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NON-EXISTENCE THEOREM EXCEPT THE OUT-OF-PHASE AND IN-PHASE SOLUTIONS IN THE COUPLED VAN DER POL EQUATION SYSTEM

ТЕОРЕМА ПРО НЕІСНУВАННЯ РОЗВ'ЯЗКІВ ЗА ВИНЯТКОМ РЕЖИМУ СИНХРОННИХ КОЛИВАНЬ ТА РЕЖИМУ КОЛИВАНЬ У ПРОТИФАЗІ ДЛЯ СИСТЕМИ З'ЄДНАНИХ РІВНЯНЬ ВАН ДЕР ПОЛЯ

We consider the coupled van der Pol equation system in this paper. Our coupled system consists of two van der Pol equations which are connected by the linear terms with each other. In this paper, we consider that two distinctive solutions (the out-of-phase and in-phase solutions) exist in the dynamical system of the coupled equations and we give the answers to some of the problems.

Розглянуто систему з'єднаних рівнянь Ван дер Поля. Ця система складається з двох рівнянь Ван дер Поля, що пов'язані між собою лінійними членами. У статті розглянуто випадок, коли динамічна система з'єднаних рівнянь має два різних розв'язки (у режимі синхронних коливань та у режимі коливань у протифазі), і дано відповіді на деякі питання.

1. Introduction. In the course of studying the periodic solution of the differential equation with nonlinear perturbed terms

$$x'' + x = \varepsilon f(x, x'),$$

where $x = x(t)$ and $'$ denotes the derivative with respect to time t (we use $'$ for the symbol of derivative hereinafter), the method of averaging (using Fourier series) was established by Kryloff and Bogoliuboff of the Kiev school of mathematics after 1930 in connection with the asymptotic methods [1 – 3]. The method of averaging was used for the first time by van der Pol, and then Kryloff and Bogoliuboff gave the full justification of the method. After that, Urabe considered more general forms than the above equation using moving coordinate system [4, 5]. Hayashi studied nonlinear oscillations mainly from the viewpoint of physics [6, 7] and Minorsky did it from the mechanical point of view [8].

On the other hand, the group around Mitropolsky and Samoilenko [9] investigated nonlinear systems of differential equations with lag and some classes of integro-differential and difference equations, and they developed a method for the solution of problems concerning the existence of periodic solutions and construction of algorithms for calculating these solutions. However, there still exist difficult problems arising in nonlinear coupled systems or in the case of a large number of degrees of freedom with nonlinearity due to the inevitable computing complexity of the system.

We treat the van der Pol equation system with coupling presented below by the positional difference. Let $y = y(t)$ and $z = z(t)$ be two real valued functions. We consider the dynamical system

$$\mathbf{\Sigma}_{\varepsilon,k} \begin{cases} y'' - \varepsilon(1 - y^2)y' + y = k(y - z), \\ z'' - \varepsilon(1 - z^2)z' + z = k(z - y), \quad t_0 \le t. \end{cases}$$

Here, $k, \varepsilon\ (> 0)$ are constants and t_0 indicates an initial time. When $k = 0$, the dynamical system $\mathbf{\Sigma}_{\varepsilon,0}$ turns into two independent van der Pol oscillators [10]. The single van der Pol oscillator is a well-known classical problem. Many studies on the van der Pol equation have been carried out, and the fact that the van der Pol equation has a unique limit cycle is known and proved by the Poincaré – Bendixson theorem (see, for example, [11]). However, the coupled van der Pol system, that is, the dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$, constructs a three-dimensional manifold. Therefore, we cannot apply the Poincaré – Bendixson theorem to the dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$, to analyze the system. "Does there exist the limit cycle in $\mathbf{\Sigma}_{\varepsilon,k}$?" [12], "If there exists the limit cycle, how many limit cycles are there?" [13] and "Are the limit cycles 'stable' or 'completely unstable' or 'semistable'?" are still *open* problems we have.

In this paper, we first show the generalized van der Pol equation and analyze it. Then the analysis of the coupled van der Pol equation system is carried out based on the formation of our method after defining the out-of-phase and in-phase solutions, which are new concepts arising when the system is coupled. We consider that there exist two distinctive solutions (out-of-phase and in-phase solutions) in the dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$. Finally, we give answers to some of the above *open* problems.

2. Preliminaries. In this section, we present the analysis of the system $\mathbf{\Sigma}_{0,k}$, that is, a coupled harmonic oscillator with linear coupling in order to reveal the features of the system $\mathbf{\Sigma}_{\varepsilon,k}$. Before discussing our problem, note that we show easily the existence and uniqueness of the solution of our dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$.

Proposition 2.1 (existence and uniqueness theorem: see [14], Section 4.6)**.** *In the system of differential equations*

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{pmatrix},$$

let each of the functions $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ *have continuous partial derivatives with respect to* $x_1, \ldots, x_n$. *Then, the initial-value problem* $\mathbf{x}' = \mathbf{f}(\mathbf{x}), \mathbf{x}(t_0) = \mathbf{x}^0$ *has one, and only one, solution* $\mathbf{x} = \mathbf{x}(t)$, *for every* $\mathbf{x}^0$ *in* $\mathcal{R}^n$.

Note that $\mathbf{\Sigma}_{\varepsilon,k}$ has a fixed point: $(0, 0, 0, 0)$ and the eigenvalues of the system are

$$r_{\pm} = \frac{\varepsilon \pm \sqrt{4 - \varepsilon^2}\, i}{2}, \qquad s_{\pm} = \frac{\varepsilon \pm \sqrt{4 - 8k - \varepsilon^2}\, i}{2}.$$

We now consider $0 < \varepsilon < 2$ and $0 < k < \dfrac{1}{2} - \dfrac{\varepsilon^2}{8}$ so that the system is unstable.

Next we consider the system $\mathbf{\Sigma}_{0,k}$ to clarify the nature of the system $\mathbf{\Sigma}_{\varepsilon,k}$. First we give the following definitions. Let $\xi_\Sigma(t) = \operatorname{col}\big(y(t), y'(t), z(t), z'(t)\big)$ be a solution of $\mathbf{\Sigma}_{\varepsilon,k}$.

Definition 2.1. *If, in the dynamical system* $\mathbf{\Sigma}_{\varepsilon,k}$, *one has*

$$y(t) + z(t) = 0,$$

where $\xi_\Sigma(t)$ *is not equivalent to* 0, *then the system is out-of-phase and the non-trivial solutions of* $y(t)$ *and* $z(t)$ *are called the out-of-phase solutions.*

Definition 2.2. *If, in the dynamical system* $\mathbf{\Sigma}_{\varepsilon,k}$, *one has*

$$y(t) - z(t) = 0,$$

where $\xi_\Sigma(t)$ *is not equivalent to* 0*, then the system is in-phase and the non-trivial solutions of* $y(t)$ *and* $z(t)$ *are called the in-phase solutions.*

Proposition 2.2. *Assume that* k *is irrational and such that* $0 < k < \dfrac{1}{2}$*. The dynamical system* $\mathbf{\Sigma}_{0,k}$

$$\mathbf{\Sigma}_{0,k} \begin{cases} y'' + y = k(y - z), \\ z'' + z = k(z - y), \quad t_0 \le t, \end{cases}$$

has only two families of periodic solutions. A family of periodic solutions is in-phase and its period is $\tau = 2\pi$*. The other is the out-of-phase solution whose period is* $\tau = \dfrac{2\pi}{\sqrt{1-2k}}$*. There exists no other family of periodic solutions.*

Proof. Omitted.

3. Analysis of the generalized van der Pol equation. In this section, we consider the differential equation $W_{\varepsilon,m,\phi}$,

$$W_{\varepsilon,m,\phi} \colon w'' - \varepsilon(w' - \phi) + mw = 0, \tag{3.1}$$

where $w = w(t)$, $\phi = \phi(w, w')$, $0 < \varepsilon < 2\sqrt{m}$, and $'$ denotes the derivative with respect to t. We call this the generalized van der Pol equation since we obtain the ordinary van der Pol equation if we set $W_{\varepsilon,1,w^2w'}$, that is, $m = 1, \phi(t) = w^2(t)w'(t)$. However, we have no restriction regarding $m \in \Re$ and $\phi = \phi(w, w')$ (but we simply write $\phi = \phi(t)$ instead of $\phi(w, w')$) in this section. We can write this in a matrix form as

$$x_w' = A_w x_w - \varepsilon\xi, \tag{3.2}$$

where

$$A_w = \begin{pmatrix} 0 & 1 \\ -m & \varepsilon \end{pmatrix}, \qquad x_w = \begin{pmatrix} w \\ w' \end{pmatrix}, \qquad \xi = \begin{pmatrix} 0 \\ \phi \end{pmatrix}.$$

We know that the solution $x_w(t)$ can be written as

$$x_w(t) = e^{A_w(t-t_0)} x_w(t_0) - \varepsilon \int\limits_{t_0}^{t} e^{A_w(t-s)} \xi(s) ds. \tag{3.3}$$

Here, A_w has two eigenvalues r and its complex conjugate $\bar{r}$:

$$r = \frac{\varepsilon + \sqrt{4m - \varepsilon^2}\, i}{2}, \qquad \bar{r} = \frac{\varepsilon - \sqrt{4m - \varepsilon^2}\, i}{2}.$$

We now see that A_w has a spectral representation

$$A_w = rP_1 + \bar{r}P_2,$$

$$E = P_1 + P_2, \ P_1P_2 = P_2P_1 = 0.$$

Hence, from the relation

$$rP_1 = A_w - \bar{r}P_2 = A_w - \bar{r}(E - P_1),$$

we obtain

$$P_1 = \frac{1}{r-\bar{r}}(A_w - \bar{r}), \qquad P_2 = \frac{1}{r-\bar{r}}(r - A_w).$$

