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ON THE UNBOUNDED ORDER PARAMETER IN LATTICE GKS-TYPE OSCILLATOR EQUILIBRIUM SYSTEMS

ПРО НЕОБМЕЖЕНИЙ ПАРАМЕТР ПОРЯДКУ У ГРАТКОВИХ РІВНОВАЖНИХ СИСТЕМАХ ОСЦИЛЯТОРІВ ТИПУ ГКШ

An unbounded order parameter (magnetization) is established to exist for a wide class of lattice Gibbs (equilibrium) systems of linear oscillators interacting via a strong pair near neighbor polynomial potential and other many-body potentials. The considered systems are characterized by a general polynomial short-range interaction potential energy generating Gibbs averages that satisfy two generalized GKS inequalities.

Встановлено існування необмеженого параметра порядку (намагніченості) для широкого класу ґраткових гіббсівських (рівноважних) систем лінійних осциляторів, що взаємодіють завдяки сильному парному поліноміальному потенціалу близьких сусідів та іншим багаточастинковим потенціалам. Розглянуті системи характеризуються загальною поліноміальною близькодійовою потенціальною енергією, що породжує середні, які підкоряються двом нерівностям ГКШ.

1. Introduction and main result. In this paper we consider Gibbs classical systems of one-dimensional oscillators (unbounded spins) on the d-dimensional hyper-cubic lattice $\mathbb{Z}^d$, with a polynomial ferromagnetic GKS (Griffiths – Kely – Sherman)-type translation-invariant potential energy $U(q_\Lambda) = U(-q_\Lambda)$ on a hypercube Λ with the finite cardinality $|\Lambda|$ centered at the origin, where q_Λ is an array of $(q_x, x \in \Lambda)$, q_x is the oscillator coordinate taking values in $\mathbb{R}$.

For a wide class of oscillator systems with a polynomial ferromagnetic n-n (near neighbor) pair potential whose strength is g there exists the unit spin long-range order (lro), that is the following inequality holds [1 – 4]

$$\langle s_x s_y \rangle_\Lambda \geq 1 - o(\lambda), \qquad s_x = \operatorname{sign} \sigma_x, \quad \sigma_x(q_\Lambda) = q_x, \tag{1.1}$$

where $\langle ., . \rangle_\Lambda$ denotes the Gibbs average, λ is either g^{-1} or the temperature and $o(\lambda)$ is a continuous function tending to zero in the limit of zero λ. Such the lro generates the bounded ferromagnetic order parameter (bounded magnetization) $m_\Lambda = |\Lambda|^{-1} \sum_{x \in \Lambda} s_x$, which due to (1.1) has a non-zero average when it is squared, that is $\langle m_\Lambda^2 \rangle_\Lambda \geq 1 - o(\lambda)$. If there is short-range order, that is the average in (1.1) decreases when the (Euclidean) distance $|x - y|$ grows, then the bounded magnetization is zero and there is no order in the system.

An existence of the unit spin lro for oscillator systems with a n-n pair (non-polynomial) potential for oscillator systems has been proven earlier in the paper [5], in which the reflection positive Pieirls argument was employed (see also [6, 7]). The non-trivial problem to derive the similar bound for $\langle \sigma_x \sigma_y \rangle_\Lambda$ was not considered in the mentioned papers. We solve this problem in this paper. Our result implies that the system is ordered and has the non-zero unbounded ferromagnetic order parameter $M_\Lambda = |\Lambda|^{-1} \sum_{x \in \Lambda} \sigma_x$ at a large g. Our technique can be characterized as a GKS-type Pieirls argument strengthened by the Ruelle superstability bound. It is based on the facts that the basic constant (independent of oscillator variables) present in the Ruelle super-stability bound [8, 9] grows polynomially in g at infinity and that the average $\left\langle e^{-(g/2)[\sigma_x^l \sigma_y^k + \sigma_x^k \sigma_y^l]} \right\rangle_\Lambda$ decreases exponentially in g at infinity, where the expression under the sign of the exponent is proportional to the pair n-n potential and x, y are nearest neighbors. Non-triviality of M_Λ at low temperatures was proved earlier in [10] for lattice systems of oscillators with the interaction generated by the pair bilinear nearest neighbor potential. The technique of [10] is not connected with a generalized Pieirls argument. A review of results concerning lro in lattice oscillator systems a reader may find in [4]. New results concerning existence of the magnetization in Ising models can be found in [11, 12].

