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ON LATTICE OSCILLATOR-TYPE KIRKWOOD – SALSBURG EQUATION WITH ATTRACTIVE MANYBODY POTENTIALS

ПРО ҐРАТКОВЕ РІВНЯННЯ КІРКВУДА – ЗАЛЬЦБУРГА ОСЦИЛЯТОРНОГО ТИПУ З ПРИТЯГУВАЛЬНИМИ БАГАТОЧАСТИНКОВИМИ ПОТЕНЦІАЛАМИ

A lattice oscillator-type Kirkwood – Salsburg (KS) equation with a general one-body phase measurable space and manybody interaction potentials is considered. For special choices of the measurable space, its solutions describe grand-canonical equilibrium states of lattice equilibrium classical and quantum linear oscillator systems. We prove the existence of a solution of the symmetrized KS equation for manybody interaction potentials which are either attractive (non-positive) and finite-range or infinite-range and repulsive (positive). The considered symmetrization of the KS equation is new and is based on the superstability of manybody potentials.

Розглядається ґраткове рівняння Кірквуда – Зальцбурга (КС) осциляторного типу з загальним фазовим одночастинковим вимірним простором та багаточастинковими потенціалами взаємодії. При певному виборі цього вимірного простору розв'язки КС рівняння описують кореляційні функції великого канонічного ансамблю ґраткових рівноважних класичних та квантових систем осциляторів. Доведено існування розв'язку симетризованого КС рівняння для багаточастинкових потенціалів взаємодії, які або притягувальні (недодатні) та мають скінченну дію, або відштовхувальні (додатні) та мають нескінченну дію. Розглядувана симетризація нова і ґрунтується на умові суперстійкості для багаточастинкових потенціалів.

1. Introduction and main result. In this paper we consider the oscillator-type lattice Kirkwood – Salsburg (KS) equation with the one-body phase measurable space (Ω, P^0), an interaction potential energy $U(\omega_X)$, $X \subset \mathbb{Z}^d$ and an external potential $u(\omega)$, where $\omega_X = (\omega_x \in \Omega, x \in X)$. It is an resolvent-type equation satisfied by a sequence of correlation functions $\rho = \{\rho(\omega_X), X \in \mathbb{Z}^d\}$ and may describe grand canonical classical and quantum oscillator systems with a potential energy generated by a pair and manybody potentials $u_Y(\omega_Y)$, $Y \subset \mathbb{Z}^d$ ($|Y| = 2$ corresponds to a pair potential, $|Y|$ is a number of sites in Y), that is

$$U(\omega_X) = \sum_{|Y|\geq 2, Y\subseteq X} u_Y(\omega_Y).$$

The potential energy is an unbounded function and $P^0(\Omega) = \infty$ for oscillator-type or abstract unbounded spin systems. The space Ω can be considered as a metric space (σ-algebra is associated with Borel sets), which is a discrete union of finite balls, and the measure P^0 is finite on them.

The KS equation is written as follows

$$\rho = zK\rho + z\alpha,$$

where $\alpha(\omega_X) = \delta_{|X|,1}$, $\delta_{k,l}$ is the Kroneker symbol. The KS operator K in its turn is given by

$$(KF)(\omega_X) = \sum_{Z\subseteq X^c} \int K(\omega_x|_{X\setminus x}; \omega_Z)\left[F(\omega_{X\setminus x\cup Z}) - \int P(d\omega_x)F(\omega_{X\cup Z})\right]P(d\omega_Z),$$

where the integrations are performed over the cartesian $|Y|$-fold product $\Omega^{|Y|}$ of the measurable space and Ω, for $X = x$ the first term in the square bracket corresponding to $Z = \varnothing$ is equal to zero and $P(d\omega_Y) = \prod_{y\in Y} e^{-\beta u(\omega_x)} P^0(d\omega_y)$. The KS kernels are connected with the potential energy $U(\omega_X)$ in the following way $(x \in X, X \cap Y = \varnothing)$

$$e^{-\beta W(\omega_x|\omega_{X\setminus x\cup Y})} = \sum_{S\subseteq Y} K(\omega_x|\omega_{X\setminus x};\omega_S), \qquad W(\omega_x|\omega_Y) = U(\omega_{Y\cup x}) - U(\omega_Y). \tag{1.1}$$

The expression for the kernel K will be derived in the beginning of the next section and is given by (2.2) or (2.3). It has the same structure as the KS kernel for particle systems [1, 2].

A derivation of the KS equation with manybody potentials is very close to its derivation in the case of lattice gas proposed in [3]. We give it in the Appendix starting from the expression of the grand canonical correlation functions in a compact set which is enlarged to the whole $\mathbb{Z}^d$. That is, the KS equation is related to the correlation functions in the thermodynamic limit.

Classical lattice oscillator systems are described by $\omega = q \in \mathbb{R} = \Omega$, $P^0(dq) = dq$ and quantum lattice oscillator systems by $\omega = w \in \Omega$, $P^0(dw) = dqP^{\beta}_{q,q}(dw)$ and

$$u(w) = \beta^{-1}\int_0^{\beta} u(w(\tau))d\tau, \qquad U(w_X) = \beta^{-1}\int_0^{\beta} U(w_X(\tau))d\tau, \qquad w_X(0) = q_X,$$

where Ω is the space of all continuous paths, $P^{\beta}_{q,q}(dw)$ is the conditional Wiener measure concentrated on continuous paths, starting from q and arriving into q at a "time" β (see [4, 5]). This Wiener measure is generated by the probability transition density $P^t_0(q - q') = (4\pi t)^{-1/2}\exp\left\{-\dfrac{(q-q')^2}{4t}\right\}$ which coincides with the kernel of $\exp\{t\partial^2\}$, where ∂ is the operator of differentiation in the oscillator variable q. Ω can be represented as the Cartesian product of $\mathbb{R}$ and the space Ω_0 of continuous paths starting from the origin due to the translation invariance (with respect to the starting point) of the conditional Wiener measure (see the remark in the end of the paper). We assume that the mass of an oscillator is equal to $\dfrac{1}{2}$ and the Plank constant is equal to the unity.

The lattice oscillator-type (or unbounded spin) KS equation is not well known even for pair interaction contrary to the case of the particle KS equation considered by Ruelle and Ginibre in [3, 5]. Classical and quantum systems of oscillators with pair interaction were usually considered in the canonical ensemble (see [6 – 9]). A short-range ternary interaction between quantum oscillators was considered in [10] also in the canonical ensemble. The lattice KS equation for unbounded spins appeared earlier for the integer-spin Ising systems with pair interaction in [11] and systems of classical and quantum oscillators with finite-range positive (repulsive) manybody potentials in [12, 13].

It is known [1, 3] that to solve the particle KS equation at low activities one needs to symmetrize it with respect to the stability condition if short-range pair potentials are not positive. In this paper we show that in order to solve the lattice oscillator-type KS equation one needs to symmetrize it with respect to a super-stability condition introduced in [14] for classical lattice oscillator systems. In a general case it can be formulated as follows: there exists a non-negative function v on Ω such that

$$|u_Y(\omega_Y)| \le J_Y \sum_{y\in Y} v(\omega_y), \qquad N_0 = \int e^{\beta\gamma v^{1+\zeta}(\omega)} P(d\omega) < \infty, \tag{1.2}$$

where the constants β, ζ are non-negative, the constant γ is positive, $\|J\|_1 = \max_x \sum_{Y,x\in Y} J_Y < \infty$ and the summation is performed over subsets of $\mathbb{Z}^d$ containing a site x. In the case of positive many-body finite-range potentials we used the symmetrization with respect to the superstability condition for a pair potential in [12]. The idea of the symmetrization was used in [11] in a special way. For finite-range manybody potentials we will be able to put $\zeta = 0$ if

$$\left|u_{x,y}(\omega_x,\omega_y)\right| \le J_{x-y}\sqrt{v(\omega_x)v(\omega_x)}. \tag{1.3}$$

Such the condition was postulated by Kunz for proving of a convergence of a polymer cluster expansion for gibbsian canonical correlation functions of a lattice system of oscillators interacting via a pair potential $u_{x,y}$. He employed this condition for an estimate of the cluster functions, satisfying the KS recursion relation, in a way reminiscent of its symmetrization.

