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A MODULAR TRANSFORMATION FOR A GENERALIZED THETA FUNCTION WITH MULTIPLE PARAMETERS

МОДУЛЯРНЕ ПЕРЕТВОРЕННЯ ДЛЯ УЗАГАЛЬНЕНОЇ ТЕТА-ФУНКЦІЇ З БАГАТЬМА ПАРАМЕТРАМИ

We obtain a modular transformation for the theta function

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2+\lambda m+\mu n+\nu}\zeta^{Am+Bn}z^{Cm+Dn}.$$

We are thus able to unify and extend several modular transformations in literature.

Одержано модулярне перетворення для тета-функції

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2+\lambda m+\mu n+\nu}\zeta^{Am+Bn}z^{Cm+Dn},$$

що дає можливість уніфікувати та узагальнити декілька модулярних перетворень, відомих з літератури.

1. Definitions. We define, for $|q|<1$ and $\zeta \neq 0 \neq z$, the function, which we call a theta function with multiple parameters,

$$a(q,\zeta,z;a,b,c;\lambda,\mu,\nu;A,B;C,D) := \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{Q(m,n;a,b,c;\lambda,\mu,\nu)}\zeta^{Am+Bn}z^{Cm+Dn}, \quad (1.1)$$

where

$$Q(m,n;a,b,c;\lambda,\mu,\nu) = \tilde{Q}(m,n;a,b,c)+\lambda m+\mu n+\nu$$

with

$$\tilde{Q}(m,n;a,b,c) = am^2+bmn+cn^2.$$

Our main concern here, however, will be (1.1) with $a=b$, that is, with the function

$$a(q,\zeta,z;a,a,c;\lambda,\mu,\nu;A,B;C,D) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2+\lambda m+\mu n+\nu}\zeta^{Am+Bn}z^{Cm+Dn}. \quad (1.2)$$

In what follows, we freely use many relations that simply follow from the above notations, for instance,

$$\tilde{Q}_m = \frac{\partial\tilde{Q}}{\partial m} = 2am+bn,$$

$$\tilde{Q}_n = \frac{\partial\tilde{Q}}{\partial n} = bm+2cn,$$

$$\tilde{Q}_m(B,-A)\varphi+\tilde{Q}_m(D,-C)\theta = \tilde{Q}_m(B\varphi+D\theta,-A\varphi-C\theta),$$

where

$$\tilde{Q}_m(m,n) := \tilde{Q}_m(m,n;a,b,c).$$

A special case of (1.2), namely,

$$a(q,\zeta,z;1,1,1;0,0,0;1,1;1,-1) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}\zeta^{m+n}z^{n-m} \tag{1.3}$$

denoted briefly by $a(q,\zeta,z)$ was introduced by S. Bhargava in [1] and was shown to have properties which unified and generalized several known properties of the Hirschhorn – Garvan – Borwein cubic analogues [2] of the classical theta function. S. Bhargava and S. N. Fathima obtained in [3] a modular transformation $a(q,\xi,z)$ of (1.3) which unified several modular transformations established by S. Cooper [4] for the Hirschhorn – Garvan – Borwein cubic theta functions. Other special cases of (1.2) with $\lambda = \mu = \nu = 0$ have been studied by various authors including S. Bhargava and N. Anitha [5], who have recently obtained a triple product for (1.3) and C. Adiga, M. S. Mahadeva Naika and J. H. Han [6]. Thus, (1.2) unifies and extends several works in literature [2 – 4, 6 – 8].

The objective of present paper is to obtain a modular transformation for (1.2). It would be interesting to try and extend our results to (1.1) which we leave as an open question. It is possible to first treat (1.1) with $\lambda = \mu = \nu = 0$, $A = B = 1 = C$, $D = -1$ and then effect suitable transformations on ζ and z to obtain our main result (4.3). However, we have preferred to present our main result and all the related lemmas directly and in detail in order to bring out the motivation and lucidity of the inter play between the various parameters all through, than would be the case in the abbreviated alternative approach.

A basic lemma is established in Section 2, which shows that one can express (1.1), under the condition $a \mid b$, in terms of the Jacobian theta function

$$f(qz,qz^{-1}) = \sum_{-\infty}^{\infty} q^{n^2}z^n,$$

where, in Ramanujan's notation [7],

$$f(a,b) := \sum_{-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.4}$$

In Section 3, we obtain modular transformations for the various Jacobian theta functions which constitute $a(q,\zeta,z;a,b,c;\lambda,\mu,\nu;A,B;C,D)$ (when $a \mid b$). Our main result is established in Section 4. The remaining sections are devoted to special cases.

2. Relation between $\boldsymbol{a(q,\zeta,z;a,b,c;\lambda,\mu,\nu;A,B;C,D)}$ and the Jacobian theta function when $\boldsymbol{a \mid b}$. The following lemma is basic to the remainder of this work.

Lemma 2.1. *Given that* $a \mid b$, *we have*

$$a(q,\zeta,z;a,b,c;\lambda,\mu,\nu;A,B;C,D) =$$

$$= q^{\nu} f(q^{a+\lambda}\zeta^{A}z^{C}, q^{a-\lambda}\zeta^{-A}z^{-C}) f(q^{a'+\lambda'}\zeta^{A'}z^{C'}, q^{a'-\lambda'}\zeta^{-A'}z^{-C'}) +$$

$$+ q^{\nu+\mu+C}\zeta^{B}z^{D} f(q^{a+b+\lambda}\zeta^{A}z^{C}, q^{a-b-\lambda}\zeta^{-A}z^{-C}) f(q^{2a'+\lambda'}\zeta^{A'}z^{C'}, q^{-\lambda'}\zeta^{-A'}z^{-C'}), \tag{2.1}$$

where

$$a' = \Delta a^{-1}, \qquad \Delta = 4ac - b^2, \qquad \lambda' = \tilde{Q}_m(\mu,-\lambda)a^{-1},$$

$$A' = \tilde{Q}_m(B,-A)a^{-1}, \qquad C' = \tilde{Q}_m(D,-C)a^{-1}.$$

Proof. We define an auxiliary function P, following the method in [8], for instance,

$$P(q,\zeta,\hat{z};t) := \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{am^2+bmn+cn^2+\lambda m+\mu n+\nu}\zeta^{Am+Bn}\hat{z}^n t^m. \tag{2.2}$$