Our potential energy is given by (see also [3, 4])

$$U(q_\Lambda) = \sum_{x\in\Lambda} u_0(q_x) - g \sum_{\langle x,y\rangle\in\Lambda} (q_x^k q_y^l + q_x^l q_y^k) + U'(q_\Lambda), \tag{1.2}$$

where $\langle x, y\rangle$ means nearest neighbors, u_0 is a bounded below polynomial of the $2n$-th degree such that $u_0 - \dfrac{1}{2}q^{2n}$ is also bounded from below, $k + l = 2n_0 < 2n$,

$$U'(q_\Lambda) = -\sum_{A\subseteq\Lambda} \phi_A(q_A), \qquad \phi_A(q_A) = \sum_{\langle n_{(|A|)}\rangle<2n} J_{A;n_{(|A|)}} S q_{[A]}^{n_{|A|}}, \quad J_{A;n_{(|A|)}} \geq 0,$$

the first sum is performed over subsets of Λ with the number of sites $|A| \leq n$ and the second one over the sequence of positive integers such that the number $\langle n_{(|A|)}\rangle = \sum_{j=1}^{|A|} n_j < 2n$ is even, $n_{(|A|)} = (n_1, \ldots, n_{|A|})$, $J_{A\cup x;n_{(|A|)},n_0} = J_{A-x\cup 0;n_{(|A|)},n_0}$ (translation invariance). If $A = (x_1, \ldots, x_k) = x_{(k)}$, $|A| = k$ then $J_{A;n_{(|A|)}} = J_{x_{(k)},n_{(k)}} = J_{n_{(k)}}\big(|x_1 - x_k|, \ldots, |x_{k-1} - x_k|\big)$, where $|x|$ is the Euclidean norm of the site x and $q_{[A]}^{n_{|A|}} = \prod_{j=1}^k q_{x_j}^{n_j}$. Here S means symmetrization. There is another representation for U, given by (3.1), in which its interacting part is zero for coinciding arguments (this part differs from U by a "boundary" term generated by an external field). We assume that the interaction is short-range, that is

$$J_l^- = \sum_{0\in A} \frac{1}{|A|} \sum_{\langle n_{(|A|)}\rangle=2l} J_{A;n_{(|A|)}} < \infty,$$

where the summation is performed over all subsets of $\mathbb{Z}^d$ which contain the origin. Our main result is formulated as follows.

Theorem 1.1. *Let $d \geq 2$ and $l > 2n_0 - l$ then there exists a positive number $g_0 > 1$ such that for $g > g_0$ the following uniform in Λ bound is valid*

$$\langle \sigma_x \sigma_y \rangle_\Lambda \geq \bar{\sigma} g^{-\theta} - \sigma^0 g^{-\alpha}, \tag{1.3}$$

where $\theta = 2 + \dfrac{d+1}{n-n_0}\dfrac{l}{2n_0 - l}$, $\bar{\sigma}$, $\sigma^0 > 0$ depend on β and α is an arbitrary positive number.

***Corollary* 1.1.** *For an arbitrary temperature there exists a positive number g_* such that for $g \geq g_*$ the left-hand side of* (1.3) *is positive uniformly in Λ, implying the existence of lro and that the unbounded order parameter M_Λ is non-zero in the thermodynamic limit.*

The proof of (1.3) demands the bounds which were not employed in [1 – 4] for the proof of (1.1), namely, the Ruelle superstability bound [8]

$$\rho^{\Lambda}(q_X) \le \exp\left\{ -\beta \sum_{x\in X} (u_\varepsilon(q_x) - \beta^{-1}c) \right\}, \quad u_\varepsilon = u - 3\varepsilon v, \tag{1.4}$$

where c is the basic superstability constant, ε is an arbitrary arbitrary small number $\left(\varepsilon < \dfrac{1}{3}\right)$ and

$$u(q) = u_0(q) - 2dgq^{2n_0} - \sum_{l=1}^{n-1} J_l^- q^{2l}, \qquad v(q) = \sum_{j=1}^{n-1} q^{2j}. \tag{1.5}$$

The Ruelle superstability bound (1.4) is proved if one establishes the following superstability and regularity conditions for the potential energy

$$U(q_X) \ge \sum_{x\in X} u(q_x), \tag{1.6}$$

$$\left|W(q_{X_1}; q_{X_2})\right| \le \frac{1}{2} \sum_{x\in X_1,\, y\in X_2} \Psi(|x-y|)\left[v(q_x) + v(q_y)\right], \quad X_1 \cap X_2 = \varnothing, \tag{1.7}$$

where

$$W(q_X; q_Y) = U(q_{X\cup Y}) - U(q_Y) - U(q_X), \quad \|\Psi\|_1 < \infty,$$

$\|\Psi\|_1 = \sum_x \Psi(|x|)$ (the summation is performed over $\mathbb{Z}^d$).

We prove Theorem 1.1 with the help of Theorem 1.2 and Proposition 1.1.