Using this, we simply write the exponential function of A_w as follows:

$$e^{A_w t} = e^{rt}P_1 + e^{\bar{r}t}P_2 = \frac{1}{r-\bar{r}}\Big(\big(e^{rt} - e^{\bar{r}t}\big)A_w + \big(re^{\bar{r}t} - \bar{r}e^{rt}\big)\Big).$$

We can easily obtain

$$\frac{e^{rt} - e^{\bar{r}t}}{r-\bar{r}} = e^{\varepsilon t/2}\frac{\sin(\vartheta t)}{\vartheta},$$

$$\frac{re^{\bar{r}t} - \bar{r}e^{rt}}{r-\bar{r}} = e^{\varepsilon t/2}\left(\cos(\vartheta t) - \frac{\varepsilon}{2}\frac{\sin(\vartheta t)}{\vartheta}\right),$$

where

$$\vartheta = \frac{\sqrt{4m - \varepsilon^2}}{2}. \tag{3.4}$$

Hence, we have

$$e^{A_w t} = e^{\varepsilon t/2}\left(\frac{\sin(\vartheta t)}{\vartheta}A_w + \cos(\vartheta t) - \frac{\varepsilon}{2}\frac{\sin(\vartheta t)}{\vartheta}\right)$$

and we see that, by virtue of equation (3.3), the solution of equation (3.2) satisfies

$$x_w(t) = e^{\frac{1}{2}\varepsilon(t-t_0)}\left\{\frac{\sin\big(\vartheta(t-t_0)\big)}{\vartheta}\begin{pmatrix} -\varepsilon/2 & 1 \\ -m & \varepsilon/2 \end{pmatrix} + \cos\big(\vartheta(t-t_0)\big)\right\}x_w(t_0) -$$

$$-\varepsilon\int\limits_{t_0}^{t} e^{\varepsilon(t-s)/2}\left\{\frac{\sin\big(\vartheta(t-s)\big)}{\vartheta}\begin{pmatrix} -\varepsilon/2 & 1 \\ -m & \varepsilon/2 \end{pmatrix} + \cos\big(\vartheta(t-s)\big)\right\}\times$$

$$\times\begin{pmatrix} 0 \\ \phi\big(s; t_0, w(t_0), w'(t_0)\big) \end{pmatrix} ds. \tag{3.5}$$

In equation (3.5), $\phi\big(s; t_0, w(t_0), w'(t_0)\big)$ means the function ϕ of s defined by the solution with the initial condition of $w(t_0), w'(t_0)$ at time t_0.

Here, we define

$$U_t(\vartheta) := \begin{pmatrix} \cos(\vartheta t) & \dfrac{\sin(\vartheta t)}{\vartheta} \\ -\vartheta\sin(\vartheta t) & \cos(\vartheta t) \end{pmatrix},$$

and give the following lemma:

Lemma 3.1. *The following relation is true:*

$$\frac{\sin(\vartheta t)}{\vartheta}\begin{pmatrix} -\varepsilon/2 & 1 \\ -m & \varepsilon/2 \end{pmatrix} + \cos(\vartheta t) = \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}^{-1} U_{-t}(\vartheta)\begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}.$$

Proof. We easily prove this by using the relation of $m = \vartheta^2 + \dfrac{\varepsilon^2}{4}$ from equation (3.4). We can rewrite equation (3.5) using Lemma 3.1 as

$$x_w(t) = e^{\varepsilon(t-t_0)/2} \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}^{-1} U_{-(t-t_0)}(\vartheta) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} x_w(t_0) -$$

$$-\varepsilon \int\limits_{t_0}^{t} e^{\varepsilon(t-s)/2} \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}^{-1} U_{-(t-s)}(\vartheta) \times$$

$$\times \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} 0 \\ \phi\big(s; t_0, w(t_0), w'(t_0)\big) \end{pmatrix} ds.$$

Multiplying both sides of the above equation by $e^{-\varepsilon(t-t_0)/2} U_{t-t_0}(\vartheta) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}$, we obtain

$$e^{-\varepsilon(t-t_0)/2} U_{t-t_0}(\vartheta) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} x_w(t) =$$

$$= \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} x_w(t_0) - \varepsilon U_{-t_0}(\vartheta) \int\limits_{t_0}^{t} e^{-\varepsilon(s-t_0)/2} U_s(\vartheta) \times$$

$$\times \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} 0 \\ \phi\big(s; t_0, w(t_0), w'(t_0)\big) \end{pmatrix} ds =$$

$$= \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} x_w(t_0) - U_{-t_0}(\vartheta) \times$$

$$\times \int\limits_{t_0}^{t} e^{-\varepsilon(s-t_0)/2} \begin{pmatrix} \dfrac{\sin(\vartheta s)}{\vartheta} \\ \cos(\vartheta s) \end{pmatrix} \phi\big(s; t_0, w(t_0), w'(t_0)\big) ds.$$

Here, we set $\alpha_{w0} = w(t_0)$ and $\beta_{w0} = w'(t_0)$ for simplicity and define the following symbols:

$$I_s(t, t_0; \alpha_{w0}, \beta_{w0}) := \int\limits_{t_0}^{t} e^{-\varepsilon(s-t_0)/2} \frac{\sin(\vartheta s)}{\vartheta} \phi\big(s; t_0, \alpha_{w0}, \beta_{w0}\big) ds,$$

$$I_c(t, t_0; \alpha_{w0}, \beta_{w0}) := \int\limits_{t_0}^{t} e^{-\varepsilon(s-t_0)/2} \cos(\vartheta s) \phi\big(s; t_0, \alpha_{w0}, \beta_{w0}\big) ds.$$

Thus, we obtain the equation

$$e^{-\varepsilon(t-t_0)/2}U_{t-t_0}(\vartheta)\begin{pmatrix}-1/\varepsilon & 0\\ -1/2 & 1/\varepsilon\end{pmatrix}x_w(t)=$$

$$=\begin{pmatrix}-1/\varepsilon & 0\\ -1/2 & 1/\varepsilon\end{pmatrix}\begin{pmatrix}\alpha_{w0}\\ \beta_{w0}\end{pmatrix}-U_{-t_0}(\vartheta)\begin{pmatrix}I_s(t,t_0;\alpha_{w0},\beta_{w0})\\ I_c(t,t_0;\alpha_{w0},\beta_{w0})\end{pmatrix}. \tag{3.6}$$

We utilize the following relations in computing equation (3.6):

$$U_t(\vartheta)U_s(\vartheta)=U_{t+s}(\vartheta),$$

$$U_t^{-1}(\vartheta)=U_{-t}(\vartheta),$$

$$U_0(\vartheta)=E.$$

***Remark* 3.1.** The multiplication of equation (3.6) by $-\varepsilon$ yields

$$e^{-\varepsilon(t-t_0)/2}U_{t-t_0}(\vartheta)\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}x_w(t)=$$

$$=\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}\begin{pmatrix}\alpha_{w0}\\ \beta_{w0}\end{pmatrix}+\varepsilon U_{-t_0}(\vartheta)\begin{pmatrix}I_s(t,t_0;\alpha_{w0},\beta_{w0})\\ I_c(t,t_0;\alpha_{w0},\beta_{w0})\end{pmatrix}. \tag{3.7}$$

We simply substitute $\varepsilon=0$ into equation (3.7) and obtain the solution of the harmonic oscillator $w''+mw=0$ as follows:

$$x_w(t)\big|_{\varepsilon=0}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}U_{t_0-t}(\sqrt{m}\,)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\widehat{\alpha_{w0}}\\ \widehat{\beta_{w0}}\end{pmatrix}=$$

$$=U_{t-t_0}(\sqrt{m})\begin{pmatrix}\widehat{\alpha_{w0}}\\ \widehat{\beta_{w0}}\end{pmatrix}. \tag{3.8}$$

Here, $\widehat{\alpha_{w0}}$ and $\widehat{\beta_{w0}}$ are arbitrary initial values, that are independent of ε and not necessarily equal to α_{w0} and β_{w0}, respectively. We establish that the period τ of a non-trivial periodic solution of equation (3.8) is $\tau=2\pi/\sqrt{m}$ from $\det\big(U_\tau(\sqrt{m})-1\big)=0$.

Theorem 3.1. *Suppose that* $\lim_{t\to\infty}e^{-\varepsilon t/2}x_w(t)=0$. *Then*

$$\lim_{t\to\infty}\begin{pmatrix}I_s(t,t_0;\alpha_{w0},\beta_{w0})\\ I_c(t,t_0;\alpha_{w0},\beta_{w0})\end{pmatrix}=U_{t_0}(\vartheta)\begin{pmatrix}-1/\varepsilon & 0\\ -1/2 & 1/\varepsilon\end{pmatrix}\begin{pmatrix}\alpha_{w0}\\ \beta_{w0}\end{pmatrix}.$$

Proof. The above equation follows from directly equation (3.6).

Before stating the next theorem, we prepare the following proposition.

Proposition 3.1 (the property of autonomous systems) (for example, see [14]). *The following statements are equivalent*:

(1) *there exists* $\tau > 0$ *such that* $x_w(t_0 + \tau) = x_w(t_0)$ *for some* t_0,

(2) *there exists* $\tau > 0$ *such that* $x_w(t + \tau) = x_w(t)$ *for any* t.

Without warning, we often use this nature hereinafter.