We immediately have that the function a of (1.1) satisfies

$$a(q,\zeta,z;a,b,c;\lambda,\mu,\nu;A,B;C,D) = P(q,\zeta,z^D,z^C). \tag{2.3}$$

Collecting coefficients of $\hat{z}^{2n}$ in (2.2) and setting $\hat{m} = m + \frac{b}{a}n$, which is valid since $a \,|\, b$ by assumption, we have, on slight manipulations,

$$P_e[\hat{z}^{2n}] = q^{\nu+a'n^2+\lambda' n}\zeta^{A'n}t^{-bn/a}f(q^{a+\lambda}\zeta^A t, q^{a-\lambda}\zeta^{-A}t^{-1}),$$

with f as in (1.4). This immediately gives, on putting back $t = z^C$, $\hat{z} = z^D$,

$$\sum_{-\infty}^{\infty} P_{\text{even}}[\hat{z}^{2n}]\hat{z}^{2n} = q^{\nu} f(q^{a+\lambda}\zeta^A z^C, q^{a-\lambda}\zeta^{-A}z^{-C}) f(q^{a'+\lambda'}\zeta^{A'}z^{C'}, q^{a'-\lambda'}\zeta^{-A'}z^{-C'}). \tag{2.4}$$

This establishes a part of (2.1). For the remaining part, we proceed similarly, by collecting coefficients of $\hat{z}^{2n+1}$, and we obtain,

$$\sum_{-\infty}^{\infty} P_{\text{odd}}[\hat{z}^{2n+1}]\hat{z}^{2n+1} = q^{\nu+\mu+C}\zeta^B z^D f(q^{a+b+\lambda}\zeta^A z^C, q^{a-b-\lambda}\zeta^{-A}z^{-C}) \times$$

$$\times f(q^{2a'+\lambda'}\zeta^{A'}z^{C'}, q^{-\lambda'}\zeta^{-A'}z^{-C'}). \tag{2.5}$$

We omit the details. Adding (2.4) and (2.5), we get the required result (2.1).

3. Some auxiliary modular transformations. In this section, we obtain modular transformations for the Jacobian theta functions occurring on the right-hand side of (2.1). For this, we need the classical theta function transformation [7] (Entry 20)

$$\sqrt{\alpha} f(e^{-\alpha^2+n\alpha}, e^{-\alpha^2-n\alpha}) = \left\{\exp\left(\frac{n^2}{4}\right)\right\}\sqrt{\beta} f(e^{-\beta^2+in\beta}, e^{-\beta^2-in\beta}) \tag{3.1}$$

provided that $\alpha\beta = \pi$ and $\operatorname{Re}(\alpha^2) > 0$. In fact, the following two special cases of (3.1) are repeatedly used:

$$f(e^{-\pi t+i\theta}, e^{-\pi t-i\theta}) = \frac{1}{\sqrt{t}}\left\{\exp\left(-\frac{\theta^2}{4\pi t}\right)\right\} f\left(e^{-\frac{\pi+\theta}{t}}, e^{-\frac{\pi-\theta}{t}}\right) \tag{3.1$'$}$$

and

$$f(e^{-\pi t+i\theta}, e^{-i\theta}) = \sqrt{\frac{2}{t}}\left\{\exp\left(\frac{\pi t}{8} - \frac{i\theta}{2} - \frac{\theta^2}{2\pi t}\right)\right\} f\left(-e^{-\frac{2\pi+2\theta}{t}}, -e^{-\frac{2\pi-2\theta}{t}}\right). \tag{3.1$''$}$$

Indeed, putting $\alpha = \sqrt{\frac{\pi}{t}}$, $\beta = \sqrt{\pi t}$, and $n = \frac{\theta}{\sqrt{\pi t}}$ in (3.1) yields (3.1)$'$ and putting $\alpha = \sqrt{\frac{\pi t}{2}}$, $\beta = \sqrt{\frac{2\pi}{t}}$ and $n = -\sqrt{\frac{\pi t}{2}} + i\sqrt{\frac{2}{\pi t}}\theta$ in (3.1) gives (3.1)$''$.

Lemma 3.1. *With* $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, *and* $z = e^{i\theta}$, *we have the modular transformations*

$$f(q^{a+\lambda}\zeta^A z^C, q^{a-\lambda}\zeta^{-A}z^{-C})q^{\frac{\lambda^2}{4a}}\exp\left(\frac{i\lambda(A\varphi + C\theta)}{2a}\right) =$$

$$= \frac{1}{\sqrt{2at}}\exp\left(-\frac{(A\varphi + C\theta)^2}{8\pi at}\right) f\left(e^{-\frac{\pi}{2at} - \frac{A\varphi + C\theta}{2at} - \frac{i\pi\lambda}{a}}, e^{-\frac{\pi}{2at} + \frac{A\varphi + C\theta}{2at} + \frac{i\pi\lambda}{a}}\right), \quad (3.2)$$

$$f(q^{a'+\lambda'}\zeta^{A'} z^{C'}, q^{a'-\lambda'}\zeta^{-A'}z^{-C'})q^{\frac{\lambda'^2}{4a'}}\exp\left(\frac{i\lambda'(A'\varphi + C'\theta)}{2a'}\right) =$$

$$= \frac{1}{\sqrt{2a't}}\exp\left(-\frac{(A'\varphi + C'\theta)^2}{8\pi a't}\right) f\left(e^{-\frac{\pi}{2a't} - \frac{A'\varphi + C'\theta}{2a't} - \frac{i\pi\lambda'}{a'}}, e^{-\frac{\pi}{2a't} + \frac{A'\varphi + C'\theta}{2a't} + \frac{i\pi\lambda'}{a'}}\right) \quad (3.3)$$