Theorem 1.2. *Let $d \ge 2$, $l > 2n_0 - l$, c be the basic superstability constant and $c \le \bar{c}g^{\kappa} + o(g^{-1})$, where $\bar{c}$ is a positive constant. Then inequality* (1.3) *holds in which either $\theta = 2 + \dfrac{\kappa}{n}\dfrac{l}{2n_0 - l}$ for $\kappa \ge \dfrac{n}{n-n_0}$ or $\theta = 2 + \dfrac{1}{n-n_0}\dfrac{l}{2n_0-l}$ for $\kappa < \dfrac{n}{n-n_0}$. If $\kappa = \dfrac{n}{n-n_0}$* (1.3) *holds for sufficiently large β.*

From the analytical structure of the basic superstability constant presented in [4, 9] one easily derives the following proposition proved in the last section.

Proposition 1.1. *Let c be the basic superstability constant then $\kappa = \dfrac{n(d+1)}{n-n_0}$.*

We shall rely, also, on the following proposition whose proof can be found in [4].

Proposition 1.2. *Let U_0 be a bounded from below even polynomial of the $2n$-th degree, $U(q) = U_0(q) - 2dgq^{2n_0}$, $n_0 < n$. Then there exists positive constants $g_0 > 1$, κ_0, $\bar{\mu}$, $\bar{e}$ such that for $g \ge g_0$ the potential U has the the unique deepest minimum e^0 and the following inequalities hold*

$$e^0 \le \bar{e}g^{1/2(n-n_0)}, \qquad \left|U(e^0)\right| \le \bar{\mu}g^{n/(n-n_0)}, \qquad \int e^{-\beta U(q)}dq \le \kappa_0 e^{-\beta U(e^0)},$$

where the integration is performed over $\mathbb{R}$.

The first two bounds in this proposition are equalities for the simplest potential $U(q) = u(q) = \eta q^{2n} - 2dgq^{2n_0}$ and its unique positive minimum

$$e^0 = \left(2d\frac{n_0}{\eta n}g\right)^{1/(2(n-n_0))}, \qquad u(e^0) = -\eta\frac{n-n_0}{n_0}\left(2d\frac{gn_0}{\eta n}\right)^{n/(n-n_0)}.$$

Our paper is organized as follows. In the next section we give a proof of Theorem 1.2. In the third section (1.5) – (1.7) and Proposition 1.1 are proved.

2. Proof of Theorem 1.2. The Gibbs averages for a measurable function F_X on $\mathbb{R}^{|X|}$ are given by

$$\langle F_X\rangle_\Lambda = Z_\Lambda^{-1}\int F_X(q_X)e^{-\beta U(q_\Lambda)}dq_\Lambda = \int F_X(q_X)\rho^\Lambda(q_X)dq_X,$$

$$\rho^\Lambda(q_X) = Z_\Lambda^{-1}\int e^{-\beta U(q_\Lambda)}dq_{\Lambda\setminus X}, \qquad Z_\Lambda = \int e^{-\beta U(q_\Lambda)}dq_\Lambda,$$

where $\int dq_X$ denotes the integral over $\mathbb{R}^{|X|}$, β is the inverse temperature. We assume that

$$u_0(q) = \eta q^{2n} + u^1(q), \qquad u^1(q) = \sum_{j=1}^{n-1}\eta_j q^{2j}, \quad \eta \ge 1.$$

The proof of Theorem 1.2 begins from a derivation of the inequality

$$\langle\sigma_x\sigma_y\rangle_\Lambda \ge r^2 - 2\Big(\big\langle\chi_x^+\chi_y^-\big\rangle_\Lambda^{1/2} + \langle\chi_x^-\chi_y^+\rangle_\Lambda^{1/2}\Big)\langle\sigma_x^4\rangle_\Lambda^{1/4}\langle\sigma_y^4\rangle_\Lambda^{1/4} -$$

$$- r^2\Big[2\Big(\langle\chi_{x,[-r,r]}\rangle_\Lambda + \langle\chi_{y,[-r,r]}\rangle_\Lambda\Big) + \langle\chi_x^+\chi_y^-\rangle_\Lambda + \langle\chi_y^+\chi_x^-\rangle_\Lambda\Big], \tag{2.1}$$

where $\chi_{x,[r,r']}(q_\Lambda) = \chi_{[r,r']}(q_x)$, $\chi_x^+ = \chi_{x,[0,\infty]}$, $\chi_x^- = \chi_{x,[-\infty,0]}$ and $\chi_{[r,r']}$ is the characterisctic function of the interval $[r,r']$.

