we can chose an open neighbourhood Z of m such that $\bar{Z} \subset U$ and $f|_{\bar{Z}}$ is closed. Since C_f is closed in M, then $f(\bar{Z} \cap C_f)$ is closed in N, and so $\mathcal{R}(f|\bar{Z}) = N \setminus f(\bar{Z} \cap C_f)$ is open in N. Since $\mathcal{R}(f|U) \subset \mathcal{R}(f|\bar{Z})$ then $\mathcal{R}(f|\bar{Z})$ is dense as well.

Theorem 4.3 is proved.

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