Theorem 3.2. *Let* $x_w(t)$ *be a solution of* $W_{\varepsilon,m,\phi}$. *Then the following statements are equivalent*:

(i) *for some* t_0, *one has*

$$\begin{pmatrix} I_s(t_0+\tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t_0+\tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix} =$$

$$= U_{t_0}(\vartheta)\left(1 - e^{-\varepsilon\tau/2}U_\tau(\vartheta)\right)\begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}\begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix}; \tag{3.9}$$

(ii) *for some* t_0, *one has*

$$\begin{pmatrix} I_s(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix} = U_{t_0}(\vartheta)\left(1 - e^{-\varepsilon\tau/2}U_\tau(\vartheta)\right)\begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}\begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix} +$$

$$+ e^{-\varepsilon\tau/2}U_\tau(\vartheta)\begin{pmatrix} I_s(t, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix} \quad \text{for any} \quad t; \tag{3.10}$$

(iii) $x_w(t)$ *is periodic with period* τ.

Proof. (ii)⇒(i). If we set $t = t_0$ in equation (3.10), we obtain equation (3.9).

(i)⇒(iii). We assume that equation (3.9) is satisfied. Setting $t = t_0 + \tau$ in equation (3.6) and using equation (3.9), we obtain $x_w(t_0 + \tau) = x_w(t_0)$. Therefore, by Proposition 3.1, we have $x_w(t + \tau) = x_w(t)$.

(iii)⇒(ii). The substitution $t + \tau$ instead of t in equation (3.6) leads to

$$e^{-\varepsilon(t+\tau-t_0)/2}U_{t+\tau-t_0}(\vartheta)\begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} x_w(t+\tau) =$$

$$= \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}\begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix} - U_{-t_0}(\vartheta)\begin{pmatrix} I_s(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix}. \tag{3.11}$$

Assuming $x_w(t+\tau) = x_w(t)$ and multiplying both sides of equation (3.11) by $e^{\varepsilon\tau/2}U_{-\tau}(\vartheta)$, we have

$$e^{-\varepsilon(t-t_0)/2}U_{t-t_0}(\vartheta)\begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} x_w(t) =$$

$$= e^{\varepsilon\tau/2}U_{-\tau}(\vartheta)\begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}\begin{pmatrix} \alpha_{w0} \\ \beta_{w0} \end{pmatrix} - e^{\varepsilon\tau/2}U_{-\tau-t_0}(\vartheta)\begin{pmatrix} I_s(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \\ I_c(t+\tau, t_0; \alpha_{w0}, \beta_{w0}) \end{pmatrix}. \tag{3.12}$$

The equation of both right-hand sides of equations (3.6) and (3.12) yields equation (3.10).

Theorem is proved.

Here, we set $m = 1$ and $\phi = w^2 w'$, that is, the ordinary van der Pol equation is considered. Then we give the next proposition, which states how the orbit of $\varepsilon \to 0$ shapes. This problem was studied in [1, 4], where the periodic orbit was obtained. For example, in [15], this orbit was reduced under the assumption that the requirements of the Poincaré expansion theorem are satisfied. However, we give a new proof without this assumption.

Proposition 3.2. *Let the periodic solution of the van der Pol equation be* $w(t,\varepsilon)$. *The orbit of* $W_{\varepsilon,1,w^2w'}$ (*the van der Pol equation*) *as* $\varepsilon \to 0$ *is presented by*

$$w^2(t,0) + w'^2(t,0) = 4,$$

where $w(t,0) = \lim_{\varepsilon\to 0} w(t,\varepsilon)$. *The period of* $W_{\varepsilon,1,w^2w'}$, *which is denoted by* $\tau(\varepsilon)$, *is represented as*

$$\tau(\varepsilon) = 2\pi + o(\varepsilon). \tag{3.13}$$

Proof. We rewrite equation (3.9) for the peridic condition of the van der Pol equation as follows:

$$U_{t_0}(\theta)\Big(1 - e^{-\varepsilon\tau(\varepsilon)/2} U_{\tau(\varepsilon)}(\theta)\Big)\begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix}\begin{pmatrix} w(t_0,\varepsilon) \\ w'(t_0,\varepsilon)\end{pmatrix} =$$

$$= \begin{pmatrix} I_s(t_0+\tau(\varepsilon), t_0; w(t_0,\varepsilon), w'(t_0,\varepsilon)) \\ I_c(t_0+\tau(\varepsilon), t_0; w(t_0,\varepsilon), w'(t_0,\varepsilon))\end{pmatrix}, \tag{3.14}$$

where $\theta = \dfrac{\sqrt{4-\varepsilon^2}}{2}$. In the van der Pol equation, the solution w and its period τ depend on the paremater ε so that we denote them by $w(t,\varepsilon)$ and $\tau(\varepsilon)$, respectively. From equation (3.14), we obtain

$$\left\{\frac{\cos\theta t_0}{\varepsilon} - e^{-(\varepsilon/2)\tau(\varepsilon)}\frac{\cos\theta\big(t_0+\tau(\varepsilon)\big)}{\varepsilon} - \right.$$

$$\left. -\frac{1}{2}\left(e^{-(\varepsilon/2)\tau(\varepsilon)}\frac{\sin\theta\big(t_0+\tau(\varepsilon)\big)}{\theta} - \frac{\sin\theta t_0}{\theta}\right)\right\} w(t_0,\varepsilon) +$$

$$+ \left\{e^{-(\varepsilon/2)\tau(\varepsilon)}\frac{\sin\theta\big(t_0+\tau(\varepsilon)\big)}{\theta\varepsilon} - \frac{\sin\theta t_0}{\theta\varepsilon}\right\} w'(t_0,\varepsilon) =$$

$$= -\int_{t_0}^{t_0+\tau(\varepsilon)} e^{-\varepsilon(s-t_0)/2}\frac{\sin\theta s}{\theta}\big(w^2(s,\varepsilon)w'(s,\varepsilon)\big)ds,$$

$$\left\{-\frac{\theta\sin\theta t_0}{\varepsilon} + e^{-(\varepsilon/2)\tau(\varepsilon)}\frac{\theta\sin\theta\big(t_0+\tau(\varepsilon)\big)}{\varepsilon} + \right.$$

$$+\frac{1}{2}\Big(-e^{(\varepsilon/2)\tau(\varepsilon)}\cos\theta\big(t_0+\tau(\varepsilon)\big)+\cos\theta t_0\Big)\Bigg\}w(t_0,\varepsilon)+$$

$$+\left\{e^{-(\varepsilon/2)\tau(\varepsilon)}\frac{\cos\theta\big(t_0+\tau(\varepsilon)\big)}{\varepsilon}-\frac{\cos\theta t_0}{\varepsilon}\right\}w'(t_0,\varepsilon)=$$

$$=-\int\limits_{t_0}^{t_0+\tau(\varepsilon)} e^{-\varepsilon(s-t_0)/2}\cos\theta s\big(w^2(s,\varepsilon)w'(s,\varepsilon)\big)ds.$$

We assume that $\varepsilon\to 0$ in the above equation. Let $\tau_1=\left.\dfrac{d\tau(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0}$. Note that $w(t,0)=$ $=\cos(t_0-t)w(t_0,0)-\sin(t_0-t)w'(t_0,0)$ and $\theta=\dfrac{\sqrt{4-\varepsilon^2}}{2}\to 1$, $\tau(\varepsilon)\to 2\pi$, $\varepsilon\to 0$. Then we have

$$(\pi\cos t_0+\tau_1\sin t_0)w(t_0,0)+(-\pi\sin t_0+\tau_1\cos t_0)w'(t_0,0)=$$

$$=-\frac{\pi}{4}\big(\sin t_0 w'(t_0,0)-\cos t_0 w(t_0,0)\big)\big(w^2(t_0,0)+w'^2(t_0,0)\big),$$

$$(-\pi\sin t_0+\tau_1\cos t_0)w(t_0,0)+(-\pi\cos t_0-\tau_1\sin t_0)w'(t_0,0)=$$

$$=-\frac{\pi}{4}\big(\sin t_0 w(t_0,0)+\cos t_0 w'(t_0,0)\big)\big(w^2(t_0,0)+w'^2(t_0,0)\big).$$

From the above relations, we have

$$w^2(t_0,0)+w'^2(t_0,0)=4,\quad \tau_1=0.$$

Since t_0 is an arbitrary initial time, we finally obtain the orbit as $\varepsilon\to 0$ considered in the proposition. We also have equation (3.13) from $\tau_1=0$.

Proposition is proved.

Remark **3.2.** Proposition 3.2 is consistent with the earlier results (see, for example, [4, p. 104] and [15, p. 133]).

4. Analysis of the coupled van der Pol equation system. ***4.1. Formation of the fundamental equations for the analysis.*** We now set

$$y(t_0)=\alpha_0,\qquad y'(t_0)=\beta_0,\qquad z(t_0)=\lambda_0,\qquad z'(t_0)=\mu_0,$$

and define some new symbols as follows:

$$\phi_\pm(t):=y^2(t)y'(t)\pm z^2(t)z'(t),$$

$$\theta_+:=\frac{\sqrt{4-\varepsilon^2}}{2},$$

$$\theta_-:=\frac{\sqrt{4-\varepsilon^2-8k}}{2},$$

$$I_s^\pm(t,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0):=$$

$$:=\int\limits_{t_0}^{t} e^{\varepsilon(t-s)/2}\frac{\sin\big(\theta_\pm(t-s)\big)}{\theta_\pm}\phi_\pm(s;t_0,\alpha_0,\beta_0,\lambda_0,\mu_0)ds,$$

$$I_c^{\pm}(t,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) :=$$

$$:= \int\limits_{t_0}^{t} e^{\varepsilon(t-s)/2}\cos\big(\theta_\pm(t-s)\big)\phi_\pm(s;t_0,\alpha_0,\beta_0,\lambda_0,\mu_0)ds,$$

where a double sign $\pm$ in equations corresponds in order.