in the notations used in (2.1),

$$f(q^{a+b+\lambda}\zeta^A z^C, q^{a-b-\lambda}\zeta^{-A}z^{-C})q^{\frac{(b+\lambda)^2}{4a}}\exp\left(\frac{i(b+\lambda)(A\varphi + C\theta)}{2a}\right) =$$

$$= \frac{1}{\sqrt{2at}}\exp\left(-\frac{(A\varphi + C\theta)^2}{8\pi at}\right) f\left(e^{-\frac{\pi}{2at} - \frac{A\varphi + C\theta}{2at} - \frac{i\pi(b+\lambda)}{a}}, e^{-\frac{\pi}{2at} + \frac{A\varphi + C\theta}{2at} + \frac{i\pi(b+\lambda)}{a}}\right), \quad (3.4)$$

and

$$f(q^{2a'+\lambda'}\zeta^{A'} z^{C'}, q^{-\lambda'}\zeta^{-A'}z^{-C'})q^{\frac{(a'+\lambda')^2}{4a'}}\exp\left(\frac{i(a'+\lambda')(A'\varphi + C'\theta)}{2a'}\right) =$$

$$= \frac{1}{\sqrt{2a't}}\exp\left(-\frac{(A'\varphi + C'\theta)^2}{8\pi a't}\right) f\left(-e^{-\frac{\pi}{2a't} - \frac{A'\varphi' + C'\theta}{2a't} - \frac{i\pi\lambda'}{a'}}, -e^{-\frac{\pi}{2a't} + \frac{A'\varphi' + C'\theta}{2a't} + \frac{i\pi\lambda'}{a'}}\right). \quad (3.5)$$

Proof. The proof is straight forward. We need only make repeated use of (3.1) or its special cases (3.1)$'$ and (3.1)$''$. We omit the details.

***Remark* 3.1.** We have already assumed that $a \mid b$. If, further, $\frac{b}{a}$ is an odd integer, then the factors $e^{\pm i\pi b/a}$ appearing in the arguments of f on the right-hand side of (3.4) both equal -1 and, hence, (3.4) takes the form

$$f(q^{a+b+\lambda}\zeta^A z^C,\ q^{a-b-\lambda}\zeta^{-A}z^{-C})q^{\frac{(b+\lambda)^2}{4a}}\exp\left(\frac{i(b+\lambda)(A\varphi + C\theta)}{2a}\right) =$$

$$= \frac{1}{\sqrt{2at}} \exp\left(-\frac{(A\varphi + C\theta)^2}{8\pi at} \right) f\left(-e^{-\frac{\pi}{2at} - \frac{A\varphi+C\theta}{2at} - \frac{i\pi\lambda}{a}}, -e^{-\frac{\pi}{2at} + \frac{A\varphi+C\theta}{2at} + \frac{i\pi\lambda}{a}} \right). \quad (3.4)'$$

Lemma 3.2. *We have*

$$f(q^{a+\lambda}\zeta^A z^C,\ q^{a-\lambda}\zeta^{-A} z^{-C}) f(q^{a'+\lambda'}\zeta^{A'} z^{C'}, q^{a'-\lambda'}\zeta^{-A'} z^{-C'}) \times$$
$$\times\, q^{\frac{\lambda^2}{4a} + \frac{\lambda'^2}{4a'}} \exp\left(\frac{i\lambda(A\varphi + C\theta)}{2a} + \frac{i\lambda'(A'\varphi + C'\theta)}{2a'} \right) =$$
$$= \frac{1}{2t\sqrt{\Delta}} \exp\left(-\frac{(A\varphi + C\theta)^2}{8\pi at} - \frac{(A'\varphi' + C'\theta)^2}{8\pi a't} \right) \times$$
$$\times\, f\left(e^{-\frac{\pi}{2at} - \frac{A\varphi+C\theta}{2at} - \frac{i\pi\lambda}{a}}, e^{-\frac{\pi}{2at} + \frac{A\varphi+C\theta}{2at} + \frac{i\pi\lambda}{a}} \right) \times$$
$$\times\, f\left(e^{-\frac{\pi}{2a't} - \frac{A'\varphi+C'\theta}{2a't} - \frac{i\pi\lambda'}{a'}}, e^{-\frac{\pi}{2a't} + \frac{A'\varphi'+C'\theta}{2a't} + \frac{i\pi\lambda'}{a'}} \right), \quad (3.6)$$

$$f(q^{a+b+\lambda}\zeta^A z^C,\ q^{a-b-\lambda}\zeta^{-A} z^{-C}) f(q^{2a'+\lambda'}\zeta^{A'} z^{C'},\ q^{-\lambda'}\zeta^{-A'} z^{-C'}) \times$$
$$\times\, q^{\frac{(b+\lambda)^2}{4a} + \frac{(a'+\lambda')^2}{4a'}} \exp\left(\frac{i(b+\lambda)(A\varphi + C\theta)}{2a} + \frac{i(a'+\lambda')(A'\varphi + C'\theta)}{2a'} \right) =$$
$$= \frac{1}{2t\sqrt{\Delta}} \exp\left(-\frac{(A\varphi + C\theta)^2}{8\pi at} + \frac{(A'\varphi + C'\theta)^2}{8\pi a't} \right) \times$$
$$\times\, f\left(e^{-\frac{i\pi b}{a}} e^{-\frac{\pi}{2at} - \frac{A\varphi+C\theta}{2at} - \frac{i\pi\lambda}{a}}, e^{\frac{i\pi b}{a}} e^{-\frac{\pi}{2at} + \frac{A\varphi+C\theta}{2at} + \frac{i\pi\lambda}{a}} \right) \times$$
$$\times\, f\left(-e^{-\frac{\pi}{2a't} - \frac{A'\varphi+C'\theta}{2a't} - \frac{i\pi\lambda'}{a}}, -e^{-\frac{\pi}{2a't} + \frac{A'\varphi'+C'\theta}{2a't} + \frac{i\pi\lambda'}{a'}} \right), \quad (3.7)$$