We will consider the following four cases: (A) finite-range potentials; (B_1) infinite-range positive potentials; (B_2) finite-range manybody potentials and infinite-range pair potentials; (C) $\zeta = 0$, finite-range manybody potentials and infinite-range pair potentials satisfying (1.3). The range of the potentials will be denoted by R.

We will find solutions of the KS equation for positive finite-range potentials and symmetrized KS equation in other cases in the Banach spaces $\mathbb{E}_\xi$, $\mathbb{E}_{\xi,f}$ ($\mathbb{E}_\xi = \mathbb{E}_{\xi,0}$), respectively. $\mathbb{E}_{\xi,f}$ is the linear space of sequences of measurable functions with the norm

$$\|F\|_{\xi,f} = \max_X \xi^{-|X|} \operatorname*{ess\,sup}_{w_X} \exp\left\{-\sum_{x\in X} f(\omega_x)\right\} |F_X(\omega_X)|, \quad f(\omega) = \gamma\beta v^{1+\zeta}(\omega).$$

We will use the following notations: $P'(d\omega) = e^{f(\omega)}P(d\omega)$, $N_0' = N_0^{-1}\tilde N_0$, $\tilde N_0 = \int v(\omega)P'(d\omega)$, $N_1 = \int e^{\beta c_0 v}P'(d\omega)$, $N_2 = \int v(\omega)e^{\beta\|J_2\|_1 v(\omega)}P'(d\omega)$; $c_0 = \|J\|_1$ ($c_0 = \|J_2\|_1$) for general (positive finite-range manybody) potentials, the norm $\|J\|_1$ will be denoted by $\|J_2\|_1$ if manybody potentials are zero and

$$|J|_l = \max_x |J|_l(x), \qquad |J|_l(x) = \sum_{Z\subseteq(x)^c} J_{x\cup Z}\sigma^{|Z|}(|Z|+1)^{l-1}, \quad l\ge 1, \quad \sigma \ge 1,$$

where the summation is performed over $\mathbb{Z}^d\backslash x$. For the symmetrization we employ the function

$$W'(x|\omega_X) = \sum_{y\in X\backslash x} v(\omega_y) \sum_{Y\subseteq(x\cup y)^c} J_{x\cup y\cup Y}\sigma^{|Y|}, \quad \sigma \ge 1.$$

We have the following inequality:

$$\sum_{x\in X} W'(x|\omega_X) = \sum_{x\in X} v(\omega_x) \sum_{y\in X\backslash x}\sum_{Y\subseteq(x\cup y)^c} J_{y\cup x\cup Y}\sigma^{|Y|} \le$$

$$\le \sum_{x\in X} v(\omega_x) \sum_{y\in(x)^c}\sum_{Y\subseteq(x\cup y)^c} J_{y\cup x\cup Y}\sigma^{|Y|} \le |J|_1 \sum_{x\in X} v(\omega_x). \tag{1.4}$$

It makes possible to symmetrize the KS equation with its respect with the help of the related inequality

$$W'(x|\omega_X) \le |J|_1 v(\omega_x) \tag{1.5}$$

in the following way. Let $\chi_x(\omega_X)$ be the characteristic(indication) function of the set D_x where this inequality holds. Then (1.4) implies that $\cup_{x\in X} D_x = \Omega^{|X|}$ or

$$\sum_{x\in X} \chi_x(\omega_X) \ge 1$$

since D_x may intersect for different x. It is more convenient to deal with χ^*_x

$$\chi^*_x(\omega_X) = \left(\sum_{y\in X} \chi_y(\omega_X)\right)^{-1} \chi_x(\omega_X), \qquad \sum_{x\in X} \chi^*_x(\omega_X) = 1. \tag{1.6}$$

The symmetrized KS operator $\tilde{K}$ is given by

$$(\tilde{K}F)(\omega_X) = \sum_{x\in X} \chi^*_x(\omega_X) \sum_{Z\subseteq X^c} \int K(\omega_x|_{X\backslash x}; \omega_Z)\times$$

$$\times \left[F(\omega_{X\backslash x\cup Z}) - \int P(d\omega_x) F(\omega_{X\cup Z})\right] P(d\omega_Z)$$

where for $X = x$ the first term in the square bracket corresponding to $Z = \varnothing$ is equal to zero. The symmetrized KS equation

$$\rho = z\tilde{K}\rho + z\alpha$$

is derived after multiplying both sides of the KS equation for fixed X, x by the characteristic functions $\chi^*_x(\omega_X)$ and applying (1.6). For all the cases except B_1 one can put $\sigma = 1$. Our main result is formulated in the following theorem.

Theorem 1.1. *Let either $\zeta > 0$ or $\zeta = 0$ and $\gamma - c_0 - |J|_1 \ge 0$, $\gamma - \xi(3|J|_1 + \|J\|_1) \ge 0$, $\gamma - c_0 - |J|_1 - \xi N_1 \|J_2\|_1 \ge 0$, $\gamma > |J|_1$ for the four cases A, B_1, B_2, C, respectively. Moreover, let $\sigma \ge 1 + N_0$ in the case B_1. Then there exists a continuous positive function $G(\xi)$ such that for the norm of the symmetrized KS operator in the Banach space $\mathbb{E}_{\xi,f}$ the following bound holds $\|\tilde{K}\|_{\xi,f} \le (\xi^{-1} + N_0)e^{G(\xi)}$ and the vector ρ from the space $\mathbb{E}_{\xi,f}$*

$$\rho = \sum_{n\ge 0} z^{n+1} \tilde{K}^n \alpha \tag{1.7}$$

determines the unique solution of the symmetrized KS equations in $\mathbb{E}_{f,\xi}$, respectively, if $|z| < \|\tilde{K}\|^{-1}_{\xi,f}$. If the potentials are positive and finite-range then these conclusions hold for $f = 0$, the KS operator and the solution of the KS equation if K is substituted instead of $\tilde{K}$ in the right-hand side of (1.7).

This theorem will be proven with the help of our basic bound for the function

$$\bar{K}_x(\omega_X) = \sum_{Y\subseteq X^c} \xi^{|Y|} \int |K(\omega_x|\omega_{X\backslash x}; \omega_Y) P'(d\omega_Y),$$

$$P'(d\omega_Y) = \exp\left\{\sum_{x\in Y} f(\omega_x)\right\} P(d\omega_Y).$$

For the norm of the KS operator we have the following inequality:

$$\|\tilde{K}\|_{\xi,f} \le (\xi^{-1} + N_0) \max_X \operatorname*{ess\,sup}_{\omega_X} \bar{K}(\omega_X|f),$$
$$\bar{K}(\omega_X|f) = \sum_{x\in X} \chi_x^*(\omega_X) e^{-f(\omega_x)} \bar{K}_x(\omega_X). \tag{1.8}$$

We will show that N_0 is divergent at zero β for classical and quantum oscillator systems with mild restrictions on the potential energy (see the proof of Proposition 4.2). This implies that the convergence radius of series in (1.7) shrinks to zero at zero β. This is a result of the facts that U, u are unbounded functions and P^0 is an unbounded measure. The results of the proposed paper generalize the results of our papers [12, 13].