Inequality (2.1) is an analog of the inequality for the two point spin Gibbs average for the bounded spin systems from [5]. It is known from [3, 4] that $\langle\chi_x^+\chi_y^-\rangle_\Lambda$ exponentially tends to zero at infinity in g. In order to derive (1.3) from (2.1) for r polynomially decreasing in g at infinity one has to establish that the Gibbs average $\langle\sigma_x^4\rangle_\Lambda$ tends only polynomially to infinity in growing g and that $\langle\chi_{x,[-r,r]}\rangle_\Lambda$ tends exponentially to zero at the same time. We will establish that $\langle\chi_{x,[-r,r]}\rangle_\Lambda$ tends to zero at infinity in g with the help of the equality

$$\langle\chi_{x,[-r,r]}\rangle_\Lambda = \langle\chi_{x,[-r,r]}\chi_{x_*,[-r,r]}\rangle_\Lambda + \langle\chi_{x,[-r,r]}\chi_{x_*,[-r,r]^c}\rangle_\Lambda =$$

$$= \langle\chi_{x,[-r,r]}\chi_{x_*,[-r,r]}\rangle_\Lambda + \langle\chi_{x,[-r,r]}\chi_{x_*;r,r'}\rangle_\Lambda + \langle\chi_{x,[-r,r]}\chi_{x_*,[-r',r']^c}\rangle_\Lambda, \tag{2.2}$$

where $x_* \in \Lambda$ is one of the nearest neighbors of x, $\chi_{x;r,r'}(q_\Lambda) = \chi_{r,r'}(q_x) = \chi_{[r,r']}(q_x) + \chi_{[-r',-r]}(q_x)$, bounds (2.3′), (2.3″), (2.4) and r, r' chosen in a special way. The last term in the right-hand side of (2.2) and $\langle\sigma_x^4\rangle$ will be estimated with the help of the superstability bound.

The following bound has been already employed by us in [3, 4] for a proof of (1.1)

$$\chi^+(q_x)\chi^-(q_y) \le e^{-\frac{g}{2}[q_x^l q_y^k + q_x^k q_y^l]}, \quad k + l = 2n_0. \tag{2.3}$$

For $l,\ k = 1$ it was proposed in [13]. An exposition of the two generalized GKS inequalities can be found in [14, 15].

In this paper we introduce the following new bounds for estimates of the summands in the right-hand side of (2.1)

$$\chi_{[-r,r]}(q_x)\chi_{[-r,r]}(q_y) \le e^{gr^{2n_0}}e^{-(g/2)[q_x^l q_y^k + q_x^k q_y^l]}, \tag{2.3′}$$

$$\chi_{[-r,r]}(q_x)\chi_{r,r'}(q_y) \le e^{(g/2)(r^l r'^k + r^k r'^l)}e^{-(g/2)[q_x^l q_y^k + q_x^k q_y^l]}, \quad r' > r, \tag{2.3″}$$

$$\left\langle \chi_{x,[-r,r]}\chi_{x_*,[-r',r']^c}\right\rangle_\Lambda \le \langle \chi_{x_*,[-r',r']^c}\rangle_\Lambda \le e^c \int\limits_{|q|\ge r'} e^{-\beta u_\varepsilon(q)}dq \le$$

$$\le e^{c-\beta r'^{2n}/4} \int\limits_{|q|\ge r'} e^{-\beta \tilde u(q)}dq, \qquad (2.4)$$

$$\langle \sigma_x^4\rangle_\Lambda \le r'^4 + e^c \int\limits_{|q|\ge r'} e^{-\beta u_\varepsilon(q)}q^4 dq \le r'^4 + \kappa_4(4\beta^{-1})^4 e^{c-\beta r'^{2n}/4}\int\limits_{|q|\ge r'} e^{-\beta\tilde u(q)}dq, \qquad (2.5)$$

where $\kappa_4 = \max\limits_{q\ge 0} q^4 e^{-q}$, $\tilde u(q) = u_\varepsilon(q) - \dfrac{1}{2}q^{2n}$. We applied the superstability bound for $\rho^\Lambda(q_y),\ y = x, x_*$ in (2.4) and (2.5) and the estimate $\int_{|q_x|\le r'} q_x^4\rho^\Lambda(q_x)dq_x \le$ $\le r'^4 \int \rho^\Lambda(q)dq = r'^4$ in (2.5). Thus all the averages containing characteristic functions in (2.1) will be estimated with the help of the average $\left\langle e^{-\frac{g}{2}[\sigma_x^l\sigma_y^k+\sigma_x^k\sigma_y^l]}\right\rangle_\Lambda$ for the nearest neighbors x, y and the superstability bound. Let's apply the third inequality in Proposition 1.2 for $U(q) = \tilde u(q)$, $e^0 = \tilde e$, $\kappa_0 = \tilde\kappa$. Then the last integral in (2.4) and (2.5) is less than

$$e^{c-\beta r'^{2n}/4}\int e^{-\beta\tilde u(q)}dq \le \exp\left\{c - \beta\left(\frac{1}{4}r'^{2n} - \left|\tilde u(\tilde e)\right|\right)\right\}\tilde\kappa \le$$

$$\le \exp\left\{c - \beta\left(\frac{1}{4}r'^{2n} - \bar\mu g^{n/(n-n_0)}\right)\right\}\tilde\kappa. \qquad (2.6)$$