Let $x_+(t) = y(t) + z(t)$ for $y, z \in \mathbf{\Sigma}_{\varepsilon,k}$. Then we have the differential equation $W_{\varepsilon,1,\phi_+}$ corresponding to equation (3.1) of the previous section, that is,

$$W_{\varepsilon,1,\phi_+} : x_+'' - \varepsilon(x_+' - \phi_+) + x_+ = 0.$$

We again define other symbols as follows:

$$I_{s\pm}(t,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) :=$$

$$:= \int\limits_{t_0}^{t} e^{-\varepsilon(s-t_0)/2}\frac{\sin\big(\theta_\pm s\big)}{\theta_\pm}\phi_\pm(s;t_0,\alpha_0,\beta_0,\lambda_0,\mu_0)ds,$$

$$I_{c\pm}(t,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) :=$$

$$:= \int\limits_{t_0}^{t} e^{-\varepsilon(s-t_0)/2}\cos\big(\theta_\pm s\big)\phi_\pm(s;t_0,\alpha_0,\beta_0,\lambda_0,\mu_0)ds.$$

Before obtaining the fundamental equations for the analysis, we prepare the next lemma.

Lemma 4.1. *The following relation is true:*

$$\begin{pmatrix} I_{s\pm}(t,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) \\ I_{c\pm}(t,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) \end{pmatrix} = U_{t_0}(\theta_\pm) \begin{pmatrix} I_{s\pm}(t-t_0,0;\alpha_0,\beta_0,\lambda_0,\mu_0) \\ I_{c\pm}(t-t_0,0;\alpha_0,\beta_0,\lambda_0,\mu_0) \end{pmatrix}.$$

Proof. For $I_{s\pm}$, we have

$$I_{s\pm}(t,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) =$$

$$= \int\limits_{t_0}^{t} e^{-\varepsilon(s-t_0)/2}\frac{\sin(\theta_\pm s)}{\theta_\pm}\phi_\pm(s;t_0,\alpha_0,\beta_0,\lambda_0,\mu_0)ds =$$

$$= \int\limits_{0}^{t-t_0} e^{-\varepsilon s'/2}\frac{\sin\big(\theta_\pm(s'+t_0)\big)}{\theta_\pm}\phi_\pm(s'+t_0;t_0,\alpha_0,\beta_0,\lambda_0,\mu_0)ds' =$$

(by virtue of the property of autonoumous systems)

$$= \int\limits_{0}^{t-t_0} e^{-\varepsilon s'/2}\frac{\sin\big(\theta_\pm(s'+t_0)\big)}{\theta_\pm}\phi_\pm(s';0,\alpha_0,\beta_0,\lambda_0,\mu_0)ds' =$$

$$= \cos(\theta_\pm t_0) \int\limits_0^{t-t_0} e^{-\varepsilon s/2} \frac{\sin(\theta_\pm s)}{\theta_\pm} \phi_\pm(s; 0, \alpha_0, \beta_0, \lambda_0, \mu_0) ds +$$

$$+ \frac{\sin(\theta_\pm t_0)}{\theta_\pm} \int\limits_0^{t-t_0} e^{-\varepsilon s/2} \cos(\theta_\pm s) \phi_\pm(s; 0, \alpha_0, \beta_0, \lambda_0, \mu_0) ds =$$

$$= \cos(\theta_\pm t_0)\ I_{s\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) +$$

$$+ \frac{\sin(\theta_\pm t_0)}{\theta_\pm}\ I_{c\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0). \tag{4.1}$$

For $I_{c\pm}$, we have

$$I_{c\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) = -\theta_\pm \sin(\theta_\pm t_0)\ I_{s\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) +$$

$$+ \cos(\theta_\pm t_0)\ I_{c\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0). \tag{4.2}$$

From equations (4.1) and (4.2) we obtain

$$\begin{pmatrix} I_{s\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix} =$$

$$= \begin{pmatrix} \cos(\theta_\pm t_0) & \dfrac{\sin(\theta_\pm t_0)}{\theta_\pm} \\ -\theta_\pm \sin(\theta_\pm t_0) & \cos(\theta_\pm t_0) \end{pmatrix} \begin{pmatrix} I_{s\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}.$$

Using the definition of the rotational matrix U, we prove the lemma.

As the fundamental equation for $x_+(t)$, that is, $y(t) + z(t)$, we have the following linear system of integral equations using integral symbols defined above, which corresponds to equation (3.6):

$$e^{-\varepsilon(t-t_0)/2} U_{t-t_0}(\theta_+) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_+(t) \\ x'_+(t) \end{pmatrix} =$$

$$= \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_+(t_0) \\ x'_+(t_0) \end{pmatrix} - U_{-t_0}(\theta_+) \begin{pmatrix} I_{s+}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c+}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}. \tag{4.3}$$

By applying Lemma 4.1 to the above equation, we obtain

$$e^{-\varepsilon(t-t_0)/2} U_{t-t_0}(\theta_+) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_+(t) \\ x'_+(t) \end{pmatrix} =$$

$$= \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_+(t_0) \\ x'_+(t_0) \end{pmatrix} - \begin{pmatrix} I_{s+}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c+}(t - t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}. \tag{4.4}$$

If we set $x_-(t) = y(t) - z(t)$ for $y, z \in \mathbf{\Sigma}_{\varepsilon,k}$, then we obtain $W_{\varepsilon,1-2k,\phi_-}$, that is,

$$W_{\varepsilon,1-2k,\phi_-} : x''_- - \varepsilon(x'_- - \phi_-) + (1-2k)x_- = 0.$$

In the same way, we obtain the linear system of integral equations for $x_-(t) = y(t) - - z(t)$:

$$e^{-\varepsilon(t-t_0)/2} U_{t-t_0}(\theta_-) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_-(t) \\ x'_-(t) \end{pmatrix} =$$

$$= \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_-(t_0) \\ x'_-(t_0) \end{pmatrix} - \begin{pmatrix} I_{s-}(t-t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c-}(t-t_0, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}.$$

4.2. Necessary and sufficient condition for the periodicity of the coupled van der Pol equation system. We give the necessary and sufficient condition for the periodicity of the solutions of the coupled van der Pol equation system in this subsection. First, the following theorem holds in the same way as Theorem 3.1.

Theorem 4.1. *Suppose that* $\lim_{t\to\infty} e^{-\varepsilon t/2} \mathrm{col}\big(x_\pm(t), x'_\pm(t)\big) = 0$. *Then*

$$\lim_{t\to\infty} \begin{pmatrix} I_{s\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix} = U_{t_0}(\theta_\pm) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_\pm(t_0) \\ x'_\pm(t_0) \end{pmatrix}.$$

In this theorem, a double sign $\pm$ corresponds in order.

Proof. Omitted.

Below, we state some properties for the case where the system has the periodicity. Remember that $\xi_\Sigma(t) = \mathrm{col}\big(y(t), y'(t), z(t), z'(t)\big)$.

Theorem 4.2. *Suppose that* $\xi_\Sigma(t+\tau) = \xi_\Sigma(t)$, *then the following relations are equivalent for a fixed* t_0:

(i) $x_+(t_0) = 0,\ x'_+(t_0) = 0$;

(ii) $I_{s+}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) = 0,\ I_{c+}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) = 0,\ n = = 1, 2, \ldots$.

Proof. (i) $\Rightarrow$ (ii). Substituting $t = t_0 + n\tau$ into equation (4.3), we obtain

$$e^{-\varepsilon n\tau/2} U_{n\tau}(\theta_+) \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_+(t_0 + n\tau) \\ x'_+(t_0 + n\tau) \end{pmatrix} =$$

$$= \begin{pmatrix} -1/\varepsilon & 0 \\ -1/2 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} x_+(t_0) \\ x'_+(t_0) \end{pmatrix} - U_{-t_0}(\theta_+) \begin{pmatrix} I_{s+}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c+}(t_0 + n\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}.$$

From the assumption of the theorem, we have

$$\begin{pmatrix} y(t_0 + n\tau) \\ y'(t_0 + n\tau) \\ z(t_0 + n\tau) \\ z'(t_0 + n\tau) \end{pmatrix} = \begin{pmatrix} y(t_0) \\ y'(t_0) \\ z(t_0) \\ z'(t_0) \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \beta_0 \\ \lambda_0 \\ \mu_0 \end{pmatrix}.$$

The substitution of this result yields

$$e^{-\varepsilon n\tau/2}U_{n\tau}(\theta_+)\begin{pmatrix}-1/\varepsilon & 0\\ -1/2 & 1/\varepsilon\end{pmatrix}\begin{pmatrix}x_+(t_0)\\ x'_+(t_0)\end{pmatrix} =$$

$$= \begin{pmatrix}-1/\varepsilon & 0\\ -1/2 & 1/\varepsilon\end{pmatrix}\begin{pmatrix}x_+(t_0)\\ x'_+(t_0)\end{pmatrix} -$$

$$- U_{-t_0}(\theta_+)\begin{pmatrix}I_{s+}(t_0+n\tau, t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c+}(t_0+n\tau, t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}. \tag{4.5}$$

Using the relation from (i), that is, $x_+(t_0) = 0$ and $x'_+(t_0) = 0$, we arrive at (ii).

(ii) $\Rightarrow$ (i). The substitution of (ii) into equation (4.5) leads to

$$x_+(t_0) = 0,$$
$$x'_+(t_0) = 0,$$

which means that (i) holds.

In the same manner, we obtain the next theorem.

Theorem 4.3. *Suppose that $\xi_\Sigma(t+\tau) = \xi_\Sigma(t)$. Then the following relations are equivalent for a fixed t_0:*

(i) $x_-(t_0) = 0$, $x'_-(t_0) = 0$;

(ii) $I_{s-}(t_0+n\tau, t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) = 0$, $I_{c-}(t_0+n\tau, t_0;\alpha_0,\beta_0,\lambda_0,\mu_0) = 0$, $n = 1, 2, \ldots$.

Proof. We can prove this theorem by the same manner as Theorem 4.2.