The case corresponding to positive infinite-range potentials (B_1) will be treated by us separately starting from the second representation of the KS kernels. Our result for this case may be considered as a nontrivial generalization of the Ruelle's result in [3] concerning the existence of a solution of the KS equation for lattice gas with many-body interaction potentials. Remark that The lattice gas is equivalent to the Ising model, i.e., the simplest lattice system of finite-valued spins. Oscillator systems are systems with unbounded spins which are more complicated than the bounded spin systems. All our results presented in this paper and previous ones in [12, 13] lead to the **conclusion:** *to have a solution in $\mathbb{E}_{\xi,f}$ the lattice oscillator-type KS equation needs a special symmetrization if potentials are not positive and finite-range.*

Our paper is organized as follows. In the next section we write down two expressions for the KS kernels and the basic bound in Theorem 2.1, which prove Theorem 1.1, and comment on an optimal choice of ξ which trivializes the expression for G. In the third section we prove the basic bound. In the fourth section we adapt our result for quantum lattice oscillator systems and establish the character of dependence of G at the zero β.

2. KS kernels and the basic bound. The first representation for the KS kernels cab be derived with the help of the purely algebraic relation

$$F(\omega_X) = \sum_{S\subseteq X} \sum_{S'\subseteq S} (-1)^{|S\backslash S'|} F(\omega_{S'}). \tag{2.1}$$

It is derived from the simple equality

$$n = |X|, \quad \sum_{S\in X} (-1)^{|S|} = \sum_{l=0}^{n} (-1)^l C_n^l = 0. \qquad C_n^l = \frac{n!}{l!(n-l)!}.$$

Indeed, let's consider the coefficient before $F(\omega_{X\backslash x})$ in the right-hand side of the previous equality for arbitrary x. It corresponds either for the case $S = X$ or $S = X\backslash x$ and $S' = X\backslash x$. The signs are different before F for these options and this coefficient is equal to zero. Further one has to take $S = X$, $S = X\backslash x_1$, $S = X\backslash x_2$, $S = X\backslash x_1 \cup x_2$, $S' = X\backslash x_1 \cup x_2$ and check that the coefficient before $F(\omega_{X\backslash x_1\cup x_2})$, i.e., the last equality for $n = 2$, is equal to zero. In the same fashion one has to calculate the coefficients before $F(\omega_{X\backslash x_1\cup x_2 \ldots \cup x_n})$, corresponding for the choice $S' = X\backslash x_1 \cup x_2 \cup \ldots \cup x_n$, and check that it coincides with the above sum with the binomial coefficients. As a result (1.1) follows from (2.1) with

$$K(\omega_x|\omega_{X\backslash x};\omega_Y)=\sum_{S\subseteq Y}(-1)^{|Y\backslash S|}e^{-\beta W(\omega_x|\omega_{X\backslash x\cup S})},\tag{2.2}$$

The second representation for the KS kernels is found from the standard arguments whose analog in the case of the lattice gas can be seen in [3]. Let

$$W(\omega_X;\omega_Y|x)=\sum_{x\in Z\subseteq X}u_{Z\cup Y}(\omega_{Z\cup Y}),$$

$$\tilde W(\omega_X;\omega_Y|x)=\sum_{x\in Z\subseteq X}\sum_{\varnothing\neq S\subseteq Y}u_{Z\cup S}(\omega_{Z\cup S})=\sum_{\varnothing\neq S\subseteq Y}W(\omega_X;\omega_S|x).$$

Then

$$W(\omega_x|\omega_{X\backslash x},\omega_Y)=W(\omega_x|\omega_{X\backslash x})+\tilde W(\omega_X;\omega_Y|x)$$

and

$$e^{-\beta\tilde W(\omega_X;\omega_Y|x)}=\prod_{\varnothing\neq S\subseteq Y}\big(1+(e^{-\beta W(\omega_X;\omega_S|x)}-1)\big)=\sum_{S\subseteq Y}K_x(\omega_X;\omega_S),$$

$$K_x(\omega_X;\omega_\varnothing)=1,$$

where

$$K_x(\omega_X;\omega_Y)=\sum_{n=1}^{|Y|}\sum_{\cup Y_j=Y,Y_j\neq\varnothing}\prod_{j=1}^{n}\big(e^{-\beta W(\omega_X;\omega_{Y_j}|x)}-1\big).$$

As a result we obtain the second representation for the KS kernels

$$K(\omega_x|\omega_{X\backslash x};\omega_Y)=e^{-\beta W(\omega_x|\omega_{X\backslash x})}K_x(\omega_X;\omega_Y)\tag{2.3}$$

which will be used only for positive infinite-range potentials.

Proposition 2.1. *Let all the potentials be finite-range except the pair one and have the range R. Then the following equality holds for $X\cap Y=\varnothing$, $x\in X$*

$$K(\omega_x|\omega_{X\backslash x};\omega_Y)=\sum_{S'\subseteq Y}K(\omega_x|\omega_{X\backslash x};\omega_{S'})\chi_{B_x(R)}(S')G(\omega_x|\omega_{Y\backslash S'})\chi_{B^c_x(R)}(Y\backslash S'),\tag{2.4}$$

where $B_x(R)$ is the hyper-ball with the radius R centered at x, $B^c_x(R)=\mathbb{Z}^d\backslash B_x(R)$,

$$G(\omega_x|\omega_S)=\sum_{S'\subseteq S}(-1)^{|S\backslash S'|}e^{-\beta W_2(\omega_x|\omega_{S'})}=\prod_{y\in S}(e^{-\beta u_{(x,y)}(\omega_x,\omega_y)}-1)$$

and $W_2(\omega_x|\omega_{S'})=\sum_{y\in S'}u_{x,y}(\omega_x,\omega_y)$.

Proof. The manybody potentials have the finite range R, that is for an arbitrary $x \in X$, $|X| > 2$ the following quality holds $u_X(\omega_X) = 0$, $|x - x'| \geq R$, $x' \in X \backslash x$ and $|x - x'|$ is the Euclidean distance between two lattice sites. We demand also

$$W(\omega_x|_{X \backslash x}, \omega_S) = W(\omega_x|\omega_{X \backslash x}, \omega_{S \backslash S_2}) + W_2(\omega_x|\omega_{S_2}), \tag{2.5}$$

where $y \notin B_x(R)$ if $y \in S_2$. Here one has to take also into account the equality

$$W_2(\omega_x|\omega_S) = W_2(\omega_x|\omega_{S_2}) + W_2(\omega_x|\omega_{S \backslash S_2}).$$

