UDC 517.9

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ASYMPTOTIC BEHAVIOR AND PERIODIC NATURE OF TWO DIFFERENCE EQUATIONS

АСИМПТОТИЧНА ПОВЕДІНКА ТА ПЕРІОДИЧНА ПРИРОДА ДВОХ РІЗНИЦЕВИХ РІВНЯНЬ

We discuss the global asymptotic stability of the solutions of the difference equations

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_n x_{n-1} x_{n-2}}, \qquad n = 0, 1, \ldots,$$

where the initial conditions x_{-2}, x_{-1}, x_0 are real numbers.

Розглянуто глобальну асимптотичну стійкість розв'язків різницевих рівнянь

$$x_{n+1} = \frac{x_{n-2}}{\pm 1 + x_n x_{n-1} x_{n-2}}, \qquad n = 0, 1, \ldots,$$

де початкові умови x_{-2}, x_{-1}, x_0 є дійсними числами.

1. Introduction and preliminaries. Difference equations, although their forms look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions. One can refer to [1, 2]. The study of nonlinear rational difference equations of higher order is of paramount importance, since we still know so little about such equations. Cinar [3, 4] examined the global asymptotic stability of all positive solutions of the rational difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}, \qquad n = 0, 1, \ldots .$$

He also discussed the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{-1 + x_n x_{n-1}}, \qquad n = 0, 1, \ldots .$$

In this paper, we discuss the global stability and periodic character of all solutions of the difference equations

$$x_{n+1} = \frac{x_{n-2}}{1 + x_n x_{n-1} x_{n-2}}, \qquad n = 0, 1, \ldots, \tag{1}$$

and

$$x_{n+1} = \frac{x_{n-2}}{-1 + x_n x_{n-1} x_{n-2}}, \qquad n = 0, 1, \ldots . \tag{2}$$

2. The difference equation $x_{n+1} = \dfrac{x_{n-2}}{1 + x_n x_{n-1} x_{n-2}}$. In this section we study the difference equation

$$x_{n+1} = \frac{x_{n-2}}{1 + x_n x_{n-1} x_{n-2}}, \qquad n = 0, 1, \ldots .$$

Theorem 1. *Let* x_{-2}, x_{-1} *and* x_0 *are positive real numbers. Then all solutions of equation* (1) *are*

$$x_n = \begin{cases} x_{-2}\prod_{j=0}^{(n-1)/3}\dfrac{1+3j\alpha}{1+(3j+1)\alpha}, & n = 1, 4, 7,\ldots, \\[2ex] x_{-1}\prod_{j=0}^{(n-2)/3}\dfrac{1+(3j+1)\alpha}{1+(3j+2)\alpha}, & n = 2, 5, 8,\ldots, \\[2ex] x_0\prod_{j=1}^{n/3}\dfrac{1+(3j-1)\alpha}{1+3j\alpha}, & n = 3, 6, 9,\ldots, \end{cases} \tag{3}$$

where $\alpha = x_{-2}x_{-1}x_0$.

Proof. Let $\alpha = x_{-2}x_{-1}x_0$. Then

$$x_1 = \frac{x_{-2}}{1+\alpha}, \qquad x_2 = x_{-1}\frac{1+\alpha}{1+2\alpha} \quad \text{and} \quad x_3 = x_0\frac{1+2\alpha}{1+3\alpha}.$$

Now assume that $m \geq 1$. Then we have

$$x_{3m-2} = x_{-2}\prod_{j=0}^{m-1}\frac{1+3j\alpha}{1+(3j+1)\alpha},$$

$$x_{3m-1} = x_{-1}\prod_{j=0}^{m-1}\frac{1+(3j+1)\alpha}{1+(3j+2)\alpha},$$

$$x_{3m} = x_0\prod_{j=0}^{m-1}\frac{1+(3j+2)\alpha}{1+(3j+3)\alpha}.$$

Now

$$\frac{x_{3m-2}}{1+x_{3m}x_{3m-1}x_{3m-2}} =$$

$$= \frac{x_{-2}\prod_{j=0}^{m-1}\dfrac{1+3j\alpha}{1+(3j+1)\alpha}}{1 + x_{-2}\prod_{j=0}^{m-1}\dfrac{1+3j\alpha}{1+(3j+1)\alpha}\, x_{-1}\prod_{j=0}^{m-1}\dfrac{1+(3j+1)\alpha}{1+(3j+2)\alpha}\, x_0\prod_{j=0}^{m-1}\dfrac{1+(3j+2)\alpha}{1+(3j+3)\alpha}} =$$