Let $\kappa \le \dfrac{n}{n-n_0}$ in Theorem 1.2 and put $r' = (8\bar\mu)^{1/2n}g^{1/2(n-n_0)}$. Then the expression in the round brackets in the right-hand side of the last inequality is equal to $\beta\bar\mu g^{n/(n-n_0)}$ and the right-hand side of (2.4) tends to zero in the limit of infinite g $\left(\text{if } \bar\mu\beta > \bar c \text{ for } \kappa = \dfrac{n}{n-n_0}\right)$. Let $\kappa > \dfrac{n}{n-n_0}$ and put $r' = (8\bar c g^\kappa\beta^{-1})^{1/2n}$. Then $4^{-1}\beta r'^{2n} - c \ge$ $\ge \bar c g^\kappa - o(g^{-1})$ and the right-hand side of (2.4) together with the second term in the right-hand side of (2.5) tends to zero in the limit of infinite g once more. Let's put

$$r = g^{-1}r'^{-l(2n_0-l)^{-1}}.$$

Then the exponents in (2.3′), (2.3″), containing r, r', are bounded in g since $k =$ $= 2n_0 - l < l$, $l \ge 1$. Hence Theorem 1.2 is proved if the inequality (2.1) is valid since the average $\left\langle e^{-(g/2)[\sigma_x^l\sigma_y^k+\sigma_x^k\sigma_y^l]}\right\rangle_\Lambda$ for the nearest neighbors x, y exponentially tends to zero in the limit of infinite g [3, 4] and r' grows as g to some finite power. Now, to prove Theorem 1.2 we have to prove (2.1).

***Proof of* (2.1):**

$$1 = \chi_{[-\infty,-r]}(q) + \chi_{[r,\infty]}(q) + \chi_{[-r,r]}(q_\Lambda) = \chi_{[-r,r]^c}(q) + \chi_{[-r,r]}(q_\Lambda).$$

Let's insert this decomposition in q_x, q_y into the two point Gibbs average. We obtain the following bound:

$$\langle \sigma_x \sigma_y \rangle_\Lambda \geq$$
$$\geq r^2 \Big[-\big\langle \chi_{x,[-r,r]}\chi_{y,[-r,r]}\big\rangle_\Lambda + \big\langle \chi_{x,[r,\infty]}\chi_{y,[r,\infty]}\big\rangle_\Lambda + \big\langle \chi_{x,[-\infty,-r]}\chi_{y,[-\infty,-r]}\big\rangle_\Lambda \Big] -$$
$$-2\Big(\langle \chi_x^+ \chi_y^- \rangle_\Lambda^{1/2} + \langle \chi_x^- \chi_y^+ \rangle_\Lambda^{1/2} \Big) \langle \sigma_x^2 \sigma_y^2 \rangle_\Lambda^{1/2}. \qquad (2.7)$$

Here we applied the inequalities

$$\chi_{x,[r,\infty]}\chi_{y,[-\infty,-r]} \leq \chi_x^+ \chi_y^-,$$
$$\sigma_x \sigma_y \Big(\chi_{x,[-\infty,-r]} + \chi_{x,[r,\infty]} \Big) \chi_{y,[-r,r]} \geq$$
$$\geq -|\sigma_x \sigma_y| \big(\chi_{x,[-\infty,-r]}\chi_{y,[0,r]} + \chi_{x,[r,\infty]}\chi_{y,[-r,0]} \big),$$
$$\chi_{x,[-\infty,-r]}\chi_{y,[0,r]} \leq \chi_x^- \chi_y^+, \qquad \chi_{x,[r,\infty]}\chi_{y,[-r,0]} \leq \chi_x^+ \chi_y^-,$$

and the Schwartz inequality

$$\big\langle |\sigma_x \sigma_y| \chi_{x,[r,\infty]}\chi_{y,[-\infty,-r]} \big\rangle_\Lambda \leq \langle \sigma_x^2 \sigma_y^2 \rangle_\Lambda^{1/2} \big\langle \chi_{x,[r,\infty]}\chi_{y,[-\infty,-r]} \big\rangle_\Lambda^{1/2},$$
$$\big\langle |\sigma_x \sigma_y| \chi_{x,[r,\infty]}\chi_{y,[-r,0]} \big\rangle_\Lambda \leq \langle \sigma_x^2 \sigma_y^2 \rangle_\Lambda^{1/2} \big\langle \chi_{x,[r,\infty]}\chi_{y,[-r,0]} \big\rangle_\Lambda^{1/2}.$$