Let's substitute the equality

$$1 = \prod_{y \in Y} (\chi_{B_x^c(R)}(y) + \chi_{B_x(R)}(y)) = \sum_{S' \subseteq Y} \chi_{B_x(R)}(S') \chi_{B_x^c(R)}(Y \backslash S')$$

into the expression for the KS kernel and apply (3.8). This results in

$$\sum_{S \subseteq Y} (-1)^{|Y \backslash S|} e^{-\beta W(\omega_x|\omega_{X \backslash x}, \omega_S)} \sum_{S' \subseteq Y} \chi_{B_x(R)}(S') \chi_{B_x^c(R)}(Y \backslash S') =$$

$$= \sum_{S' \subseteq Y} \sum_{S \subseteq Y} (-1)^{|Y \backslash S|} e^{-\beta W(\omega_x|\omega_{X \backslash x}, \omega_S)} \chi_{B_x(R)}(S') \chi_{B_x^c(R)}(Y \backslash S') =$$

$$= \sum_{S' \subseteq Y} \sum_{S_2 \subseteq Y \backslash S'} \sum_{S_1 \subseteq S'} (-1)^{(|Y| - |S_1| - |S_2|)} \times$$

$$\times e^{-\beta [W(\omega_x|\omega_{X \backslash x}, \omega_{S_1}) + W_2(x|\omega_{S_2})]} \chi_{B_x(R)}(S') \chi_{B_x^c(R)}(Y \backslash S') =$$

$$= \sum_{S' \subseteq Y} \chi_{B_x(R)}(S') \chi_{B_x^c(R)}(Y \backslash S') \sum_{S_1 \subseteq S'} (-1)^{(|S'| - |S_1|)} e^{-\beta W(\omega_x|\omega_{X \backslash x}, \omega_{S_1})} \times$$

$$\times \sum_{S_2 \subseteq Y \backslash S'} (-1)^{(Y - |S'| - |S_2|)} e^{-\beta W_2(\omega_x|\omega_{S_2})}.$$

Proposition 2.1 is proved.

Theorem 1.1 will be proven with the help of (1.5), (1.6) and the following theorem.

Theorem 2.1. *The following inequality holds*:

$$\bar{K}_x(\omega_X) \leq \exp\{\xi c_1 + \beta c_2 v(\omega_x) + \xi c_3 \sqrt{\beta v(\omega_x)} + \beta c_4 W'(x|\omega_X)\}, \quad c_j \geq 0, \tag{2.6}$$

where $c_2 = c_3 = c_4 = 0$, $c_1 = 2|B_0(R)|N_0$ *hold for positive finite-range potentials*; $c_3 \neq 0$ *except for C*; $c_4 = 2\xi$, $c_2 = \xi(|J|_1 + \|J\|_1)$, *for B*$_1$ *and* $c_4 = 1$ *for the rest three cases. The constant* c_1 *takes the following values in the four remaining cases, respectively,*

$$2|B_0(R)|N_1, \quad \beta N_0'|J|_2, \quad 2|B_0(R)|N_1 + N_2\beta\|J_2\|_1, \quad 2|B_0(R)|N_1.$$

For the constant c_2 *the following expression is true* $c_2 = c^0 + \xi c'$, *where* $c^0 = c_0$, *for A*, *B*$_2$; $c' = 0$ *for A*, *C*; $c' = N_1\|J_2\|_1$ *for B*$_2$ *and* $\gamma - |J|_1 > c^0$ *if* $\gamma - |J|_1 > 0$ *for C.*

For positive finite-range potentials Theorem 2.1 and (1.8) yield immediately that $G(\xi) = \xi c_1$ in Theorem 1.1. That is, the KS operator is bounded in $\mathbb{E}_\xi$. The inequalities (2.6) and (1.5), (1.6) yield the bound

$$\bar{K}(\omega_X|f) \leq e^{c_1\xi} \operatorname*{ess\,sup}_{\omega} \exp\left\{-f(\omega) + \beta(c_2 + c_4|J|_1)v(\omega) + \xi c_3\sqrt{\beta v(\omega)}\right\}. \quad (2.7)$$

This bound and (1.8) imply that the symmetrized KS operator is bounded in the Banach space $\mathbb{E}_{\xi,f}$ for $\zeta, \gamma > 0$ and G in Theorem 1.1 is given by

$$G(\xi) = \xi c_1 + \zeta\beta\gamma^{-1/\zeta}\left(\frac{c_5}{1+\zeta}\right)^{(1+\zeta)/\zeta}, \qquad c_5 = c_2 + c_4|J|_1.$$

Here we used the formula $\max_{v\geq 0} e^{-v^{1+\zeta}+av} = \exp\left\{\zeta\left(\frac{a}{1+\zeta}\right)^{(1+\zeta)/\zeta}\right\}$. If $\zeta = 0$ then the condition $\gamma - c_2 - c_4|J|_1 \geq 0$ corresponds to the conditions stated in Theorem 1.1 for the three first cases and then

$$G(\xi) = \xi c_1, \quad \zeta = 0. \quad (2.8)$$

For the fourth case C we have $\gamma - c_2 - c_4|J|_1 = \gamma - c^0 - |J|_1$ and derive, taking into account the last statement in Theorem 2.1 (c^0 depends on γ), for $\gamma > |J|_1$ (this condition is required in Theorem 1.1 for $\zeta = 0$)

$$G(\xi) = \xi c_1 + \xi^2 c_6. \quad (2.9)$$

The optimal choice of ξ will be such that $G(\xi)$ is less than a constant or a function with a simple dependence on β. Such a choice is obvious if $G(\xi)$ is given either by (2.8) or (2.9). Then either $\xi = c_1^{-1}$ and $G(\xi) = 1$ or $\xi = \left(c_1 + \sqrt{c_6}\right)^{-1}$ and $G(\xi) \leq 2$. The same choice is possible for $\zeta > 0$ and A with G proportional to β (c_5 does not depend on ξ). For B$_1$, $\zeta > 0$ one can put $\xi = \left(c_1 + 3|J|_1 + \|J\|_1\right)^{-1}$ and obtain $G(\xi) \leq$ $\leq 1 + \zeta\beta\gamma^{-1/\zeta}\left(\frac{1}{1+\zeta}\right)^{(1+\zeta)/\zeta}$. For B$_2$, $\zeta > 0$ one can put $\xi = \left(N_1\|J_2\|_1 + c_1\right)^{-1}$ and this means that $G(\xi) \leq 1 + \zeta\beta\gamma^{-1/\zeta}\left(\frac{c_0 + |J|_1}{1+\zeta}\right)^{(1+\zeta)/\zeta}$.

3. Proof of Theorem 2.1. We have the following inequalities which are analogs of the inequalities for the KS kernels for the lattice gas systems from [1]

$$\sum_{Y\subseteq X^c} \xi^{|Y|} \int |K_x(\omega_X;\omega_Y)|P'(d\omega_Y) \leq$$

$$\leq \sum_{Y\subseteq X^c} \xi^{|Y|} \sum_{l=1}^{|Y|} \sum_{\cup Y_j = Y} \prod_{j=1}^{l} \int |e^{-\beta W(\omega_X;\omega_{Y_j}|x)} - 1|P'(d\omega_{Y_j}) =$$

$$= \sum_{n\geq 0} \xi^n \sum_{|Y|=n, Y\subseteq X^c} \sum_{l=1}^{n} \sum_{\cup Y_j = Y} \prod_{j=1}^{l} \int |e^{-\beta W(\omega_X;\omega_{Y_j}|x)} - 1|P'(d\omega_{Y_j}) \leq$$

$$\leq \sum_{n\geq 0} \frac{\xi^n}{n!} \left[\sum_{Y\subseteq X^c} \int |e^{-\beta W\left(\omega_X;\omega_Y|x\right)} - 1| P'(d\omega_Y) \right]^n =$$

$$= \exp\left\{ \xi \sum_{Y\subseteq X^c} \int |e^{-\beta W\left(\omega_X;\omega_Y|x\right)} - 1| P'(d\omega_Y) \right\}.$$