$$= \frac{x_{-2}\prod_{j=0}^{m-1}\dfrac{1+3j\alpha}{1+(3j+1)\alpha}}{1+\alpha\prod_{j=0}^{m-1}\dfrac{1+3j\alpha}{1+(3j+3)\alpha}} = \frac{x_{-2}\prod_{j=0}^{m-1}\dfrac{1+3j\alpha}{1+(3j+1)\alpha}}{1+\alpha\dfrac{1}{1+3m\alpha}} =$$

$$= \frac{1+3m\alpha}{1+(3m+1)\alpha}\, x_{-2}\prod_{j=0}^{m-1}\frac{1+3j\alpha}{1+(3j+1)\alpha} = x_{-2}\prod_{j=0}^{m}\frac{1+3j\alpha}{1+(3j+1)\alpha} = x_{3m+1}.$$

This completes the proof.

Remark. If $\alpha = x_{-2}x_{-1}x_0 \neq -1/n$, for all $n \geq 1$, then formula (3) also represents solutions of equation (1) when x_{-2}, x_{-1} and x_0 are real numbers.

Theorem 2. *Equation* (1) *has a period*-3 *solution* $\{\ldots, \varphi_1, \varphi_2, \varphi_3, \varphi_1, \varphi_2, \varphi_3, \ldots\}$ *with* $\varphi_1\varphi_2\varphi_3 = \alpha = 0$.

Proof. Let $\alpha = 0$. Using formula (3) it is sufficient to see that

$$x_n = \begin{cases} x_{-2}, & n = 1, 4, 7, \ldots, \\ x_{-1}, & n = 2, 5, 8, \ldots, \\ x_0, & n = 3, 6, 9, \ldots, \end{cases}$$

therefore, for $n = 0, 1, \ldots$ we have

$$x_{3m} = x_0, \qquad x_{3m+1} = x_{-1} \quad \text{and} \quad x_{3m+2} = x_{-2}.$$

Now suppose that $x_{-2} = \varphi_1$, $x_{-1} = \varphi_2$, $x_0 = \varphi_3$. It follows that

$$\{\ldots, \varphi_1, \varphi_2, \varphi_3, \varphi_1, \varphi_2, \varphi_3, \ldots\}$$

is a periodic solution with $\varphi_1\varphi_2\varphi_3 = \alpha = 0$.

This completes the proof.

Theorem 3. *The unique equilibrium point* $\bar{x} = 0$ *of equation* (1) *is nonhyperbolic point.*

Theorem 4. *Assume that* $\alpha \neq 0$ *and* $\alpha \neq -1/n$. *Then every solution of equation* (1) *converges to zero.*

Proof. Let $\{x_n\}$ be arbitrary solution of equation (1). We consider only the case $\alpha < 0$, the case $\alpha > 0$ is similar and will be omitted. From formula (3) we have

$$x_{3m+1} = x_{-2} \prod_{j=0}^{m} \frac{1+3j\alpha}{1+(3j+1)\alpha} = x_{-2} \exp \prod_{j=0}^{m} \ln \frac{1+3j\alpha}{1+(3j+1)\alpha} =$$

$$= x_{-2} \exp\left(-\prod_{j=0}^{m} \ln \frac{1+(3j+1)\alpha}{1+3j\alpha} \right) = x_{-2} \exp\left(-\sum_{j=0}^{m} \ln\left(1 + \frac{\alpha}{1+3j\alpha} \right) \right) =$$

$$= x_{-2}\, c(n_0) \exp - \alpha \left(\sum_{j=n_0}^{m} \left(\frac{1}{1+3j\alpha} + O\left(\frac{1}{j^2} \right) \right) \right) \to 0, \qquad n \to \infty,$$

since $\sum_{j=n_0}^{m} \dfrac{1}{1+3j\alpha} \to -\infty$ as $n \to \infty$ and $\sum_{j=n_0}^{m} O\left(\dfrac{1}{j^2} \right)$ is convergent.