Further

$$\langle \chi_{x,[r,\infty]}\chi_{y,[r,\infty]} \rangle_\Lambda = \big\langle \chi_{x,[r,\infty]}(1 - \chi_{y,[-r,r]} - \chi_{y,[-\infty,-r]}) \big\rangle_\Lambda \geq$$
$$\geq \langle \chi_{x,[r,\infty]} \rangle_\Lambda - \langle \chi_{y,[-r,r]} \rangle_\Lambda - \langle \chi_{x,[r,\infty]}\chi_{y,[-\infty,-r]}) \rangle_\Lambda \geq$$
$$\geq \langle \chi_{x,[r,\infty]} \rangle_\Lambda - \langle \chi_{y,[-r,r]} \rangle_\Lambda - \langle \chi_x^+ \chi_y^- \rangle_\Lambda. \qquad (2.8)$$

Since our systems are invariant under the transformation of changing of all oscillator variables signs we have

$$\langle \chi_{x,[r,\infty]} \rangle_\Lambda = \langle \chi_{x,[-\infty,-r]} \rangle_\Lambda, \qquad \langle \chi_{x,[-\infty,-r]}\chi_{y,[-\infty,-r]} \rangle_\Lambda = \langle \chi_{x,[\infty,r]}\chi_{y,[\infty,r]} \rangle_\Lambda.$$

As a result the first equality and the equality

$$\langle \chi_{x,[r,\infty]} \rangle_\Lambda + \langle \chi_{x,[-\infty,-r]} \rangle_\Lambda + \langle \chi_{x,[-r,r]} \rangle_\Lambda = 1$$

give two equalities

$$\langle \chi_{x,[r,\infty]} \rangle_\Lambda = \frac{1}{2} - \frac{1}{2} \langle \chi_{x,[-r,r]} \rangle_\Lambda, \qquad \langle \chi_{x,[-r,-\infty]} \rangle_\Lambda = \frac{1}{2} - \frac{1}{2} \langle \chi_{x,[-r,r]} \rangle_\Lambda.$$

Substituting the first equality into (2.8) one obtains

$$\langle \chi_{x,[r,\infty]}\chi_{y,[r,\infty]} \rangle_\Lambda \geq \frac{1}{2} - \frac{1}{2} \langle \chi_{x,[-r,r]} \rangle_\Lambda - \langle \chi_{y,[-r,r]} \rangle_\Lambda - \langle \chi_x^+ \chi_y^- \rangle_\Lambda.$$

The same inequality holds for the second term in the first square bracket in (2.7) with the permuted x, y . Hence (2.7), the last inequality, the inequality

$$\chi_{x,[-r,r]}\chi_{y,[-r,r]} \leq \frac{1}{2} \big(\chi_{x,[-r,r]} + \chi_{y,[-r,r]} \big)$$

and the Schwartz inequality complete the proof.

3. Estimates for potential energy. Let us derive the following new representation for the potential energy from which (1.6) is easily derived

$$U(q_\Lambda) = \sum_{x\in\Lambda} u(q_x) + g \sum_{\langle x,y\rangle\in\Lambda} (q_x - q_y)^2 Q(q_x, q_y) + U^-(q_\Lambda) + U_{\partial\Lambda}(q_\Lambda), \tag{3.1}$$

where $U^-(q,\ldots,q) = 0$, $U^- \geq 0$, $Q \geq 0$, $U_{\partial\Lambda}(q_\Lambda) \geq 0$, $U_{\partial\Lambda}(q_\Lambda)$ is a boundary term, generated by a boundary external field, and u is determined by (1.5). It will be derived with the help of the following proposition.

Proposition 3.1. *Let $|A|$ be an arbitrary positive integer and $\langle n_{(|A|)}\rangle = \sum_{j=1}^{|A|} n_j = 2l$. Then there exists a positive polynomial $Q_{x,y}$ such that the following equality holds*

$$Sq_{[A]}^{n_{|A|}} = \frac{1}{|A|}\sum_{x\in A} q_x^{2l} - \sum_{x\neq y\in A} (q_x - q_y)^2 Q_{x,y}(q_A). \tag{3.2}$$

Proof. We will use induction. Let $A = (1,\ldots,k)$ and $P_{n_{(k)}}(q_{(k)}) = Sq_{[k]}^{n_{(k)}} = S\prod_{j=1}^{n} q_j^{n_j}$, $n_k < n_j, n_{k-1} = n' - n_k$. Let's introduce the function

$$P_{n_{(k-2)};r} = P_{n_{(k-2)},n'-r,r},$$

where $r \in \mathbb{R}^+$. Then the following equalities are true

$$P_{n_{(k-2)};n_k} = P_{n_{(k)}}, \qquad n' + \sum_{j=1}^{k-2} n_j = 2l, \tag{3.3}$$

$$P_{n_{(k-2)};0}(q_{(k)}) = P_{n_{(k-2)};n'}(q_{(k)}) = \frac{1}{k}\sum_{j=1}^{k} P_{n_{(k-2)};n'}(q_{(k\backslash j)}), \tag{3.4}$$