Hence for positive potentials $u_Y \geq 0$ one derives from (2.3) the following estimate:

$$\sum_{Y\subseteq X^c} \xi^{|Y|} \int |K(\omega_x|\omega_{X\setminus x};\omega_Y)| P'(d\omega_Y) \leq$$

$$\leq \exp\left\{ \xi\beta \sum_{Y\subseteq X^c} \int |W(\omega_X;\omega_Y|x)| P'(d\omega_Y) \right\}. \tag{3.1}$$

For positive short-range potentials (case B_1) we have to make estimates of $W(\omega_x|\omega_{X\setminus x})$ and the integral in (3.1) using (1.2)

$$\big|W(\omega_X;\omega_Y|x)\big| \leq \sum_{x\in Z\subseteq X} |u_{Z\cup Y}(\omega_{Z\cup Y})| \leq \sum_{x\in Z\subseteq X} J_{Z\cup Y} \sum_{y\subseteq Z\cup Y} v(\omega_y) =$$

$$= \sum_{x\in Z\subseteq X} J_{Z\cup Y} \left[\sum_{y\in Y} v(\omega_y) + \sum_{y\in Z\setminus x} v(\omega_y) + v(\omega_x) \right]. \tag{3.2}$$

The last inequality yields

$$\int |W(\omega_X;\omega_Y|x)| P'(d\omega_Y) \leq$$

$$\leq N_0^{|Y|} \sum_{Z\subseteq X\setminus x} J_{x\cup Z\cup Y} \left[N_0'|Y| + \sum_{y\in Z\setminus x} v(\omega_y) + v(\omega_x) \right] =$$

$$= N_0^{|Y|} \sum_{Z\subseteq X\setminus x} J_{x\cup Z\cup Y} \big[N_0'|Y| + v(\omega_x) \big] +$$

$$+ N_0^{|Y|} \sum_{y\in X\setminus x} v(\omega_y) \sum_{Z\subseteq X\setminus(x\cup y)} J_{y\cup x\cup Z\cup Y}.$$

Here we utilized the equality

$$\sum_{Y\subseteq\Lambda} \sum_{y\in Y} F(Y;y) = \sum_{y\in\Lambda\setminus X} \sum_{Y\subseteq\Lambda\setminus y} F(Y\cup y;y). \tag{3.2'}$$

As a result

$$\sum_{Y\subseteq X^c} \int |W(\omega_X;\omega_Y \mid x)| P'(d\omega_Y) \leq N_0'|J|_2 + |J|_1 v(\omega_x) + W'(x \mid \omega_X), \quad \sigma \geq 1 + N_0. \tag{3.3}$$

We also obtain from (1.2) and (3.2′)

$$\left|W(\omega_x \mid \omega_{X\backslash x})\right| \le \sum_{x\in Z\subseteq X} |u_Z(\omega_Z)| \le$$

$$\le \sum_{Z\subseteq X\backslash x} J_{Z\cup x} \sum_{y\in Z\cup x} v(\omega_y) \le W'(\omega_x \mid \omega_{X\backslash x}) + \|J\|_1 v(\omega_x). \tag{3.4}$$

The previous two inequalities prove Theorem 2.1 for the case B_1. The proof of the remaining part of Theorem 1.1 will be based on the inequality

$$\bar{K}_x(\omega_X) \le e^{\beta W'(\omega_x|\omega_X)+c_0 v(\omega_x)} \prod_{y,y\neq x} (1+\xi\bar{K}_{x,y}), \tag{3.5}$$

where he product is taken over $\mathbb{Z}^d\backslash x$. If all the potentials have range R then

$$K(\omega_x \mid \omega_{X\backslash x};\omega_Y) = K(\omega_x \mid \omega_{X\backslash x};\omega_Y)\chi_{B_x(R)}(Y). \tag{3.6}$$

Here one has to apply (2.5) with $y = S_2$, $W_2(\omega_x|\omega_y) = 0$ and check that the left-hand side of the last equality is zero since the term with $S = S'$ in (2.2) has the opposite sign to the term with $S = S'\backslash y$. For positive finite-range potentials (3.6) and (2.2) result in

$$K(\omega_x \mid \omega_{X\backslash x};\omega_Y) \le 2^{|Y|}\chi_{B_x(R)}(Y),$$

$$\bar{K}_x(\omega_X) \le \sum_{Y,x\notin Y} (2\xi N_0)^{|Y|}\chi_{B_x(R)}(Y) =$$

$$= \prod_{y,x\neq y} (1+2\xi N_0\chi_{B_x(R)}(y)) \le e^{2|B_0(R)|\xi N_0}.$$

Hence Theorem 2.1 holds for positive finite-range manybody potentials potentials (the simplest subcase of A).

For positive (infinite-range) many-body potentials we have

$$\left|K(\omega_x \mid \omega_{X\backslash x};\omega_Y)\right| \le e^{\beta|W_2(\omega_x|\omega_{X\backslash x})|+\|J_2\|_1 v(\omega_x)} \prod_{y\in Y} (1+e^{\beta\|J_2\|_1 v(\omega_y)}). \tag{3.6′}$$

It is a consequence of the inequality

$$\left|K(\omega_x \mid \omega_{X\backslash x};\omega_Y)\right| \le \sum_{S\subseteq Y} e^{-\beta W_2(\omega_x|\omega_{X\backslash x\cup S})} \le$$

$$\le e^{-\beta W_2(\omega_x|\omega_{X\backslash x\cup S})} \prod_{y\in Y} (1+e^{-\beta u_{x,y}(\omega_x,\omega_y)})$$

derived from the first representation (2.2) for the KS kernels and (1.2) for the pair potential.

Now let the potentials be non-positive. From (2.2) it follows that

$$\left|K(\omega_x \mid \omega_{X\backslash x};\omega_Y)\right| \le \sum_{S\subseteq Y} e^{\beta|W(\omega_x|\omega_{X\backslash x},\omega_S)|}. \tag{3.7}$$

Let's estimate the function under the sign of the exponent of the last inequality starting from the equality

$$W(\omega_x \mid \omega_{X\backslash x\cup Y}) = \sum_{Z\subseteq X\cup Y} u_Z(\omega_Z) - \sum_{x\notin Z\subseteq X\cup Y} u_Z(\omega_Z) = \sum_{x\in Z\subseteq X\cup Y} u_Z(\omega_Z).$$

An employment of (1.2) and changing orders of summations yield

$$\big|W(\omega_x \mid \omega_{X\backslash x\cup Y})\big| \le \sum_{x\in Z\subseteq X\cup Y} |u_Z(\omega_Z)| \le$$

$$\le \sum_{x\in Z\subseteq X\cup Y} J_Z \sum_{y\subseteq Z} v(\omega_y) = \sum_{Z\subseteq X\cup Y\backslash x} J_{x\cup Z} \sum_{y\subseteq Z\cup x} v(\omega_y) =$$

$$= \sum_{y\in Y\cup X\backslash x} v(\omega_y) \sum_{Z\subseteq X\cup Y\backslash(x\cup y)} J_{x\cup y\cup Z} + v(\omega_x) \sum_{Z\subseteq X\cup Y\backslash x} J_{x\cup Z} =$$

$$= \sum_{y\in X\backslash x} v(\omega_y) \sum_{Z\subseteq X\cup Y\backslash(x\cup y)} J_{x\cup y\cup Z} + \sum_{y\in Y} v(\omega_y) \sum_{Z\subseteq X\cup Y\backslash(x\cup y)} J_{x\cup y\cup Z} +$$

$$+ v(\omega_x) \sum_{Z\subseteq X\cup Y\backslash x} J_{x\cup Z}.$$

For the third summand in the right-hand side of the last inequality we have

$$v(\omega_x) \sum_{Z\subseteq X\cup Y\backslash x} J_{x\cup Z} \le v(\omega_x) \sum_{Z\subseteq (x)^c} J_{x\cup Z} \le v(\omega_x)\|J\|_1.$$