and

$$q^{\frac{\lambda^2}{4a} + \frac{\lambda'^2}{4a'}} \exp\left[\frac{i}{2} \left\{ \frac{\lambda(A\varphi + C\theta)}{a} + \frac{\lambda'(A'\varphi + C'\theta)}{a'} \right\} \right] \times$$
$$\times\, [f(q^{a+\lambda}\zeta^A z^C, q^{a-\lambda}\zeta^{-A} z^{-C}) f(q^{a'+\lambda'}\zeta^{A'} z^{C'}, q^{a'-\lambda'}\zeta^{-A'} z^{-C'}) +$$
$$+\, q^{\mu+c}\zeta^B z^D f(q^{a+b+\lambda}\zeta^A z^C, q^{a-b-\lambda}\zeta^{-A} z^{-C}) f(q^{2a'+\lambda'}\zeta^{A'} z^{C'}, q^{-\lambda'}\zeta^{-A'} z^{-C'})] =$$
$$= \frac{1}{2t\sqrt{\Delta}} \exp\left(-\frac{(A\varphi + C\theta)^2}{8\pi at} - \frac{(A'\varphi + C'\theta)^2}{8\pi a't} \right) (\alpha\beta + \alpha'\beta'), \quad (3.8)$$

where

$$\alpha = f\left(e^{-\frac{\pi}{2at} - \frac{A\varphi+C\theta}{2at} - \frac{i\pi\lambda}{a}}, e^{-\frac{\pi}{2at} + \frac{A\varphi+C\theta}{2at} + \frac{i\pi\lambda}{a}} \right),$$

$$\beta = f\left(e^{-\frac{\pi}{2a't} - \frac{A'\varphi + C'\theta}{2a't} - \frac{i\pi\lambda'}{a'}}, e^{-\frac{\pi}{2a't} + \frac{A'\varphi' + C'\theta}{2a't} + \frac{i\pi\lambda'}{a'}} \right),$$

$$\alpha' = f\left(e^{-\frac{i\pi b}{a}} e^{-\frac{\pi}{2at} - \frac{A\varphi + C\theta}{2at} - \frac{i\pi\lambda}{a}}, e^{\frac{i\pi b}{a}} e^{-\frac{\pi}{2at} + \frac{A\varphi + C\theta}{2at} + \frac{i\pi\lambda}{a}} \right),$$

$$\beta' = f\left(e^{-\frac{\pi}{2a't} - \frac{A'\varphi + C'\theta}{2a't} - \frac{i\pi\lambda'}{a'}}, e^{-\frac{\pi}{2a't} + \frac{A'\varphi' + C'\theta}{2a't} + \frac{i\pi\lambda'}{a'}} \right).$$

Proof. Equations (3.6) follows on multiplying (3.2) and (3.3). Similarly, (3.7) follows from (3.4) and (3.5). Equation (3.8) follows from (3.6) and (3.7) once we see

$$\frac{b^2 + 2b\lambda}{4a} + \frac{a'^2 + 2a'\lambda'}{4a'} = \mu + c$$

and

$$\frac{b(A\varphi + C\theta)}{2a} + \frac{A'\varphi + C'\theta}{2} = B + D.$$

However, these are simple consequences of the definitions of Δ, a', λ', A' and C' given in (2.1).

4. Main result. We first obtain a lemma by employing the following result from the classical theory of theta functions [7]:

$$f(X, Y) f(Z, W) + f(-X, -Y) f(-Z, -W) =$$

$$= 2\left[f(X^3 Y, XY^3) f(Z^3 W, ZW^3) + XZ f\left(\frac{Y}{X}, \frac{X}{Y} X^4 Y^4 \right) f\left(\frac{W}{Z}, \frac{Z}{W} Z^4 W^4 \right) \right]. \quad (4.1)$$

Lemma 4.1. *Assuming* $\frac{b}{a}$ *to be an odd integer, we have, in the notations of* (3.8),

$$\frac{1}{2}(\alpha\beta + \alpha'\beta') = f\left(e^{-\frac{2\pi}{at} - \frac{A\varphi + C\theta}{at} - \frac{2i\pi\lambda}{a}}, e^{-\frac{2\pi}{at} + \frac{A\varphi + C\theta}{at} + \frac{2i\pi\lambda}{a}} \right) \times$$

$$\times f\left(e^{-\frac{2\pi}{a't} - \frac{A'\varphi + C'\theta}{a't} - \frac{2i\pi\lambda'}{a'}}, e^{-\frac{2\pi}{a't} + \frac{A'\varphi + C'\theta}{a't} + \frac{2i\pi\lambda'}{a'}} \right) +$$

$$+ e^{-\frac{2\pi}{2t}\left(\frac{1}{a} + \frac{1}{a'}\right) - \frac{1}{2t}\left\{\left(\frac{A}{a} + \frac{A'}{a'}\right)\varphi + \left(\frac{C}{a} + \frac{C'}{a'}\right)\theta\right\} - i\pi\left(\frac{\lambda}{a} + \frac{\lambda'}{a'}\right)} \times$$

$$\times f\left(e^{\frac{A\varphi + C\theta}{at} + \frac{2i\pi\lambda}{a}}, e^{-\frac{4\pi}{at} - \frac{A\varphi + C\theta}{at} - \frac{2i\pi\lambda}{a}} \right) \times$$

$$\times f\left(e^{\frac{A'\varphi + C'\theta}{a't} + \frac{2i\pi\lambda'}{a'}}, e^{-\frac{4\pi}{a't} - \frac{A'\varphi + C'\theta}{a't} - \frac{2i\pi\lambda'}{a'}} \right). \quad (4.2)$$

Proof. The proof follows from (4.1) on employing for X and Y the arguments in α defined in (3.8) and for Z and W those of β also defined in (3.8).

We are now finally in a position to establish our main result.