Lemma 4.2. *The following relations are equivalent*:

(i) $\xi_\Sigma(t+\tau) = \xi_\Sigma(t)$,

(ii) $x_\pm(t+\tau) = x_\pm(t)$.

Theorem 4.4 (necessary and sufficient condition for the periodicity). *The solution of the dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$ with the initial condition*

$$y(t_0) = \alpha_0, \qquad y'(t_0) = \beta_0, \qquad z(t_0) = \lambda_0, \qquad z'(t_0) = \mu_0$$

has a period τ if and only if

$$\mathbf{F}_\pm(\varepsilon) = 0, \tag{4.6}$$

where

$$\mathbf{F}_{\pm}(\varepsilon) = \Big(1 - e^{-\varepsilon\tau/2} U_\tau(\theta_\pm)\Big) \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_\pm(t_0) \\ x'_\pm(t_0) \end{pmatrix} +$$

$$+\varepsilon \begin{pmatrix} I_{s\pm}(\tau, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(\tau, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}. \tag{4.7}$$

Proof. *Necessity*. $\mathbf{\Sigma}_{\epsilon,k}$ has a period, that is, $\xi_\Sigma(t+\tau) = \xi_\Sigma(t)$ for some $\tau > 0$, because $x_\pm(t+\tau) = x_\pm(t)$ and $x'_\pm(t+\tau) = x'_\pm(t)$ from Lemma 4.2. Therefore, we have the following equation by the same procedure which yields equation (3.9):

$$\Big(1 - e^{-\varepsilon\tau/2} U_\tau(\theta_\pm)\Big) \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_\pm(t_0) \\ x'_\pm(t_0) \end{pmatrix} +$$

$$+ \varepsilon U_{-t_0}(\theta_\pm) \begin{pmatrix} I_{s\pm}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix} = 0. \tag{4.8}$$

The second term is computed by Lemma 4.1 as

$$U_{-t_0}(\theta_\pm) \begin{pmatrix} I_{s\pm}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix} = \begin{pmatrix} I_{s\pm}(\tau, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(\tau, 0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}, \tag{4.9}$$

and the substitution of this result into eqaution (4.8) leads to equations (4.6) and (4.7).

Sufficiency. Here, we prove that $\mathbf{F}_\pm = 0 \Rightarrow x_\pm(t_0+\tau) = x_\pm(t_0)$, which is equivalent to $x_\pm(t+\tau) = x_\pm(t)$. Using equation (4.9) in equation (4.7), we have

$$e^{-\varepsilon\tau/2} U_{t_0+\tau}(\theta_\pm) \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_\pm(t_0) \\ x'_\pm(t_0) \end{pmatrix} =$$

$$= U_{t_0}(\theta_\pm) \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_\pm(t_0) \\ x'_\pm(t_0) \end{pmatrix} +$$

$$+\varepsilon \begin{pmatrix} I_{s\pm}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c\pm}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix}. \tag{4.10}$$

On the other hand, the substitution of $t = t_0 + \tau$ into equation (4.3) yields

$$e^{-\varepsilon\tau/2} U_\tau(\theta_+) \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_+(t_0+\tau) \\ x'_+(t_0+\tau) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_+(t_0) \\ x'_+(t_0) \end{pmatrix} +$$

$$+\varepsilon U_{-t_0}(\theta_+) \begin{pmatrix} I_{s+}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \\ I_{c+}(t_0+\tau, t_0; \alpha_0, \beta_0, \lambda_0, \mu_0) \end{pmatrix},$$

that is,

$$e^{-\varepsilon\tau/2}U_{t_0+\tau}(\theta_+)\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}\begin{pmatrix}x_+(t_0+\tau)\\ x'_+(t_0+\tau)\end{pmatrix}=$$

$$=U_{t_0}(\theta_+)\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}\begin{pmatrix}x_+(t_0)\\ x'_+(t_0)\end{pmatrix}+$$

$$+\varepsilon\begin{pmatrix}I_{s+}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c+}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}. \qquad (4.11)$$

Similarly, we have

$$e^{-\varepsilon\tau/2}U_{t_0+\tau}(\theta_-)\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}\begin{pmatrix}x_-(t_0+\tau)\\ x'_-(t_0+\tau)\end{pmatrix}=$$

$$=U_{t_0}(\theta_-)\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}\begin{pmatrix}x_-(t_0)\\ x'_-(t_0)\end{pmatrix}+$$

$$+\varepsilon\begin{pmatrix}I_{s-}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c-}(t_0+\tau,t_0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}. \qquad (4.12)$$

The subtraction of equation (4.10) from equations (4.11) and (4.12) leads to

$$e^{-\varepsilon\tau/2}U_{t_0+\tau}(\theta_\pm)\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}\left\{\begin{pmatrix}x_\pm(t_0+\tau)\\ x'_\pm(t_0+\tau)\end{pmatrix}-\begin{pmatrix}x_\pm(t_0)\\ x'_\pm(t_0)\end{pmatrix}\right\}=0.$$

Therefore, we obtain

$$x_\pm(t_0+\tau)=x_\pm(t_0),$$
$$x'_\pm(t_0+\tau)=x'_\pm(t_0).$$

Consequently, $\mathbf{F}_\pm(\varepsilon)=0\Rightarrow x_\pm(t_0+\tau)=x_\pm(t_0)$ is proved.

Theorem is proved.

5. Non-existence theorem of periodic solutions except the out-of-phase and in-phase solutions in $\boldsymbol{\Sigma_{\varepsilon,k}}$. Let $y=y(t,\varepsilon)$ and $z=z(t,\varepsilon)$ be two real-valued functions depending on the parameter ε and $0<\varepsilon<2$, $0<k<\dfrac{1}{2}-\dfrac{\varepsilon^2}{8}$. Our objective equation system $\mathbf{\Sigma}_{\varepsilon,k}$ is as follows:

$$\mathbf{\Sigma}_{\varepsilon,k}\begin{cases}y''-\varepsilon(1-y^2)y'+y=k(y-z),\\ z''-\varepsilon(1-z^2)z'+z=k(z-y), \quad t_0\le t,\end{cases}$$

with the initial condition

$$y(t_0,\varepsilon)=\alpha_0(\varepsilon),\qquad y'(t_0,\varepsilon)=\beta_0(\varepsilon),$$
$$z(t_0,\varepsilon)=\lambda_0(\varepsilon),\qquad z'(t_0,\varepsilon)=\mu_0(\varepsilon),$$

where the initial condition also depends on the parameter ε because we write $\alpha_0(\varepsilon)$, $\beta_0(\varepsilon)$, $\lambda_0(\varepsilon)$ and $\mu_0(\varepsilon)$ deliberately.

Here, we give the assumption on periodic solutions of the dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$.

Assumption 5.1 (periodic solutions of $\mathbf{\Sigma}_{\varepsilon,k}$). *Periodic solutions of $\mathbf{\Sigma}_{\varepsilon,k}$ satisfy*

$$y\big(t+\tau(\varepsilon),\varepsilon\big)=y(t,\varepsilon),\qquad z\big(t+\tau(\varepsilon),\varepsilon\big)=z(t,\varepsilon),\qquad |\tau(\varepsilon)|<T, \tag{5.1}$$

where τ indicates a period of $\mathbf{\Sigma}_{\varepsilon,k}$ and T is independent of the parameter ε. Moreover, periodic solutions and their derivatives satisfy

$$|y(t,\varepsilon)|<M,\qquad |y'(t,\varepsilon)|<M,\qquad |z(t,\varepsilon)|<M,\qquad |z'(t,\varepsilon)|<M, \tag{5.2}$$

where M is independent of the parameter ε and t.

Hereinafter, we consider only periodic solutions restricted by Assumption 5.1. Before stating the main theorem, we prepare the following lemma.

Lemma 5.1. *Let $y(t,\varepsilon), z(t,\varepsilon)$ be a periodic solution of $\mathbf{\Sigma}_{\varepsilon,\mathbf{k}}$ satisfying Assumption* 5.1. *Assume that there exists $\lim\limits_{\varepsilon\to 0}x_\pm(t_0,\varepsilon)$ and $\lim\limits_{\varepsilon\to 0}x_\pm(t_0,\varepsilon)=x_\pm(t_0,0)$. Then there exists a solution $y(t)$ and $z(t)$ of the degenerated system $\mathbf{\Sigma}_{0,k}$ such that $\lim\limits_{\varepsilon\to 0}x_\pm(t,\varepsilon)=x_\pm(t,0)=y(t)\pm z(t)$ and $\lim\limits_{\varepsilon\to 0}x'_\pm(t,\varepsilon)=x'_\pm(t,0)=y'(t)\pm z'(t)$. Let $\tau_\pm(\varepsilon)$ and $\tau_\pm(0)$ be periods of $x_\pm(t,\varepsilon)$ created by $\mathbf{\Sigma}_{\varepsilon,k}$ and $x_\pm(t,0)$ by $\mathbf{\Sigma}_{0,k}$, respectively. Then $\lim\limits_{\varepsilon\to 0}\tau_\pm(\varepsilon)=\tau_\pm(0)$.*