For the second and first summands we have, respectively

$$\sum_{y\in Y} v(\omega_y) \sum_{Z\subseteq X\cup Y\backslash(x\cup y)} J_{x\cup y\cup Z} \le \sum_{y\in Y} v(\omega_y) \sum_{Z\subseteq (y\cup x)^c} J_{x\cup y\cup Z} \le \|J\|_1 \sum_{y\in Y} v(\omega_y),$$

$$\sum_{y\in X\backslash x} v(\omega_y) \sum_{Z\subseteq X\cup Y\backslash(x\cup y)} J_{x\cup y\cup Z} \le \sum_{y\in X\backslash x} v(\omega_y) \sum_{Z\subseteq (x\cup y)^c} J_{x\cup y\cup Z}.$$

As a result

$$\big|W(\omega_x \mid \omega_{X\backslash x\cup Y})\big| \le \|J\|_1 v(\omega_x) + W'(\omega_x|\omega_X) + \|J\|_1 \sum_{y\in Y} v(\omega_y). \tag{3.8}$$

From (3.7), (3.8) one deduces the analog of (3.6′)

$$\big|K(\omega_x \mid \omega_{X\backslash x}; \omega_Y)\big| \le e^{\beta W'(\omega_x|\omega_X) + \|J\|_1 v(\omega_x)} \prod_{y\in Y} (1 + e^{\beta\|J\|_1 v(\omega_y)}). \tag{3.9}$$

So, let the potentials have the range R and be non-positive (case A). From (3.6) and (3.9) it follows that Theorem 2.1 is true (the series in the expression for $\bar{K}_x$ into a product as in the formula before (3.6′)) since (3.5) is true with

$$\bar{K}_{x,y} = 2\chi_{B_x(R)}(y)N_1.$$

Let the pair potential be infinite-range. Then taking absolute values of the summands and dropping $\chi_{B_x^c(R)}(Y\backslash S')$ in (2.4) one deduce from it and (3.9) that

$$\left|K(\omega_x \mid \omega_{X\backslash x}, \omega_Y)\right| \le$$

$$\le e^{\beta W'(\omega_x|\omega_X)+\|J\|_1 v(\omega_x)} \prod_{y\in Y}[G(\omega_x|\omega_y) + \chi_{B_x(R)}(y)(1+e^{\beta\|J\|_1 v(\omega_y)})]. \qquad (3.10)$$

From (3.6′) it follows that for positive manybody potentials the analogous inequality holds with J_2 substituted instead of J. The last inequality gives rise to (3.5) with

$$\bar{K}_{x,y} = K_{x,y} + 2\chi_{B_x(R)}(y)N_1, \qquad (3.11)$$

where $K_{x,y} = \int |e^{-\beta u_{x,y}(\omega_x,\omega_y)} - 1|P'(d\omega_y)$. But using (1.2) for the pair potential with $J_{x,y} = J_{x-y}$ one obtains

$$K_{x,y} \le \beta \int |u_{x,y}(\omega_x,\omega_y)|e^{\beta|u_{x,y}(\omega_x,\omega_y)|}P'(d\omega_y) \le$$

$$\le \beta J_{x-y}(N_1 v(w_x) + N_2)e^{\beta J_{x-y}v(\omega_x)}. \qquad (3.12)$$

The last inequality and (3.5) show that Theorem 2.1 is true for B_2. Here one has to use the inequality $1 + \xi\beta J_{x-y}(N_1 v(w_x) + N_2) \le \exp\left\{\xi\beta J_{x-y}(N_1 v(w_x) + N_2)\right\}$.

Let the pair potential satisfy (1.3) (case C). This inequality implies

$$K_{x,y} \le \int P'(d\omega)(e^{\beta J_{x-y}\sqrt{v(\omega_x)}\sqrt{v(\omega)}} - 1)d\omega.$$

For arbitrary $a > 0$ we have

$$K_{x,y} \le \int \left(e^{\beta J_{x-y}\sqrt{v(\omega_x)}\sqrt{v(\omega)}} - e^{-\beta Jx-y\sqrt{v(\omega_x)}\sqrt{v(\omega)}}\right)P'(d\omega) =$$

$$= e^{\beta(2a)^{-2}J_{x-y}^2 v(\omega_x)} \int e^{\beta a^2 v(\omega)}\Big(e^{-\beta(a\sqrt{v(\omega)}-(2a)^{-1}J_{x-y}\sqrt{v(\omega_x)})^2} -$$

$$-e^{-\beta(a\sqrt{v(\omega)}+(2a)^{-1}J_{x-y}\sqrt{v(\omega_x)})^2}\Big)P'(d\omega).$$

For the function in the round brackets we have the bound $\left(b = (2a)^{-1}J_{x-y}\sqrt{v(\omega_x)}\right)$

$$\left|e^{-\beta(a\sqrt{v(\omega)}-b)^2} - e^{-\beta(a\sqrt{v(\omega)}+b))^2}\right| =$$

$$= \left|\pi^{-1/2}\int e^{-k^2}e^{2ika\sqrt{\beta v(\omega)}}\left(e^{2ib\sqrt{\beta}} - e^{-2ib\sqrt{\beta}}\right)\right|dk \le$$

$$\le \pi^{-1/2}\int e^{-k^2}\left(\left|e^{2ibk\sqrt{\beta}} - 1\right| + \left|e^{-2ibk\sqrt{\beta}} - 1\right|\right)dk \le \pi^{-1/2}8b\sqrt{\beta}\kappa^0,$$

$$\kappa^0 = \int e^{-k^2}|k|dk.$$

Here we used the inequality $|e^{iz}-1| \leq 2|z|$, $z \in \mathbb{R}$. Let $B^0(a) = \int e^{\beta a^2 v(\omega)} P'(d\omega)$. Then the last and previous bounds yield the generalized IN bound [11]

$$K_{x,y} \leq 8\kappa^0 B^0(a)(2a)^{-1} e^{\beta(2a)^{-2} J^2_{x-y} v(\omega_x)} J_{x-y}\sqrt{\beta v(\omega_x)},$$

$$1 + \xi(K_{x,y} + 2\chi_{B_x(R)}(y)N_1) \leq$$

$$\leq e^{\beta(2a)^{-2} J^2_{x-y} v(\omega_x) + 8\kappa^0 (2a)^{-1} J_{x-y}\sqrt{\beta v(\omega_x)} B^0(a)\xi + 2\chi_{B_x(R)}(y) N_1 \xi}.$$

(3.5), (3.11) and the last bound prove the basic bound (2.6) with

$$c_2 = c^0 = (2a)^{-2}\|J_2^2\|_1, \qquad c_3 = 8\kappa^0 (2a)^{-1}\|J_2\|_1 B^0(a). \tag{3.13}$$

We determine a from the equality

$$\gamma - |J|_1 - c^0 = \gamma - |J|_1 - (2a)^{-2}\|J_2^2\|_1 = \theta(2a)^{-2}, \quad \theta > 0,$$

and put $B_0(a) = N'$ for this choice. This means that for $\gamma - |J|_1 > 0$

$$(2a)^2 = (\gamma - |J|_1)^{-1}(\theta + \|J_2^2\|_1), \qquad c_6 = \theta^{-1}(4\kappa^0\|J_2\|_1 N')^2. \tag{3.14}$$

If in addition manybody finite-range potentials are positive then all the above equalities for c_j are true if one substitutes $\|J_2\|_1$ instead of $\|J\|_1$. Note that for classical systems, the choice $\theta = \beta^{-s}$, $\frac{1}{n} \leq s \leq 1 - \frac{n_0}{n}$, where $2n$, $2n_0$ are the degrees of the polynomials u, v, respectively, allows one to prove that c_6 tends to a finite limit (zero) if β tends to zero $\left(s > \frac{1}{n}\right)$.