Theorem 4.1. *If* $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, *and* $z = e^{i\theta}$, *we have*

$$a(q,\zeta,z;a,a,c;\lambda,\mu,\nu;A,B;C,D) :=$$

$$:= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2+\lambda m+\mu n+\nu}\zeta^{Am+Bn}z^{Cm+Dn} =$$

$$= \frac{1}{t\sqrt{\Delta}}\, q^{-\left(\frac{\lambda^2}{4a}+\frac{\lambda'^2}{4a'}\right)+\nu} \exp\left[-\frac{i}{2}\left\{\frac{\lambda(A\varphi+C\theta)}{a}+\frac{\lambda'(A'\varphi+C'\theta)}{a'}\right\}\right]\times$$

$$\times \exp\left[-\frac{1}{8\pi t}\left\{\frac{(A\varphi+C\theta)^2}{a}+\frac{(A'\varphi+C'\theta)^2}{a'}\right\}\right]\times$$

$$\times\, a(e^{-\frac{2\pi}{\Delta t}}, e^{i\tilde{\varphi}}, e^{i\tilde{\theta}}; a,a,c; 0,0,0; 1,1; 1,-1) \tag{4.3}$$

with

$$a(e^{-\frac{2\pi}{\Delta t}}, e^{i\tilde{\varphi}}, e^{i\tilde{\theta}}; a,a,c; 0,0,0; 1,1; 1,-1) :=$$

$$:= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} \tilde{q}^{a(m^2+mn)+cn^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}, \qquad \tilde{q} = e^{-\frac{2\pi}{\Delta t}}, \qquad \tilde{t} = \frac{1}{\Delta t}, \tag{4.4}$$

$$i\tilde{\varphi} := -\frac{A\varphi+C\theta}{4at}-\frac{i\pi\lambda}{2a}-\frac{3}{4}\frac{A'\varphi+C'\theta}{a't}-\frac{3i\pi\lambda'}{2a'}, \qquad \tilde{\zeta} = e^{i\tilde{\varphi}}, \tag{4.5}$$

$$i\tilde{\theta} := \frac{A\varphi+C\theta}{4at}+\frac{i\pi\lambda}{2a}-\frac{1}{4}\frac{A'\varphi+C'\theta}{a't}-\frac{i\pi\lambda'}{2a'}, \qquad \tilde{z} = e^{i\tilde{\theta}}. \tag{4.6}$$

Proof. From (2.1), (3.8), and (4.2), we need to prove that the right-hand side of (4.2) equals (4.4). To do this, we make another use of (2.1) with $(q,\zeta,z;A,B;C,D;\lambda,\mu,\nu)$ replaced by $(\tilde{q},\tilde{\zeta},\tilde{z};\tilde{A},\tilde{B};\tilde{C},\tilde{D};\tilde{\lambda},\tilde{\mu},\tilde{\nu})$, where $\tilde{q}$, $\tilde{\zeta}$, $\tilde{z}$ are as in (4.4) – (4.6) above and

$$(\tilde{A},\tilde{B}) = (1,1), \qquad (\tilde{C},\tilde{D}) = (1,-1), \qquad (\tilde{\lambda},\tilde{\mu},\tilde{\nu}) = (0,0,0). \tag{4.7}$$

We however retain $(\tilde{a},\tilde{b},\tilde{c}) = (a,b,c) = (a,c,c)$, so that, from the notations of (2.1),

$$\tilde{\Delta} = \Delta, \qquad \tilde{a}' = a' = \frac{\Delta}{a}, \qquad \tilde{\lambda}' = \tilde{Q}_m(0,0)a'^{-1} = 0,$$

$$\tilde{A}' = \frac{\tilde{Q}_m(1,-1)}{a} = 1, \qquad \tilde{C}' = \frac{\tilde{Q}_m(-1,-1)}{a} = \frac{-2a-a}{a} = -3. \tag{4.8}$$

In fact, on using the various definitions in (4.7), (4.8) and in (2.1), we at once have

$$\tilde{q}^{\tilde{a}\pm\tilde{\lambda}}\tilde{\zeta}^{\pm\tilde{A}}\tilde{z}^{\pm\tilde{C}} = e^{\left(-\frac{2\pi}{\Delta t}\right)a\pm\left(-\frac{A'\varphi'+C'\theta}{a't}-\frac{2\pi i\lambda'}{a'}\right)} = e^{-\frac{2\pi a}{\Delta t}\mp\left(\frac{A'\varphi'+C'\theta}{a't}+\frac{2\pi i\lambda'}{a'}\right)}, \tag{4.9}$$

$$\tilde{q}^{\tilde{a}'\pm\tilde{\lambda}'}\tilde{\zeta}^{\pm\tilde{A}'}\tilde{z}^{\pm\tilde{C}'} = e^{-\frac{2\pi}{\Delta t}\frac{\Delta}{a}\pm(i\tilde{\varphi}-3i\tilde{\theta})} = e^{-\frac{2\pi}{at}\mp\left(\frac{A\varphi+C\theta}{at}+\frac{2\pi i\lambda}{a}\right)}, \tag{4.10}$$

$$\tilde{q}^{\tilde{a}\pm\tilde{b}\pm\tilde{\lambda}}\tilde{\zeta}^{\pm\tilde{A}}\tilde{z}^{\pm\tilde{C}} = e^{-\frac{2\pi}{a't}\pm\left(-\frac{2\pi a}{\Delta t}\right)\mp\left(\frac{A'\varphi'+C'\theta}{a't}+\frac{2\pi i\lambda'}{a'}\right)} \text{ (as in (4.9))} =$$

$$= \begin{cases} e^{-\frac{4\pi}{a't} - \frac{A'\varphi' + C'\theta}{a't} - \frac{2\pi i\lambda'}{a'}}, \\ e^{\frac{A'\varphi' + C'\theta}{a't} + \frac{2\pi i\lambda'}{a'}}, \quad \text{respectively,} \end{cases} \tag{4.11}$$

$$\tilde{q}^{2\tilde{a}' + \tilde{\lambda}'}\tilde{\zeta}^{\tilde{A}'}\tilde{z}^{\tilde{C}'} = e^{-\frac{4\pi}{\Delta t}\frac{\Delta}{a} + (i\tilde{\varphi} - 3i\tilde{\theta})} = e^{-\frac{4\pi}{at} - \frac{A\varphi + C\theta}{at} - \frac{2\pi i\lambda}{a}} \quad \text{(as in (4.10))} \tag{4.12}$$

and, similarly,

$$\tilde{q}^{-\tilde{\lambda}'}\tilde{\zeta}^{-\tilde{A}'}\tilde{z}^{-\tilde{C}'} = e^{\frac{A\varphi + C\theta}{at} + \frac{2\pi i\lambda}{a}}. \tag{4.12$'$}$$