Proof. We only show that $\lim\limits_{\varepsilon\to 0}x_+(t,\varepsilon)=x_+(t,0)=y(t)+z(t)$ and $\lim\limits_{\varepsilon\to 0}\tau_+(\varepsilon)=$ $=\tau_+(0)$. From equation (4.4), $x_+(t,\varepsilon)$ and $x'_+(t,\varepsilon)$ are represented as

$$\begin{pmatrix}x_+(t,\varepsilon)\\ x'_+(t,\varepsilon)\end{pmatrix}=e^{\varepsilon(t-t_0)/2}\begin{pmatrix}1&0\\ \varepsilon/2&-1\end{pmatrix}^{-1}\times$$
$$\times U_{t_0-t}(\theta_+)\begin{pmatrix}1&0\\ \varepsilon/2&-1\end{pmatrix}\begin{pmatrix}x_+(t_0,\varepsilon)\\ x'_+(t_0,\varepsilon)\end{pmatrix}+$$
$$+\varepsilon e^{\varepsilon(t-t_0)/2}\begin{pmatrix}1&0\\ \varepsilon/2&-1\end{pmatrix}^{-1}U_{t_0-t}(\theta_+)\begin{pmatrix}I_{s+}(t-t_0,0;\alpha_0,\beta_0,\lambda_0,\mu_0)\\ I_{c+}(t-t_0,0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}=$$
$$=e^{\varepsilon(t-t_0)/2}\begin{pmatrix}\cos\theta_+(t_0-t)+\dfrac{\varepsilon}{2}\dfrac{\sin\theta_+(t_0-t)}{\theta_+} & -\dfrac{\sin\theta_+(t_0-t)}{\theta_+}\\ \theta_+\sin\theta_+(t_0-t)+\dfrac{\varepsilon^2}{4}\dfrac{\sin\theta_+(t_0-t)}{\theta_+} & \cos\theta_+(t_0-t)-\dfrac{\varepsilon}{2}\dfrac{\sin\theta_+(t_0-t)}{\theta_+}\end{pmatrix}\times$$
$$\times\begin{pmatrix}x_+(t_0,\varepsilon)\\ x'_+(t_0,\varepsilon)\end{pmatrix}+\varepsilon e^{\varepsilon(t-t_0)/2}\begin{pmatrix}1&0\\ \varepsilon/2&-1\end{pmatrix}^{-1}\times$$

$$\times U_{t_0-t}(\theta_+)\begin{pmatrix} I_{s+}(t-t_0,0;\alpha_0,\beta_0,\lambda_0,\mu_0) \\ I_{c+}(t-t_0,0;\alpha_0,\beta_0,\lambda_0,\mu_0)\end{pmatrix}.$$

We take $\varepsilon \to 0$ in both sides of the above equation. By virtue of Assumption 5.1, i.e., by virtue of the relations $|y(t,\varepsilon)| < M$, $|y'(t,\varepsilon)| < M$, $|z(t,\varepsilon)| < M$, and $|z'(t,\varepsilon)| < M$, the second term vanishes. Therefore, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0} x_+(t,\varepsilon) = x_+(t,0) = \cos(t_0-t)x_+(t_0,0) - \sin(t_0-t)x'_+(t_0,0), \\ &\lim_{\varepsilon\to 0} x'_+(t,\varepsilon) = x'_+(t,0) = \sin(t_0-t)x_+(t_0,0) + \cos(t_0-t)x'_+(t_0,0). \end{aligned} \tag{5.3}$$

In the same manner, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} x_-(t,\varepsilon) &= x_-(t,0) = \cos\sqrt{1-2k}(t_0-t)x_-(t_0,0) - \\ &\quad - \frac{\sin\sqrt{1-2k}(t_0-t)}{\sqrt{1-2k}}x'_-(t_0,0), \\ \lim_{\varepsilon\to 0} x'_-(t,\varepsilon) &= x'_-(t,0)\sqrt{1-2k}\sin\sqrt{1-2k}(t_0-t)x_-(t_0,0) + \\ &\quad + \cos\sqrt{1-2k}(t_0-t)x'_-(t_0,0). \end{aligned} \tag{5.4}$$

From equations (5.3) and (5.4), we construct y and z as follows:

$$y(t) = \frac{x_+(t,0)+x_-(t,0)}{2}, \qquad z(t) = \frac{x_+(t,0)-x_-(t,0)}{2}. \tag{5.5}$$

We easily find that y and z satisfy $\mathbf{\Sigma}_{0,k}$. From equations (5.3), (5.4) and (5.5), we obtain

$$\begin{aligned} &\lim_{\varepsilon\to 0} x_\pm(t,\varepsilon) = x_\pm(t,0) = y(t) \pm z(t), \\ &\lim_{\varepsilon\to 0} x'_\pm(t,\varepsilon) = x'_\pm(t,0) = y'(t) \pm z'(t). \end{aligned} \tag{5.6}$$

Furthermore, using the assumption on periodic solutions, we have

$$x_\pm(t+\tau_\pm(\varepsilon),\varepsilon) = x_\pm(t,\varepsilon), \tag{5.7}$$

$$x_\pm(t+\tau_\pm(0),0) = x_\pm(t,0). \tag{5.8}$$

From equations (5.6), (5.7) and (5.8), we get

$$\lim_{\varepsilon\to 0}\tau_\pm(\varepsilon) = \tau_\pm(0).$$

Also we obtain $\tau_+(0) = 2\pi$ and $\tau_-(0) = \dfrac{2\pi}{\sqrt{1-2k}}$.

We give the next main theorem for $\mathbf{\Sigma}_{\varepsilon,k}$.

Theorem 5.1 (non-existence of periodic solutions except the out-of-phase and in-phase solutions)**.** *Let $y(t,\varepsilon)$ and $z(t,\varepsilon)$ be a periodic solution of $\mathbf{\Sigma}_{\varepsilon,k}$, which is analytic with respect to ε on the segment $[0,\varepsilon_0)$, where $0<\varepsilon_0<2$, $0<k<\dfrac{1}{2}-\dfrac{\varepsilon_0^2}{8}$, and k is irrational. Then this solution is either out-of-phase or in-phase.*

Preparations for the proof. We assume that the periodicity is built up and a period (but unknown) is $\tau(\varepsilon)$ depending on ε. Then we have the following relation from Theorem 4.4:

$$\mathbf{F}_\pm(\varepsilon) = 0,$$

where

$$\mathbf{F}_\pm(\varepsilon) = \Big(1 - e^{-\varepsilon\tau(\varepsilon)/2} U_{\tau(\varepsilon)}(\theta_\pm)\Big) \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_\pm(t_0,\varepsilon) \\ x'_\pm(t_0,\varepsilon) \end{pmatrix} +$$

$$+\varepsilon \begin{pmatrix} I_{s\pm}(\tau(\varepsilon), 0; \alpha_0(\varepsilon), \beta_0(\varepsilon), \lambda_0(\varepsilon), \mu_0(\varepsilon)) \\ I_{c\pm}(\tau(\varepsilon), 0; \alpha_0(\varepsilon), \beta_0(\varepsilon), \lambda_0(\varepsilon), \mu_0(\varepsilon)) \end{pmatrix}.$$

First, we take $\varepsilon \to 0$ in $\mathbf{F}_+(\varepsilon) = 0$. Then we have

$$\begin{pmatrix} 1 - \cos(\tau(0)) & \sin(\tau(0)) \\ -\sin(\tau(0)) & 1 - \cos(\tau(0)) \end{pmatrix} \begin{pmatrix} x_+(t_0, 0) \\ x'_+(t_0, 0) \end{pmatrix} = 0. \tag{5.9}$$

Here, $\tau(0) = \lim_{\varepsilon\to 0} \tau(\varepsilon)$.

On the other hand, taking $\varepsilon \to 0$ in $\mathbf{F}_-(\varepsilon) = 0$, we have

$$\begin{pmatrix} 1 - \cos(\sqrt{1-2k}\,\tau(0)) & \dfrac{\sin(\sqrt{1-2k}\,\tau(0))}{\sqrt{1-2k}} \\ -\sqrt{1-2k}\sin(\sqrt{1-2k}\,\tau(0)) & 1 - \cos(\sqrt{1-2k}\,\tau(0)) \end{pmatrix} \begin{pmatrix} x_-(t_0, 0) \\ x'_-(t_0, 0) \end{pmatrix} = 0. \tag{5.10}$$

Equations (5.9) and (5.10) must hold simultaneously because we have the following results for each t_0:

(i) Equation (5.9) implies $\begin{pmatrix} x_+(t_0, 0) \\ x'_+(t_0, 0) \end{pmatrix} = 0$ or $\tau(0) = 2\pi$. In the latter case, we set $\tau_-(0) = 2\pi$ for the sake of convenience.

(ii) Similarly, equation (5.10) implies $\begin{pmatrix} x_-(t_0, 0) \\ x'_-(t_0, 0) \end{pmatrix} = 0$ or $\tau(0) = \dfrac{2\pi}{\sqrt{1-2k}}$. In the latter case, we set $\tau_+(0) = \dfrac{2\pi}{\sqrt{1-2k}}$ for the sake of convenience.

(iii) If k is irrational and satisfies $0 < k < \dfrac{1}{2} - \dfrac{\varepsilon^2}{8}$, then $j\tau_+(0) \neq l\tau_-(0)$, $j, l = 1, 2, 3, \ldots,\ j \neq l$. Therefore, we obtain following two conditions: a condition is $\begin{pmatrix} x_+(t_0, 0) \\ x'_+(t_0, 0) \end{pmatrix} = 0$ and $\tau_+(0) = \dfrac{2\pi}{\sqrt{1-2k}}$ and another condition is $\begin{pmatrix} x_-(t_0, 0) \\ x'_-(t_0, 0) \end{pmatrix} = 0$ and $\tau_-(0) = 2\pi$, since (i) and (ii) must hold simultaneously. We take some t_0 in the above consideration, but we find that t_0 can be taken arbitrary in this stage. Consequently, the former condition means out-of-phase and the latter in-phase.

Note that the condition of $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$ and $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$ is $\alpha_0(0) = \beta_0(0) = \lambda_0(0) = \mu_0(0)$, that is, the origin.

Summarizing above, when $\varepsilon = 0$, there exists no periodic solutions except the out-of-phase and in-phase solutions, in which periods are $\tau_+(0) = \dfrac{2\pi}{\sqrt{1-2k}}$ and $\tau_-(0) = 2\pi$, respectively. This fact is consistent with Proposition 2.2. Before proving the main theorem, we prepare two propositions and give the following definitions in order to prove the propositions using the inductive method.