4. Quantum systems. In this section we show that all the integrals in our theorems are well defined for quantum lattice oscillator systems.

Proposition 4.1. *Let* (1.2), (1.3) *hold for classical systems. Then* (1.2), (1.3) *hold for quantum systems and the norms* N_j, N_0', N' *are finite.*

Proof. Obviously, the first inequality in (1.2) for quantum systems is satisfied with $\beta v(w) = \int_0^\beta v(w(\tau))d\tau$. From the Schwartz inequality it follows that (1.3) holds for quantum systems. From the Helder inequality

$$\beta v^{1+\zeta}(w) = \beta^{-\zeta}\left(\int_0^\beta v(w(\tau))d\tau\right)^{1+\zeta} \leq \int_0^\beta v^{1+\zeta}(w(\tau))d\tau$$

and the Golden – Thompson inequality $\operatorname{Tr} e^{A+B} \leq \operatorname{Tr} e^A e^B$ one derives the bound

$$N_0 = \int dq \int e^{-\beta u(w) - f(w)} P^\beta_{q,q}(dw) =$$

$$= \operatorname{Tr} e^{\beta[\partial^2 - u + \gamma v^{1+\zeta}]} \leq (4\pi\beta)^{-1/2}\int e^{-\beta[u(q) - \gamma v^{1+\zeta}(q)]}dq. \tag{4.1}$$

Here we took into account that $e^{\beta\partial^2}(q,q) = (4\pi\beta)^{-1/2}$. By the same arguments

$$N_1 \leq (4\pi\beta)^{-1/2} \int e^{-\beta[u(q)-\gamma v^{1+\zeta}(q)-c_0 v(q)]} dq,$$

$$N_2 \leq (4\pi\beta)^{-1/2} \int e^{-\beta[u(q)-\gamma v^{1+\zeta}(q)-c_0 v(q)]+v(q)} dq, \tag{4.2}$$

$$B^0(a) \leq (4\pi\beta)^{-1/2} \int e^{-\beta[u(q)-(\gamma+a^2)v(q)]} dq, \tag{4.3}$$

where we used the elementary inequality $v(w) \leq e^{v(w)}$. The bound $N_0 > 0$ is derived from the bounds $v \geq 0$, $u(w(\tau)) \leq \bar{u}_R$, $|w(\tau)| \leq R$, $0 \leq \tau \leq \beta$, where $\bar{u}_R$ is a constant. These bounds show that N_0, $N_1 < \infty$, $\tilde{N_0} \leq N_2 < \infty$ and N_0', $N' < \infty$.

Proposition 4.1 is proved.

The reduced density matrices $\rho(q_X|q_X')$ are given by

$$\rho(q_X|q_X') = \int e^{-\beta \sum_{x\in X} u(w_x)} \rho(w_X) P^\beta_{q_X, q_X'}(dw_X),$$

where the sequence $\{\rho(w_X), X \subset \mathbb{Z}^d\}$ is the solution of the (symmetrized) KS equation determined in Theorem 1.1. The following bound is valid for the them

$$\rho(q_X|q_X') \leq \xi^{|X|} \|\rho\|_{\xi,f} \prod_{x\in X} \int e^{-\beta u(w_x)+f(w_x)} P^\beta_{q_x, q_x'}(dw_x) =$$

$$= \xi^{|X|} \|\rho\|_{\xi,f} \prod_{x\in X} P^\beta(q_x; q_x'),$$

where $P^\beta(q_x; q_x')$ is the kernel of the semigroup whose infinitesimal generator is $\partial^2 - - u + \gamma v^{1+\zeta}$.

The following proposition clarifies a dependence of the above norms on β in a neighborhood of the origin.

Proposition 4.2. *Let* $\eta_- q^{2n} - \bar{\eta} \leq u(q) \leq \eta_+ q^{2n} + \bar{\eta}$, $v(q) = q^{2n_0} + 1$, $1+\zeta < \dfrac{n}{n_0}$. *Let also* $\theta = \beta^{-s}$ *and* $s \leq 1 - \dfrac{n_0}{n}$. *Then*

$$N_j \leq \beta^{-(1+n)/2n} \bar{N}_j, \quad N_2 \leq \beta^{-1/2-n_0/n-1/2n} \bar{N}_2,$$

$$N_0' \leq \beta^{-n_0/n} \bar{N}_0', \quad N' \leq \beta^{-(1+n)/2n} \bar{N}',$$

where $j = 0, 1$ *and all the norms in the right-hand sides of the inequalities are finite on a finite interval in* β.

Proof. The proof is based on an application of the Helder inequality

$$\int_0^\beta w^{2n_0}(\tau) d\tau \leq \beta^{1-n_0/n} \left(\int_0^\beta w^{2n}(\tau) d\tau \right)^{n_0/n}.$$

It and the bound $|x|^r \leq c_r^0 e^{|x|}$ give

$$N_2 - N_1 \leq \beta^{-n_0/n} (4^{-1}\eta_-)^{-n_0/n} c^0_{n_0/n} \int e^{-\beta \tilde{u}(w)} dq P^\beta_{q,q}(dw),$$

where $\tilde{u}(w) = u(w) - c_0 v(w)) - f(w) - \frac{\eta_-}{4}\int_0^\beta w^{2n}(\tau)d\tau$. By rescaling the variables by $\beta^{-1/2n}$ in the integrals in the bounds (4.1), (4.2) for N_0, N_1 one sees the inequalities for N_0, N_1 in the proposition are true. They imply also the similar inequality for N_2 after an application for the Golden – Thompson inequality. The inequality for N' follows from (4.3) and the inequality (4.1) for N_1.

To estimate accurately N'_0 on has to obtain an accurate bound from above for N_0 starting from the inequalities

$$u(w) \le \eta'\beta^{-1}\int_0^\beta w^{2n}(\tau)d\tau + a', \quad 0 < \eta' < \eta_+,$$

$$N_0 \ge e^{-\beta a'}\int dq \int \exp\left\{-\eta'\int_0^\beta w^{2n}(\tau)d\tau\right\} P^\beta_{q,q}(dw),$$

where a' is a constant. The first inequality is a result of the inequality $q^l \le \epsilon q^k + \epsilon^{-l}$, $k > l$, $q \ge 0$, $\epsilon \le 1$. From the Jensen inequality it follows that

$$N_0 \ge e^{-\beta a'}(4\pi\beta)^{-1/2}\int dq \exp\left\{-\eta'\sqrt{4\pi\beta}\int P^\beta_{q.q}(dw)\int_0^\beta w^{2n}(\tau)d\tau\right\}.$$

Here we took into account $\int P^\beta_{q,q}(dw) = (4\pi\beta)^{-1/2}$. Further

$$\int P^\beta_{q.q}(dw)\int_0^\beta w^{2n}(\tau)d\tau = \int_0^\beta \int P^\tau_0(q-q')q'^{2n}P^{\beta-\tau}_0(q'-q)dq'd\tau \le$$

$$\le 2^{2n}\int_0^\beta \int P^\tau_0(q')(q'^{2n}+q^{2n})P^{\beta-\tau}_0(q')dq'd\tau = (4\pi)^{-1/2}\sqrt{\beta}(2q)^{2n} + c'_n.$$