where $q_{(k\backslash j)}$ is the sequence $(1,\ldots,k)$ without the positive integer $j \leq k$. Let

$$\frac{1}{k-1}\sum_{j=1}^{k-1} q_j^{2l} - P_{n_{(k-1)}}(q_{(k-1)}) \geq 0.$$

The following equality is easily derived

$$\frac{1}{k}\sum_{j=1}^{k} q_j^{2l} - \frac{1}{k}\sum_{j=1}^{k} P_{n_{(k-2)};n'}(q_{(n\backslash j)}) =$$

$$= \frac{1}{k}\sum_{j=1}^{k}\left[\frac{1}{k-1}\sum_{l=1,l\neq j}^{k} q_l^{2l} - P_{n_{(k-2)};n'}(q_{(k\backslash j)})\right] \geq 0.$$

The same inequality holds with 0 substituted instead of n'. This and (3.4) mean that the function $\frac{1}{k}\sum_{j=1}^{k} q_j^{2l} - P_{n_{(k-2)};r}(q_k))$ is positive at the end points of the interval $[0, n']$. Its second derivative is negative. This and the inequality $n_k \leq n'$ imply that

$$\frac{1}{k}\sum_{j=1}^{k} q_j^{2l} - P_{n_{(k)}}(q_{(k)}) \geq 0.$$

The first derivative in q_j of the left-hand side of this inequality is equal to zero for coinciding variables. This proves the proposition.

***Proof of* (3.1)**. The expression for U' is rewritten as

$$U'(q_\Lambda) = -\sum_{l \le n-1} \sum_{A \subseteq \Lambda} \phi_{A;l}(q_A), \qquad \phi_{A;l}(q_A) = \sum_{\langle n_{(|A|)} \rangle = 2l} J_{A;n_{(|A|)}} S q_{[A]}^{n_{|A|}},$$

Let

$$\phi^-_{n_{(|A|)}}(q_A) = \sum_{x \ne y \in A} (q_x - q_y)^2 Q_{x,y}(q_A).$$

Substituting (3.2) into (1.2) we obtain (3.1) with

$$U^-(q_\Lambda) = \sum_{A \subseteq \Lambda} \phi^-_A(q_A), \qquad \phi^-_A(q_A) = \sum_{\langle n_{(|A|)} \rangle < 2n} J_{A;n_{(|A|)}} \phi^-_{n_{(|A|)}}(q_A),$$

$$U_{\partial\Lambda}(q_\Lambda) = \sum_{x \in \Lambda} u_{l;\partial\Lambda}(q_x) + g \sum_{x \in \partial\Lambda} q_x^{2n_0}, \qquad u_{l;\partial\Lambda}(q_x) = \sum_{l=1}^{n-1} J_{l;\partial\Lambda} q_x^{2l},$$

$$J_{l;\partial\Lambda} = \sum_{x \in A \subset \Lambda^c} \frac{1}{|A|} \sum_{\langle n_{(|A|)} \rangle = 2l} J_{A;n_{(|A|)}},$$

where $\Lambda^c = \mathbb{X}^d \backslash \Lambda$. Here we took into account that every boundary point has $2d-1$ nearest neighbors.

***Proof of* (1.7)**. Let $\langle n_{(|A|)} \rangle = 2l$ then following bound is valid

$$\left| S q_{[A]}^{n_{|A|}} \right| \le \frac{1}{|A|!} \left(\sum_{x \in A} |q_x| \right)^{2l} \le \frac{|A|^{2l-1}}{|A|!} \sum_{x \in A} |q_x|^{2l}. \tag{3.5}$$

From the definition of W we obtain

$$W(q_X; q_Y) = \sum_{A_1 \in X, A_2 \in Y, A_j \ne \emptyset} \phi_{A_1 \cup A_2}(q_{A_1}, q_{A_2}) = \sum_{l \le n-1} W_l(q_X; q_Y),$$

where

$$W_l(q_X; q_Y) = \sum_{A_1 \in X, A_2 \in Y, A_j \ne \emptyset} \phi_{A_1 \cup A_2; l}(q_{A_1}, q_{A_2}). \tag{3.6}$$