Definition 5.1. *The statement* $\mathbf{P}_+(\nu)$, $\nu = 1, 2, 3, \ldots$, *is defined as follows*:

If $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$, *then there exist derivatives* $\dfrac{\partial^\nu x_+(t,\varepsilon)}{\partial\varepsilon^\nu}$ *and* $\dfrac{\partial^\nu x'_+(t,\varepsilon)}{\partial\varepsilon^\nu}$, *and* $\dfrac{\partial^\nu x_+(t,\varepsilon)}{\partial\varepsilon^\nu} = 0$ *and* $\dfrac{\partial^\nu x'_+(t,\varepsilon)}{\partial\varepsilon^\nu} = 0$ *at* $\varepsilon = 0$.

Definition 5.2. *The statement* $\mathbf{P}_-(\nu), \nu = 1, 2, 3, \ldots$, *is defined as follows*:

If $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$, *then there exist derivatives* $\dfrac{\partial^\nu x_-(t,\varepsilon)}{\partial\varepsilon^\nu}$ *and* $\dfrac{\partial^\nu x'_-(t,\varepsilon)}{\partial\varepsilon^\nu}$, *and* $\dfrac{\partial^\nu x_-(t,\varepsilon)}{\partial\varepsilon^\nu} = 0$ *and* $\dfrac{\partial^\nu x'_-(t,\varepsilon)}{\partial\varepsilon^\nu} = 0$ *at* $\varepsilon = 0$.

Proposition 5.1. $\mathbf{P}_+(\nu)$ *is true for* $\nu = 1, 2, 3, \ldots$.

Proposition 5.2. $\mathbf{P}_-(\nu)$ *is true for* $\nu = 1, 2, 3, \ldots$.

Proof. We prove only Proposition 5.1 using the inductive method because Proposition 5.2 can be proved by the same manner.

(i) $x_+(t,0)$ defined in equation (5.3) satisfies the differential equations $x''_+(t,0) + x_+(t,0) = 0$ with the initial conditions $x_+(t_0,0)$ and $x'_+(t_0,0)$. By uniqueness of the solution, we must have $\begin{pmatrix} x_+(t,0) \\ x'_+(t,0) \end{pmatrix} \equiv 0$ for $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$. Hence, $\lim\limits_{\varepsilon\to 0} \begin{pmatrix} x_+(t,\varepsilon) \\ x'_+(t,\varepsilon) \end{pmatrix} = 0$ from Lemma 5.1. Then we have

$$y^2(s,\varepsilon)y'(s,\varepsilon) + z^2(s,\varepsilon)z'(s,\varepsilon) =$$
$$= \big(x_+(s,\varepsilon) - z(s,\varepsilon)\big)^2 x'_+(s,\varepsilon) -$$
$$-z'(s,\varepsilon)\big(y(s,\varepsilon) - z(s,\varepsilon)\big)x_+(s,\varepsilon) \to 0, \quad \varepsilon \to 0. \tag{5.11}$$

Since we have $\mathbf{F}_+(\varepsilon) = 0$ by the periodicity condition, i.e.,

$$\Big(1 - e^{-\varepsilon\tau(\varepsilon)/2} U_{\tau(\varepsilon)}(\theta_+)\Big) \begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix} \begin{pmatrix} x_+(t_0,\varepsilon) \\ x'_+(t_0,\varepsilon) \end{pmatrix} +$$

$$+\varepsilon\begin{pmatrix}\displaystyle\int\limits_0^{\tau(\varepsilon)} e^{-\varepsilon s/2}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+ \\ \qquad + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds \\ \displaystyle\int\limits_0^{\tau(\varepsilon)} e^{-\varepsilon s/2}\cos(\theta_+ s)\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+ \\ \qquad + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds\end{pmatrix}=0. \tag{5.12}$$

Dividing equation (5.12) by ε, we obtain

$$\Big(1-e^{-\varepsilon\tau(\varepsilon)/2}U_{\tau(\varepsilon)}(\theta_+)\Big)\begin{pmatrix}1 & 0\\ \varepsilon/2 & -1\end{pmatrix}\begin{pmatrix}\dfrac{x_+(t_0,\varepsilon)}{\varepsilon}\\ \dfrac{x'_+(t_0,\varepsilon)}{\varepsilon}\end{pmatrix}+$$

$$+\begin{pmatrix}\displaystyle\int\limits_0^{\tau(\varepsilon)} e^{-\varepsilon s/2}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+ \\ \qquad + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds \\ \displaystyle\int\limits_0^{\tau(\varepsilon)} e^{-\varepsilon s/2}\cos(\theta_+ s)\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+ \\ \qquad + z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds\end{pmatrix}=0. \tag{5.13}$$

We take $\varepsilon\to 0$ in equation (5.13). Then the second term vanishes from equation (5.11) and there exist the derivatives $\dfrac{\partial x_+(t_0,0)}{\partial\varepsilon}=\lim\limits_{\varepsilon\to 0}\dfrac{x_+(t_0,\varepsilon)-x_+(t_0,0)}{\varepsilon}$ and $\dfrac{\partial x'_+(t_0,0)}{\partial\varepsilon}=\lim\limits_{\varepsilon\to 0}\dfrac{x'_+(t_0,\varepsilon)-x'_+(t_0,0)}{\varepsilon}$. Here we can take arbitrary t_0, therefore, we have the derivatives $\dfrac{\partial x_+(t,0)}{\partial\varepsilon}$ and $\dfrac{\partial x'_+(t,0)}{\partial\varepsilon}$. Furthermore, we obtain $\dfrac{\partial x_+(t,0)}{\partial\varepsilon}=0$ and $\dfrac{\partial x'_+(t,0)}{\partial\varepsilon}=0$.

Note that, in the computation of the limit, we can exchange the limit and the integral. We show below this fact. The integral of equation (5.13) is written as follows using T defined in equation (5.1):

$$\int\limits_0^{\tau(\varepsilon)} e^{-\varepsilon s/2}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds=$$

$$=\int\limits_0^{T} 1_{\tau(\varepsilon)}(s)e^{-\varepsilon s/2}\frac{\sin(\theta_+ s)}{\theta_+}\big(y^2(t_0+s,\varepsilon)y'(t_0+s,\varepsilon)+$$

$$+z^2(t_0+s,\varepsilon)z'(t_0+s,\varepsilon)\big)ds,$$

where

$$1_{\tau(\varepsilon)}(s) = \begin{cases} 1, & \text{for } s \le \tau(\varepsilon), \\ 0, & \text{for } s > \tau(\varepsilon). \end{cases}$$

Now we find

$$\left| 1_{\tau(\varepsilon)}(s) e^{-\varepsilon s/2} \frac{\sin(\theta_+ s)}{\theta_+} \big(y^2(t_0+s,\varepsilon) y'(t_0+s,\varepsilon) + \right.$$

$$\left. + z^2(t_0+s,\varepsilon) z'(t_0+s,\varepsilon) \big) \right| \le$$

$$\le \frac{1}{\theta_+} \Big| y^2(t_0+s,\varepsilon) y'(t_0+s,\varepsilon) + z^2(t_0+s,\varepsilon) z'(t_0+s,\varepsilon) \Big| \le M$$

$$\text{for} \quad 0 < s < T.$$

Here, for the sake of convenience, we use the same symbol M as in relations (5.2), but they are different from each other. Then we can apply the bounded convergence theorem and we obtain

$$\lim_{\varepsilon \to 0} \int\limits_0^{\tau(\varepsilon)} e^{-\varepsilon s/2} \frac{\sin(\theta_+ s)}{\theta_+} \big(y^2(t_0+s,\varepsilon) y'(t_0+s,\varepsilon) +$$

$$+ z^2(t_0+s,\varepsilon) z'(t_0+s,\varepsilon) \big) ds =$$

$$= \int\limits_0^T \lim_{\varepsilon \to 0} 1_{\tau(\varepsilon)}(s) e^{-\varepsilon s/2} \frac{\sin(\theta_+ s)}{\theta_+} \big(y^2(t_0+s,\varepsilon) y'(t_0+s,\varepsilon) +$$

$$+ z^2(t_0+s,\varepsilon) z'(t_0+s,\varepsilon) \big) ds =$$

$$= \int\limits_0^T 1_{\tau(0)}(s) \sin s \lim_{\varepsilon \to 0} \big(y^2(t_0+s,\varepsilon) y'(t_0+s,\varepsilon) +$$

$$+ z^2(t_0+s,\varepsilon) z'(t_0+s,\varepsilon) \big) ds = 0.$$

In the above equation, we use the relation $\tau(\varepsilon) \to \tau(0)$ as $\varepsilon \to 0$. In fact, we have $\lim\limits_{\varepsilon \to 0} x_\pm\big(t + \tau(\varepsilon), \varepsilon\big) = x_\pm\big(t + \tau(0), 0\big)$ from the assumption $\lim\limits_{\varepsilon \to 0} x_\pm(t,\varepsilon) = x_\pm(t,0) = = y(t) \pm z(t)$ and the periodicity conditions $\lim\limits_{\varepsilon \to 0} x_\pm\big(t + \tau(\varepsilon), \varepsilon\big) = x_\pm(t,0)$ and $x_\pm\big(t + + \tau(0), 0\big) = x_\pm(t,0)$.