Further,

$$\tilde{q}^{\tilde{\mu} + \tilde{C}}\tilde{\zeta}^{\tilde{B}}\tilde{z}^{\tilde{D}} = e^{-\frac{i\pi C}{\Delta t}} + i(\tilde{\varphi}\tilde{B} + \tilde{\theta}\tilde{D}) =$$

$$= e^{\frac{-i\pi(\Delta + a^2)}{4a\Delta t} + i(\tilde{\varphi} - \tilde{\theta})} \text{(since} \quad 4ac - a^2 = \Delta, \ \tilde{B} = -\tilde{D} = 1) =$$

$$= e^{-\frac{i\pi}{4t}\left(\frac{1}{a} + \frac{a}{\Delta}\right) - \left(\frac{A\varphi + C\theta}{2at} + \frac{\pi i\lambda}{a} + \frac{A'\varphi + C'\theta}{2a't} + \frac{\pi i\lambda'}{a'}\right)}, \tag{4.13}$$

on using (4.5) – (4.6). Now using (4.9) – (4.13) in (4.2), (3.8) and (2.1) sequentially with $q, \zeta, z; \ldots$ changed to $\tilde{q}$, $\tilde{\zeta}$, $\tilde{z}; \ldots$ we have (4.3) read with (4.4) – (4.6) and in the notations of (2.1).

The theorem is proved.

In the following sections we specialize our Theorem 4.1 into various known cases. Before that, we will record a slightly more "balanced" form of (4.3) as follows:

$$q^{\frac{\lambda^2}{4a} + \frac{\lambda'^2}{4a'} - \nu} \exp\left[\frac{i}{2}\left\{\frac{(A\varphi + C\theta)\lambda}{a} + \frac{(A'\varphi + C'\theta)\lambda'}{a'}\right\}\right] \times$$

$$\times \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2 + mn) + cn^2 + \lambda m + \mu n + \nu}\zeta^{Am + Bn}z^{Cm + Dn} =$$

$$= \frac{1}{t\sqrt{\Delta}} \exp\left[-\frac{1}{8\pi t}\left\{\frac{(A\varphi + C\theta)^2}{a} + \frac{(A'\varphi + C'\theta)^2}{a'}\right\}\right] \times$$

$$\times \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} \tilde{q}^{a(m^2 + mn) + cn^2}\tilde{\zeta}^{m + n}\tilde{z}^{m - n}. \tag{4.3$'$}$$

Or, what is the same,

$$q^{\frac{1}{4}\left(\frac{\lambda^2}{4a} + \frac{\lambda'^2}{4a'}\right) - \nu} \exp\left[\frac{i}{2}\left\{\frac{(A\varphi + C\theta)\lambda}{a} + \frac{(A'\varphi + C'\theta)\lambda'}{a'}\right\}\right] \times$$

$$\times \ a(q, \zeta, z; a, a, c; \lambda, \mu, \nu; A, B; C, D) =$$

$$= \frac{1}{t\sqrt{\Delta}} \exp\left[-\frac{1}{8\pi t}\left\{\frac{(A\varphi + C\theta)^2}{a} + \frac{(A'\varphi + C'\theta)^2}{a'}\right\}\right] \times$$

$$\times \ a(\tilde{q}, \tilde{\zeta}, \tilde{z}; a, a, c; 0, 0, 0; 1, 1; 1, -1). \tag{4.3$''$}$$

5. The *a*-functions of [1, 6]. We have the following special case of Theorem 4.1.

Theorem 5.1. *If* $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, *and* $z = e^{i\theta}$, *then*

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2}\zeta^{m+n}z^{m-n} =$$

$$= \frac{1}{t\sqrt{a(4c-a)}}\exp\left\{-\frac{(2a+c)\theta^2+2(c-a)\theta\varphi+c\varphi^2}{2\pi ta(4c-a)}\right\}\times$$

$$\times\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{a(m^2+mn)+cn^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}, \tag{5.1}$$

where

$$\tilde{q} = e^{-\frac{2\pi}{ta(4c-a)}}, \qquad \tilde{\zeta} = \exp\left\{\frac{(5a-2c)\theta-(a+2c)\varphi}{2a(4c-a)t}\right\},$$

$$\tilde{z} = \exp\left\{\frac{(2c+a)\theta-(a-2c)\varphi}{2a(4c-a)t}\right\}.$$

Proof. The proof is a direct consequence of Theorem 4.1 on putting $\lambda = \mu = \nu = 0$ and $A = B = C = 1$, $D = -1$, there.

***Corollary* 5.1.** *Putting* $a = 1$, *we have* (5.2) *of* [6].

***Remark* 5.1.** We note that (5.1) with $a = c = 1$ is the same as (1.4) of [3], taking into account that the left-hand side of (5.1) is unchanged with θ and φ changed to $-\theta$ and $-\varphi$ and using the simple fact that

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}(x^3y)^{m+n}\left(\frac{x}{y}\right)^{m-n} = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}(y^2)^{m+n}(x^2)^{m-n}.$$

6. The *a′*-functions of [1, 6]. We have the following theorem.

Theorem 6.1. *If* $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, *and* $z = e^{i\theta}$, *then*

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2}\zeta^{m}z^{n} = \frac{1}{t\sqrt{a(4c-a)}}\exp\left\{-\frac{a(\theta^2-\theta\varphi)+c\varphi^2}{2\pi a(4c-a)t}\right\}\times$$

$$\times\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{a(m^2+mn)+cn^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}, \tag{6.1}$$

where

$$\tilde{q} = e^{-\frac{2\pi}{a(4c-a)t}}, \qquad \tilde{\zeta} = \exp\left\{\frac{2(a-c)\varphi-3a\theta}{2a(4c-a)t}\right\}, \qquad \tilde{z} = \exp\left\{\frac{2c\varphi-a\theta}{2a(4c-a)t}\right\}.$$

Proof. To prove the theorem, it is enough to put $\lambda = \mu = \nu = 0$, $A = 1$, $B = 0$, $C = 0$, $D = 1$ in Theorem 4.1.