Here we used the inequality $(q'-q+q)^{2n} \le 2^{2n}(q^{2n} + (q-q')^{2n})$, the semigroup property of $P^t_0(q)$ and the equality $P^t_0(0) = (4\pi\beta)^{-1/2}$. c'_n does not depend on q and is finite for a finite β since

$$c'_n = 2^{2n}\int_0^\beta \int P^\tau_0(q')q'^{2n}P^{\beta-\tau}_0(q')dq'd\tau \le$$

$$\le 2^{2n}\int_0^\beta \left(\int (P^\tau_0(q'))^2 q'^{2n}dq'\right)^{1/2}\left(\int (P^{\beta-\tau}_0(q'))^2 q'^{2n}dq'\right)^{1/2}d\tau =$$

$$= (4\pi)^{-1}2^{2n}\left(\int e^{q^2/2}q^{2n}dq\right)\int_0^\beta \left[\tau(\beta-\tau)\right]^{n/2}d\tau.$$

Here we applied the Schwartz inequality. A a result

$$N_0^{-1} \le \sqrt{4\pi\beta} e^{\sqrt{4\pi\beta}c_n'+\beta a'} \left(\int dq e^{-\eta'\beta(2q)^{2n}}\right)^{-1} =$$

$$= \sqrt{4\pi}\beta^{1/2+1/2n} e^{\sqrt{4\pi\beta}c_n'+\beta a'} \left(\int dq e^{-\eta'(2q)^{2n}}\right)^{-1}.$$

Taking into account that $\tilde{N}_0 \le N_2$, the bound for N_2 and the last bound one derives $N_0' \le \beta^{-n_0/n}\bar{N}_0'$. This concludes the proof of the proposition.

Remark. For the choice $P^0(dw) = dqP_q(dw)$, where $P_q(dw)$ is the conditional Wiener measure, concentrated on continuous paths starting from q, solutions of the KS equation may correspond to correlation functions of a stochastic dynamics of lattice oscillators. The result of the proposed paper may be applied without difficulty for a proof of an existence of solutions of the BBGKY-type hierarchy for the stochastic dynamics of oscillators interacting via manybody potentials (in [15] only pair interaction was considered). A scheme for a proof of the local convergence of the finite-volume grand-canonical correlation functions to the solution of the KS equation can be found in [15].

5. Appendix. To derive the KS equation one has to start from the expressions for the grand canonical correlation functions in a compact set Λ

$$\rho^\Lambda(\omega_X) = \chi_\Lambda(X)\Xi_\Lambda^{-1} \sum_{Y\subseteq\Lambda\backslash X} z^{|Y\cup X|} \int P(d\omega_Y) e^{-\beta U(\omega_{X\cup Y})}, \tag{5.1}$$

where the grand partition function Ξ_Λ coincides with the numerator of the right-hand side of (5.1) for empty set X. Substituting $U(\omega_{X\cup Y}) = U(\omega_{X\cup Y\backslash x}) + W(\omega_x \mid \omega_{X\backslash x\cup Y})$ and the first equality in (1.1) into the expression of the finite volume grand canonical correlation functions one obtains

$$\rho^\Lambda(\omega_X) =$$

$$= \Xi_\Lambda^{-1}\chi_\Lambda(X) \sum_{Y\subseteq\Lambda\backslash X} z^{|Y\cup X|} \int P(d\omega_Y) e^{-\beta U(\omega_{X\cup Y\backslash x})} \sum_{S\subseteq Y} K(\omega_x \mid \omega_{X\backslash x}; \omega_S) =$$

$$= \Xi_\Lambda^{-1}\chi_\Lambda(X) \sum_{Y\subseteq\Lambda\backslash X} z^{|Y\cup X|} \sum_{S\subseteq Y} \int P(d\omega_Y) K(\omega_x \mid \omega_{X\backslash x}; \omega_S) e^{-\beta U(\omega_{X\cup Y\backslash x})} =$$

$$= z \sum_{Z\subseteq\Lambda\backslash X} \int P(d\omega_Z) K(\omega_x \mid \omega_{X\backslash x}; \omega_Z) \Xi_\Lambda^{-1}\chi_\Lambda(X\cup Z)\times$$

$$\times \sum_{Y\subseteq\Lambda\backslash(Z\cup X)} z^{|Y\cup X\cup Z|-1} \int P(d\omega_Y) e^{-\beta U(\omega_{X\backslash x}, \omega_Y)}.$$

The equality

$$\rho^\Lambda(\omega_{X\backslash x}) = \Xi_\Lambda^{-1}\chi_\Lambda(X\backslash x) \sum_{Y\subseteq(\Lambda\backslash X)\cup x} z^{|Y\cup X|-1} \int P(d\omega_Y) e^{-\beta U(\omega_{X\backslash x}, \omega_Y)}$$

leads to

$$\Xi_\Lambda^{-1}\chi_\Lambda(X\cup Z)\sum_{Y\subseteq\Lambda\backslash(Z\cup X)} z^{|Y\cup X\cup Z|-1}\int P(d\omega_Y)e^{-\beta U(\omega_{X\backslash x},\omega_Y)} =$$

$$= \chi_\Lambda(x)(\rho^\Lambda(\omega_{X\backslash x\cup Z}) - \int P(d\omega_x)\rho^\Lambda(\omega_{X\cup Z})).$$

It's clear that the terms with $x\in Y$ in the sum, representing the first term in the round brackets, are canceled by the same terms in the sum representing the second term in the brackets. This concludes the derivation of the KS equation if one takes also into account that $\rho^\Lambda(\omega_\varnothing)=1$. That, is the KS equation is given for $x\in X$, $|X|>1$ by

$$\rho^\Lambda(\omega_X) = z\chi_\Lambda(x)\sum_{Z\subseteq\Lambda\backslash X}\int K(\omega_x|\omega_{X\backslash x};\omega_Z)[\rho^\Lambda(\omega_{X\backslash x\cup Z}) -$$

$$-\int P(d\omega_x)\rho^\Lambda(\omega_{X\cup Z})]P(d\omega_Z) \tag{5.2}$$

and for $X=x$ by

$$\rho^\Lambda(\omega_x) = z\chi_\Lambda(x)\Bigg\{1-\int\rho^\Lambda(\omega_x)P(d\omega_x)+$$

$$+\sum_{|Z|\geq 1, Z\subseteq\Lambda\backslash x}\int K(\omega_x|\omega_Z)\left[\rho^\Lambda(\omega_Z)-\int P(d\omega_x)\rho^\Lambda(\omega_{Z\cup x})\right]P(d\omega_Z)\Bigg\}.$$

Let $\alpha(\omega_X)=\delta_{|X|,1}$. Let, also, the KS operator K be given for $\Lambda=\mathbb{Z}^d$ by the right-hand side of (5.2), if $|X|>1$ and the right-hand side of without the unity if $X=x$. As a result the finite volume and infinite volume KS equations in an abstract form look like

$$\rho_\Lambda = zK_\Lambda\rho_\Lambda + z\chi_\Lambda\alpha, \qquad \rho = zK\rho + z\alpha,$$

where $K_\Lambda=\chi_\Lambda K\chi_\Lambda$, χ_Λ is the operator of multiplication by the characteristic function of Λ: $(\chi_\Lambda F)_X(\omega_X)=\chi_\Lambda(X)F_X(\omega_X)$.

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