(ii) We assume that $\mathbf{P}_+(\nu)$, $\nu \le n$, is true, i.e., there exist $\dfrac{\partial^\nu x_+(t_0,0)}{\partial \varepsilon^\nu}$ and $\dfrac{\partial^\nu x'_+(t_0,0)}{\partial \varepsilon^\nu}$ and $\dfrac{\partial^\nu x_+(t_0,0)}{\partial \varepsilon^\nu} = 0$, $\dfrac{\partial^\nu x'_+(t_0,0)}{\partial \varepsilon^\nu} = 0$, $\nu = 0, 1, 2, \ldots, n$. Then we show that $\mathbf{P}_+(n+1)$ is true. Dividing equation (5.12) by ε^{n+1}, we obtain

$$\Big(1 - e^{-\varepsilon\tau(\varepsilon)/2}U_{\tau(\varepsilon)}(\theta_+)\Big)\begin{pmatrix} 1 & 0 \\ \varepsilon/2 & -1 \end{pmatrix}\begin{pmatrix} \dfrac{x_+(t_0,\varepsilon) - \displaystyle\sum_{\nu=1}^{n} \frac{\partial^\nu x_+(t_0,0)}{\partial\varepsilon^\nu}\varepsilon^\nu}{\varepsilon^{n+1}} \\ \dfrac{x'_+(t_0,\varepsilon) - \displaystyle\sum_{\nu=1}^{n} \frac{\partial^\nu x'_+(t_0,0)}{\partial\varepsilon^\nu}\varepsilon^\nu}{\varepsilon^{n+1}} \end{pmatrix} +$$

$$+\begin{pmatrix} \displaystyle\int_0^{\tau(\varepsilon)} e^{-\varepsilon s/2}\frac{\sin(\theta_+ s)}{\theta_+}\Big\{\big(x_+(t_0+s,\varepsilon) - z(t_0+s,\varepsilon)\big)^2\frac{x'_+(t_0+s,\varepsilon)}{\varepsilon^n} - \\ -z'(t_0+s,\varepsilon)\big(y(t_0+s,\varepsilon) - z(t_0+s,\varepsilon)\big)\dfrac{x_+(t_0+s,\varepsilon)}{\varepsilon^n}\Big\}ds \\ \displaystyle\int_0^{\tau(\varepsilon)} e^{-\varepsilon s/2}\cos(\theta_+ s)\Big\{\big(x_+(t_0+s,\varepsilon) - z(t_0+s,\varepsilon)\big)^2\frac{x'_+(t_0+s,\varepsilon)}{\varepsilon^n} - \\ -z'(t_0+s,\varepsilon)\big(y(t_0+s,\varepsilon) - z(t_0+s,\varepsilon)\big)\dfrac{x_+(t_0+s,\varepsilon)}{\varepsilon^n}\Big\}ds \end{pmatrix} = 0.$$

Here, if we take $\varepsilon \to 0$, then the second term vanishes since $\lim\limits_{\varepsilon\to 0}\dfrac{x_+(t_0+s,\varepsilon)}{\varepsilon^n} = 0$, $\lim\limits_{\varepsilon\to 0}\dfrac{x'_+(t_0+s,\varepsilon)}{\varepsilon^n} = 0$. Therefore, we find that there exist the derivatives $\dfrac{\partial^{n+1}x_+(t_0,0)}{\partial\varepsilon^{n+1}}$ and $\dfrac{\partial^{n+1}x'_+(t_0,0)}{\partial\varepsilon^{n+1}}$. Since t_0 is arbitrary, we have the existence of $\dfrac{\partial^{n+1}x_+(t,0)}{\partial\varepsilon^{n+1}}$ and $\dfrac{\partial^{n+1}x'_+(t,0)}{\partial\varepsilon^{n+1}}$. Furthermore, we obtain $\dfrac{\partial^{n+1}x_+(t,0)}{\partial\varepsilon^{n+1}} = 0$, $\dfrac{\partial^{n+1}x'_+(t,0)}{\partial\varepsilon^{n+1}} = 0$.

(iii) From (i) and (ii), we obtain that $\mathbf{P}_+(\nu)$ is true for any $\nu \in \mathcal{N}$.

The fact that the – part, i.e., $\mathbf{P}_-(\nu)$, is true for any $\nu \in \mathcal{N}$ can also be proved in the same way using the relation

$$y^2(s,\varepsilon)y'(s,\varepsilon) - z^2(s,\varepsilon)z'(s,\varepsilon) =$$

$$= \big(x_-(s,\varepsilon) + z(s,\varepsilon)\big)^2 x'_-(s,\varepsilon) +$$

$$+z'(s,\varepsilon)\big(y(s,\varepsilon) + z(s,\varepsilon)\big)x_-(s,\varepsilon).$$

We obtain the following lemma from Propositions 5.1 and 5.2.

Lemma 5.2. *We assume that $y(t,\varepsilon)$ and $z(t,\varepsilon)$ are analytic with respect to the parameter ε. If* $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} = 0$, *then* $\begin{pmatrix} x_+(t_0,\varepsilon) \\ x'_+(t_0,\varepsilon) \end{pmatrix} = 0$. *Moreover, If* $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} = 0$, *then* $\begin{pmatrix} x_-(t_0,\varepsilon) \\ x'_-(t_0,\varepsilon) \end{pmatrix} = 0$.

***Proof of Theorem* 5.1.** From Lemma 5.2, if $\begin{pmatrix} x_+(t_0,\varepsilon) \\ x'_+(t_0,\varepsilon) \end{pmatrix} \neq 0$ and $\begin{pmatrix} x_-(t_0,\varepsilon) \\ x'_-(t_0,\varepsilon) \end{pmatrix} \neq$ $\neq 0$, then $\begin{pmatrix} x_+(t_0,0) \\ x'_+(t_0,0) \end{pmatrix} \neq 0$ and $\begin{pmatrix} x_-(t_0,0) \\ x'_-(t_0,0) \end{pmatrix} \neq 0$. However, this is inconsistent with Proposition 2.2 under the assumption that k is irrational, which states that the dynamical system $\mathbf{\Sigma}_{0,k}$ does not have solutions except the out-of-phase and in-phase ones. Therefore, we have $\begin{pmatrix} x_+(t,\varepsilon) \\ x'_+(t,\varepsilon) \end{pmatrix} = 0$ or $\begin{pmatrix} x_-(t,\varepsilon) \\ x'_-(t,\varepsilon) \end{pmatrix} = 0$. Consequently, the dynamical system $\mathbf{\Sigma}_{\varepsilon,k}$ does not have any other periodic solutions except the out-of-phase and in-phase solutions.

We give the following consideration for Theorem 5.1.

***Remark* 5.1.** We consider the averaged system of $\mathbf{\Sigma}_{\varepsilon,k}$. First, using the symbols x_+ and x_-, we transform $\mathbf{\Sigma}_{\varepsilon,k}$ into

$$x''_+ + x_+ = \frac{\varepsilon}{4}\Big(x'_+(4 - x_+^2 - x_-^2) - 2x'_-x_+x_-\Big),$$

$$x''_- + (1-2k)x_- = \frac{\varepsilon}{4}\Big(x'_-(4 - x_+^2 - x_-^2) - 2x'_+x_+x_-\Big).$$

Passing to polar coordinates $x_+ = a_+\sin\theta_+$, $x'_+ = a_+\cos\theta_+$, $x_- = a_-\sin\theta_-$, $x'_- =$ $= a_-\sqrt{1-2k}\cos\theta_-$ and averaging the right-hand side of the obtained system with respect to the phase variable θ_+, θ_-, we obtain the following averaged system:

$$\begin{aligned} a'_1 &= \frac{\varepsilon}{32}(16a_1 - 2a_1a_2^2 - a_1^3), \\ a'_2 &= \frac{\varepsilon}{32}(16a_2 - 2a_2a_1^2 - a_2^3), \\ \theta'_1 &= 1, \\ \theta'_2 &= \sqrt{1-2k}, \end{aligned} \tag{5.14}$$

where a_1, a_2, θ_1, θ_2 denote the averaged counterparts of a_+, a_-, θ_+, θ_-. From the two dimensional system given by the first two equations of equation (5.14), we find four fixed points: $(0,0)$, $(4,0)$, $(0,4)$, $\left(\frac{4}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$. The first three are focuses, one unstable and two stable, while the last one is a saddle. Paying attention to the phase variables, we conclude that the averaged system has four invariant tori: one unstable zero-dimensional (the zero solution, i.e., the origin), two stable one-dimensional (the limit of the out-of-phase and in-phase solutions), one semistable two-dimensional.

According to a theorem from [16], for small enough ε, in proximity of the above-listed invariant tori of the averaged system, the corresponding analytically smooth invariant tori of the system $\Sigma_{\varepsilon,k}$ lie, which have the same dimensions and stability. For small enough ε, periodic trajectories on the semistable two-dimensional torus cannot be put in the form of analytic functions in ε satisfying the condition of periodicity.

We also present the next theorem, which shows that the orbits of $\mathbf{\Sigma}_{\varepsilon,k}$ as $\varepsilon \to 0$ become the specific orbits in $\mathbf{\Sigma}_{0,k}$.

Theorem 5.2. *Let k be irrational and let $0 < k < \dfrac{1}{2} - \dfrac{\varepsilon^2}{8}$. The orbit of $\mathbf{\Sigma}_{\varepsilon,k}$ as $\varepsilon \to 0$ is presented by*

$$y^2(t,0) + y'^2(t,0) = 4,$$

$$z^2(t,0) + z'^2(t,0) = 4$$

at in-phase, i.e., $y(t,0) - z(t,0) = 0$, and

$$y^2(t,0) + \frac{y'^2(t,0)}{1-2k} = 4,$$

$$z^2(t,0) + \frac{z'^2(t,0)}{1-2k} = 4$$

at out-of-phase, i.e., $y(t,0) + z(t,0) = 0$.

The period of in-phase and out-of-phase, which are denoted by $\tau_+(\varepsilon)$ and $\tau_-(\varepsilon)$, are represented as

$$\tau_+(\varepsilon) = 2\pi + o(\varepsilon), \qquad \tau_-(\varepsilon) = \frac{2\pi}{1-2k} + o(\varepsilon).$$

Proof. We prove this theorem as the same procedure of Proposition 3.2 without any special assumptions.

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