Inequality (3.5) yields

$$\begin{gathered}
\phi_{A;l}(q_A) \le \frac{|A|^{2l-1}}{|A|!} \sum_{x \in A} q_x^{2l} \sum_{\langle n_{(|A|)} \rangle = 2l} J_{A;n_{(|A|)}}, \\
\left| W_l(q_X; q_Y) \right| \le \frac{1}{2} \sum_{x \in X, y \in Y} \Psi'\big(|x-y|\big)\big(q_x^{2l} + q_y^{2l}\big), \\
\Psi'\big(|x-y|\big) = \sum_{l \le n-1} \Psi_l(|x-y|), \\
\Psi_l\big(|x-y|\big) = 4 \sum_{x,y \in A} \frac{|A|^{2l-1}}{|A|!} \sum_{\langle n_{(|A|)} \rangle = 2l} J_{A;n_{(|A|)}} < \infty,
\end{gathered} \tag{3.7}$$

where the first summation is performed over all subsets of $\mathbb{Z}^d$ which contain x, y. From (3.5) one derives

$$q_x^k q_y^l + q_x^l q_y^k \leq 2^{2(n_0-1)}(q_x^{2n_0} + q_y^{2n_0}).$$

Hence, (3.6), (3.7) prove (1.7) with $\Psi(|x-y|) = 2^{2n_0} g\delta_{1,|x-y|} + \Psi'(|x-y|)$, where $\delta_{1,k}$ is the Kronecker symbol and $|x|$ is the Euclidean norm of the lattice site x. $\|\Psi\|_1 < \infty$ due to the condition $J_l^- < \infty$ and the fact that summations in A are always performed over sets whose numbers of sites are less than n.

***Proof of Proposition* 1.1**. The basic constant c is a function of an arbitrary positive number r and a number $\varepsilon < \frac{1}{3}$. That is, $c = c(\varepsilon, I_r^{-1}, I(\varepsilon))$ (see [8, 9]) and the integrals $I(\varepsilon), I_r$ are determined as follows:

$$I_r = e^{-\frac{1}{2}\beta\|\Psi\|_1 \bar{v}_r} I_0, \qquad I_0 = \int\limits_{|q|\leq r} e^{-\beta\bar{u}(q)}dq, \qquad I(\varepsilon) = \int \exp\big\{-\beta u_\varepsilon(q)\big\}dq,$$

where $\bar{u}(q) = u(q) + \|\Psi\|_1 v(q)$, $\bar{v}_r = \sup\limits_{|q|\leq r} v(q)$. Moreover,

$$c(\varepsilon, z', z) = c^0 + \ln\big(1 + \xi z' + f(\varepsilon, zz')\big),$$
$$f(\varepsilon, z) = \sum_{j\geq 0} e^{-\varepsilon l_j(1+2l_j)^d}(2z)^{(1+2l_j)^d}, \quad z \geq 1, \tag{3.8}$$

where positive constants c^0, ξ may depend on ε, $l_j = (1+2\alpha)^j$, α is proportional to ε to some positive power.

From the bounds $(1+2l_j)^d \geq 1 + (2l_j)^d$, $(1+2l_j)^d \leq 2^d(1+(2l_j)^d)$ we obtain

$$f\Big(\varepsilon, \frac{z}{2}\Big) \leq z^{2^d} \sum_{j\geq 0} e^{-\varepsilon(2l_j)^{d+1}} z^{2^d(2l_j)^d} \leq$$

$$\leq z^{2^d} \sup_{x\geq 0} e^{-(1/2)\varepsilon x^{d+1} + 2^d(\ln z)x^d} \sum_{j\geq 0} e^{-(1/2)\varepsilon(2l_j)^{d+1}} \leq$$

$$\leq z^{2^d} \exp\left\{\frac{\varepsilon}{2}\left(\frac{d+1}{d\varepsilon}2^{d+1}\ln z\right)^{d+1}\right\} \sum_{j\geq 0} e^{-(1/2)\varepsilon(2l_j)^{d+1}}. \tag{3.9}$$

Here we found the maximum of the function $-\frac{1}{2}\varepsilon x^{d+1} + 2^d(\ln z)x^d$ equating its derivative in x to zero. Further, the following simple bound is true:

$$I_r \geq \exp\left\{-\beta\left[\frac{3}{2}\|\Psi\|_1\bar{v}_r - g2^{-2n_0}r^{2n_0}\right]\right\} \int\limits_{2^{-1}r\leq|q|\leq r} e^{-\beta(u_0(q)+u^1(q))}dq,$$

where $-u^1$ coincides with the third summand in the expression for u in (1.5). Since $\|\Psi\| > g$, $\bar{v}_r > r^{2n}$ the coefficient in front of the last integral decreases exponentially in g and the integral does not depend on g. That is, taking into account the second and third bounds from Proposition 1.2 with $U(q) = u_\varepsilon(q)$ for the estimate of $I(\varepsilon)$ one sees that there exists a positive numbers $\bar{I}$, $\bar{\mu}$ independent of g such that

$$I_r^{-1} I(\varepsilon) \leq \bar{I} \exp\big\{g^{n/(n-n_0)}\bar{\mu}\big\}.$$

This bound, (3.8) and (3.9) yield that there exists a positive number $\bar{c}$ independent of g such that

$$c \leq \bar{c} g^{n(d+1)/(n-n_0)} + o(g^{-1}).$$

The proposition is proved.

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