Here $c(n_0)$ is a positive constant depending on $n_0 \in \mathbb{N}$.

Similarly $x_{3m+2} \to 0$ as $n \to \infty$ and $x_{3m+3} \to 0$ as $n \to \infty$.

This completes the proof.

3. The difference equation $x_{n+1} = \dfrac{x_{n-2}}{-1 + x_n x_{n-1} x_{n-2}}$.

In this section we introduce the following results.

Theorem 5. *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (2). *Assume that* $\alpha = x_{-2}x_{-1}x_0 \neq 1$. *Then we have*

$$x_{3m+i} = \begin{cases} x_{-2}\beta_m, & i = 1, \\ \dfrac{x_{-1}}{\beta_m}, & i = 2, \\ x_0\beta_m, & i = 3, \end{cases} \tag{4}$$

where

$$\beta_m = \begin{cases} 1, & m \quad odd, \\ \dfrac{1}{-1+\alpha}, & m \quad even. \end{cases}$$

Proof. For $m = 0$ the following results hold

$$x_1 = \frac{x_{-2}}{-1+\alpha}, \qquad x_2 = x_{-1}(-1+\alpha) \quad \text{and} \quad x_3 = \frac{x_0}{-1+\alpha}.$$

Assume that $m > 0$. Then if m is even, we have

$$\frac{x_{3m-2}}{-1+x_{3m}x_{3m-1}x_{3m-2}} = \frac{x_{-2}\beta_{m-1}}{-1+x_{-2}\beta_{m-1}\dfrac{x_{-1}}{\beta_{m-1}}x_0\beta_{m-1}} =$$

$$= \frac{x_{-2}\beta_{m-1}}{-1+\alpha\beta_{m-1}} = \frac{x_{-2}}{-1+\alpha} = x_{-2}\beta_m = x_{3m+1}.$$

If m is odd, then

$$\frac{x_{3m-2}}{-1+x_{3m}x_{3m-1}x_{3m-2}} = \frac{x_{-2}\beta_{m-1}}{-1+x_{-2}\beta_{m-1}\dfrac{x_{-1}}{\beta_{m-1}}x_0\beta_{m-1}} =$$

$$= \frac{x_{-2}\beta_{m-1}}{-1+\alpha\beta_{m-1}} = \frac{x_{-2}(-1+\alpha)^{-1}}{-1+\alpha(-1+\alpha)^{-1}} = x_{-2} = x_{-2}\beta_m = x_{3m+1}.$$

This completes the proof.

Theorem 6. *The equilibrium points* $\bar{x} = 0$ *and* $\bar{x} = \sqrt[3]{2}$ *of equation* (2) *are nonhyperbolic points.*

Theorem 7. *Every solution of equation* (2) *is periodic with period* 6.

Proof. Let $\{x_n\}_{n=-2}^{\infty}$ be a solution of equation (2) then we have

$$x_{(3m+i)+6} = \begin{cases} x_{-2}\beta_{m+2}, & i = 1, \\ \dfrac{x_{-1}}{\beta_{m+2}}, & i = 2, \\ x_0\beta_{m+2}, & i = 3, \end{cases} = \begin{cases} x_{-2}\beta_m, & i = 1, \\ \dfrac{x_{-1}}{\beta_m}, & i = 2, \\ x_0\beta_m, & i = 3, \end{cases} = x_{3m+i}.$$

This completes the proof.

***Corollary* 1.** *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (2) *with* $\alpha = 2$. *Then* $\{x_n\}_{n=-2}^{\infty}$ *is periodic with period* 3.

***Corollary* 2.** *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (2) *where* x_{-2}, x_{-1} *and* x_0 *are positive real numbers such that* $\alpha = x_{-2}x_{-1}x_0 > 1$. *Then the solution* $\{x_n\}_{n=-2}^{\infty}$ *is positive.*

***Corollary* 3.** *Let* $\{x_n\}_{n=-2}^{\infty}$ *be a solution of equation* (2) *where* x_{-2}, x_{-1} *and* x_0 *are negative real numbers. Then the solution* $\{x_n\}_{n=-2}^{\infty}$ *oscillates with semicycles of length* 3.

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