***Corollary* 6.1.** *By rewriting the series on the right-hand side of* (6.1) *as*

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{a(m^2+mn)+cn^2}\tilde{\zeta}'^{m}\tilde{z}'^{n},$$

where

$$\tilde{\zeta}' = \tilde{\zeta}\tilde{z} = \frac{\varphi - 2\theta}{(4c-a)t} \quad \text{and} \quad \tilde{z}' = \frac{\tilde{\zeta}}{\tilde{z}} = \frac{(a-2c)\varphi - a\theta}{a(4c-a)t},$$

and then putting $a = 1$, *we get main result, namely, Theorem* 4.1, *of* [6].

***Remark* 6.1.** We note that by putting $a = c = 1$, we have

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}\zeta^m z^n =$$

$$= \frac{1}{t\sqrt{3}}\exp\left\{-\frac{\theta^2 - \theta\varphi + \varphi^2}{6t}\right\}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{m^2+mn+n^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}, \tag{6.2}$$

where

$$\tilde{q} = e^{-\frac{2\pi}{3t}}, \qquad \tilde{\zeta} = \exp\left(-\frac{\theta}{2t}\right), \qquad \tilde{z} = \exp\left(\frac{2\varphi - \theta}{6t}\right),$$

which is the same as (1.16) of [3] on noting the simple facts that

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}\zeta^m z^n = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}(\zeta^{-1})^m (z^{-1})^n =$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2} z^m \zeta^n .$$

Further, (6.2) is also equivalent to (1.15) of [3].

One could also obtain a slightly more general result than (6.1), namely, the following theorem.

Theorem 6.2. *If* $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, *and* $z = e^{i\theta}$, *then*

$$\left(\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2+\lambda m+\mu n+\nu}\zeta^m z^n\right)\exp\left[\frac{i(-2\mu+\lambda)\theta + i(\mu - 2\lambda)\varphi}{4a-c}\right]\times$$

$$\times\exp\left[2\pi\left(\nu - \frac{2\mu - \lambda}{4a-c}\right)t\right] =$$

$$= \frac{1}{t\sqrt{a(4c-a)}}\exp\left\{-\frac{a(\theta^2 - \theta\varphi) + c\varphi^2}{2\pi a(4c-a)t}\right\}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{a(m^2+mn)+cn^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}, \tag{6.3}$$

where

$$\tilde{q} = e^{-\frac{2\pi}{a(4c-a)t}}, \qquad \tilde{\zeta} = \exp\left[\frac{2(a-c)\varphi - 3a\theta}{2a(4c-a)t} + \frac{\pi i\{2\lambda(c-a) - 3a\mu\}}{a(4c-a)}\right],$$

and

$$\tilde{z} = \exp\left[\frac{2c\varphi - a\theta}{2a(4c-a)t} + \frac{\pi i(2c\lambda - a\mu)}{a(4c-a)}\right]. \tag{6.4}$$

7. The *b*-functions of [1, 6]. The series on the left-hand side of the first formula in the next theorem reduces to the b-series of [6] on putting $a = 1$ and $\lambda = \mu = \nu = 0$ and changing ζ to $\Omega\zeta/\omega$ and z to $z\omega/\Omega^c$ with $\Omega = \exp\left(\frac{2\pi i}{4c-1}\right)$.

Theorem 7.1. *If* $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, *and* $z = e^{i\theta}$, *then*

$$\exp\left[2\pi\left(-\nu + \frac{a\mu^2 - a\mu\lambda + c\lambda^2}{a(4a-c)}\right)t\right] \times$$

$$\times \exp\left[\frac{i}{a(4c-a)}\{(2\varphi-\theta)\lambda + (2\theta-\varphi+2\pi)\mu\}\right] \times$$

$$\times \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2+\lambda m+\mu n+\nu}\zeta^m z^n \omega^{m-n} =$$

$$= \frac{1}{t\sqrt{a(4c-a)}}\exp\left[-\frac{\left\{a(\theta^2-\theta\varphi)+c\varphi^2+2\pi a\theta+\frac{4\pi}{3}(c-a)\varphi+\frac{4\pi^2}{9}(2a+c)\right\}}{2\pi a(4c-a)t}\right] \times$$

$$\times \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} \tilde{q}^{a(m^2+mn)+cn^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}, \tag{7.1}$$

where

$$\tilde{q} = e^{-\frac{2\pi}{a(4c-a)t}},$$

$$\tilde{\zeta} = \exp\left\{-\left[\frac{2(a-c)\varphi+3a\theta}{2a(4c-a)t} + \frac{2\pi(2a+c)}{3a(4c-a)t} + \frac{\pi i\{2(c-a)\lambda+3a\mu\}}{a(4c-a)}\right]\right\},$$

$$\tilde{z} = \exp\left[\frac{2c\varphi-a\theta}{2a(4c-a)t} + \frac{2\pi(c-a)}{3a(4c-a)t} + \frac{\pi i(2c\lambda-a\mu)}{a(4c-a)}\right]. \tag{7.2}$$

Proof. We need to put in Theorem 4.1 $A = 1 = D$ and $B = 0 = C$ and change φ to $\varphi + \frac{2\pi}{3}$ and θ to $\theta + \frac{4\pi}{3}$. Routine calculations then give (7.1) read with (7.2).

***Remark* 7.1.** By putting $a = c = 1$ and $\lambda = \mu = \nu = 0$ in the above, we see that the series on the left-hand side of (7.1) would be the same as the b-function of (1.27) in [1] and the series on the right-hand side of (7.1) would be the same as the a-function of (1.25) there. The equation (7.1) itself will become

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}\zeta^m z^n \omega^{m-n} =$$

$$= \frac{1}{t\sqrt{3}}\exp\left\{-\frac{\theta^2-\theta\varphi+\varphi^2}{6\pi t} - \frac{\theta}{3t} - \frac{2\pi}{9t}\right\}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{m^2+mn+n^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n} \tag{7.3}$$

with

$$\tilde{q} = e^{-\frac{2\pi}{3t}}, \quad \tilde{\zeta} = \exp\left(-\frac{\theta}{2t} - \frac{2\pi}{3t}\right), \quad \tilde{z} = \exp\left(\frac{2\varphi-\theta}{6t}\right). \tag{7.4}$$

Further, since the series on the left-hand side of (7.3) also equals (on first changing m to $-m$ and then n to $-n$ and then interchanging m and n)

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}(\zeta^{-1})^m (z^{-1})^n \omega^{m-n},$$

we have that

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+n^2}\zeta^m z^n \omega^{m-n} =$$

$$= \frac{1}{t\sqrt{3}}\exp\left[-\frac{\theta^2-\theta\varphi+\varphi^2}{6\pi t}+\frac{\varphi}{3t}-\frac{2\pi}{9t}\right]\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{m^2+mn+n^2}\tilde{\zeta}'^{m+n}\tilde{z}'^{m-n}, \qquad (7.3)'$$

where

$$\tilde{q} = e^{-\frac{2\pi}{3t}}, \qquad \tilde{\zeta}' = \exp\left(\frac{\varphi}{2t}-\frac{2\pi}{3t}\right), \qquad \tilde{z}' = \exp\left(\frac{\varphi-2\theta}{6t}\right) =$$

$$= \frac{1}{t\sqrt{3}}\exp\left[-\frac{\theta^2-\theta\varphi+\varphi^2}{6\pi t}+\frac{\varphi}{3t}\right]\tilde{q}^{1/3}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{m^2+mn+n^2+m+n}\tilde{\zeta}''^{m+n}\tilde{z}''^{m-n}, \qquad (7.4)'$$

$$\tilde{\zeta}'' = \exp\left(\frac{\varphi}{2t}\right), \qquad \tilde{z}'' = \exp\left(\frac{2\theta-\varphi}{6t}\right).$$

Thus, we have shown that

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}(e^{-2\pi t})^{m^2+mn+n^2}(e^{i\varphi})^m(e^{i\theta})^n\omega^{m-n} = \frac{1}{t\sqrt{3}}\exp\left[-\frac{\theta^2-\theta\varphi+\varphi^2}{6\pi t}+\frac{\varphi}{3t}\right]\times$$

$$\times\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left(e^{-\frac{2\pi}{3t}}\right)^{m^2+mn+n^2+m+n+\frac{1}{3}}\left(e^{\frac{\varphi}{2t}}\right)^{m+n}\left(e^{\frac{2\theta-\varphi}{6t}}\right)^{m-n},$$

which is the same as (1.16) of [3].

8. The *c* -functions of [1, 6]. The series on the left-hand side of the first formula in the next theorem reduces to c-series of [6] by putting $a = 1$.

Theorem 8.1. *If* $q = e^{-2\pi t}$, $\zeta = e^{i\varphi}$, *and* $z = e^{i\theta}$, *then*

$$\exp\left(\frac{2i\varphi}{a(4c-a)}\right)\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{a(m^2+mn)+cn^2+\frac{3am}{\Delta}+\frac{2c+a}{\Delta}n+\frac{2a+c}{\Delta^2}}\zeta^{m+n}z^{m-n} =$$

$$= \frac{1}{t\sqrt{\Delta}}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{a(m^2+mn)+cn^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}\times$$

$$\times\exp\left[-\left\{\frac{c\varphi^2+2(2a+c)\theta^2+2(c-a)\varphi\theta}{2\pi a(4c-a)t}\right\}\right], \qquad (8.1)$$

where

$$\tilde{q} = e^{-\frac{2\pi}{\Delta t}}, \qquad \Delta = a(4c-a),$$

$$\tilde{\zeta} = \exp\left[-\left\{\frac{2(c+a)\varphi+\theta(2c-5a)}{2a(4c-a)t}+\frac{3\pi i}{a(4c-a)}\right\}\right],$$

$$\tilde{z} = \exp\left[\frac{(2c-a)\varphi+(2c+a)\theta}{2a(4c-a)t}+\frac{\pi i}{a(4c-a)}\right].$$

Proof. The proof is straight forward. We need only do some routine calculations on putting $\lambda = \dfrac{3a}{\Delta}$, $\mu = \dfrac{2c+a}{\Delta}$, $\nu = \dfrac{2a+c}{\Delta^2}$ in Theorem 4.1.

***Remark* 8.1.** By putting $a = c = 1$, we see that the series on the left-hand side of (8.1) is the same as the c-function of (1.28) in [3] and (8.1) reduces to

$$\exp\left(\frac{2i\varphi}{3}\right)\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} q^{m^2+mn+cn^2+m+n+\frac{1}{3}}\zeta^{m+n}z^{m-n} =$$

$$= \frac{\exp\left[-\dfrac{\varphi^2+3\theta^2}{6\pi t}\right]}{t\sqrt{3}}\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{m^2+mn+n^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n}$$

with

$$\tilde{q} = e^{-\frac{2\pi}{3t}}, \qquad \tilde{\zeta} = -\exp\left\{\frac{\theta-\varphi}{2t}\right\}, \qquad \tilde{z} = -\exp\left\{\frac{\varphi+3\theta}{6t}+\frac{\pi i}{3}\right\}.$$

This is the same as (1.18) of [3] with φ changed to $-\varphi$ and on realizing that

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{m^2+mn+n^2}\tilde{\zeta}^{m+n}\tilde{z}^{m-n} = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\tilde{q}^{m^2+mn+n^2}\left(\frac{\tilde{\zeta}}{\omega\tilde{z}}\right)^m(\omega^2\tilde{\zeta}\tilde{z})\omega^{m-n}.$$

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