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Wei-feng Xia, Yu-ming Chu (Huzhou Teachers College, China)

THE SCHUR CONVEXITY AND SCHUR MULTIPLICATIVE CONVEXITY FOR A CLASS OF SYMMETRIC FUNCTIONS WITH APPLICATIONS*

ОПУКЛІСТЬ ЗА ШУРОМ І МУЛЬТИПЛІКАТИВНА ОПУКЛІСТЬ ЗА ШУРОМ ДЛЯ ОДНОГО КЛАСУ СИМЕТРИЧНИХ ФУНКЦІЙ ТА ЇХ ЗАСТОСУВАННЯ

For $x = (x_1, x_2, \ldots, x_n) \in (0, 1]^n$ and $r \in \{1, 2, \ldots, n\}$, the symmetric function $F_n(x, r)$ is defined by

$$F_n(x, r) = F_n(x_1, x_2, \ldots, x_n; r) = \sum_{1 \le i_1 < i_2 \ldots i_r \le n} \prod_{j=1}^{r} \frac{1 - x_{i_j}}{x_{i_j}},$$

where $i_1, i_2, \ldots, i_n$ are positive integers.

This paper deals with the Schur convexity and Schur multiplicative convexity of $F_n(x, r)$. As applications, some inequalities are established by use of the theory of majorization.

Для $x = (x_1, x_2, \ldots, x_n) \in (0, 1]^n$ та $r \in \{1, 2, \ldots, n\}$ симетрична функція $F_n(x, r)$ визначається співвідношенням

$$F_n(x, r) = F_n(x_1, x_2, \ldots, x_n; r) = \sum_{1 \le i_1 < i_2 \ldots i_r \le n} \prod_{j=1}^{r} \frac{1 - x_{i_j}}{x_{i_j}},$$

де $i_1, i_2, \ldots, i_n$ — додатні цілі числа.

У статті розглянуто властивості опуклості за Шуром та мультиплікативної опуклості за Шуром для функції $F_n(x, r)$. Як застосування, встановлено деякі нерівності з використанням теорії мажорування.

1. Introduction. The Schur convex function was introduced by I. Schur in 1923 [16]. It has many important applications in analytic inequalities [7 – 9, 11, 15, 19, 26, 28, 31, 32], extended mean values [4, 23, 24, 27], theory of statistical experiments [29]. graphs and matrices [6], combinational optimization [13], reliability [14], stochastic orderings [25] and other related fields. G. H. Hardy, J. E. Littlewood and G. Pólya were also interested in some inequalities that are related to the Schur convexity [12]. The following definition for Schur convex or concave function can be found in many references such as [4, 9, 16, 20, 22].

Definition 1.1. *Let $E \subseteq R^n$, $n \ge 2$, be a set. A real-valued function F on E is called a Schur convex function if*

$$F(x_1, x_2, \ldots, x_n) \le F(y_1, y_2, \ldots, y_n)$$

for each pair of n-tuples $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in E such that x is majorized by y (in symbols $x \prec y$), i.e.,

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1$$

and

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$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ *denotes the* i*-th largest component of* x. F *is called Schur concave if* $-F$ *is Schur convex.*

Recall that the following so-called Schur's condition is very useful for determining whether or not a given function is Schur convex or Schur concave.

Theorem 1.1 [8, 9, 11, 15, 16]. *Let* $f\colon (0, 1]^n \to R$, $n \geq 2$, *be a continuous symmetric function. If* f *is differentiable in* $(0, 1]^n$, *then* f *is Schur convex on* $(0, 1]^n$ *if and only if*

$$(x_i - x_j)\left(\frac{\partial f(x)}{\partial x_i} - \frac{\partial f(x)}{\partial x_j}\right) \geq 0 \tag{1.1}$$

for all $i, j = 1, 2, \ldots, n$ *and* $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$. *And* f *is Schur concave if and only if inequality* (1.1) *is reversed. Here,* f *is a symmetric function on* $(0, 1]^n$ *which means that* $f(Px) = f(x)$ *for all* $x \in (0, 1]^n$ *and any* $n \times n$ *permutation matrix* P.

***Remark* 1.1.** Since f is symmetric, the Schur's condition in Theorem 1.1, i.e., (1.1) can be reduced as

$$(x_1 - x_2)\left(\frac{\partial f(x)}{\partial x_1} - \frac{\partial f(x)}{\partial x_2}\right) \geq 0.$$

Recently, C. P. Niculescu [21] introduced the multiplicatively convex function, which reveals an entire new world of beautiful inequalities. And the Schur multiplicative convexity was introduced and investigated by K. Z. Guan [9, 10], and Y. M. Chu, X. M. Zhang and G. D. Wang [5].

Definition 1.2 [5, 9, 10]. *Let* I *be a subinterval of* $(0, \infty)$. *A positive real-valued function* F *on* I^n, $n \geq 2$, *is called a Schur multiplicatively convex function if*

$$F(x_1, x_2, \ldots, x_n) \leq F(y_1, y_2, \ldots, y_n)$$

for each pair of n*-tuples* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$ *in* I^n *such that* x *is logarithmically majorized by* y (*in symbols* $\log x \prec \log y$), *i.e.,*

$$\prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n-1,$$

and

$$\prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}.$$

F *is called Schur multiplicatively concave if* $\dfrac{1}{F}$ *is Schur multiplicatively convex.*

Theorem 1.2 [5, 9, 10]. *Let* $f\colon (0, 1]^n \to (0, \infty)$, $n \geq 2$, *be a continuous symmetric function. If* f *is differentiable in* $(0, 1)^n$, *then* f *is Schur multi-*

plicatively convex on $(0, 1]^n$ *if and only if*

$$(x_1 - x_2)\left(x_1 \frac{\partial f(x)}{\partial x_1} - x_2 \frac{\partial f(x)}{\partial x_2} \right) \geq 0 \tag{1.2}$$

for all $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$. *And* f *is Schur multiplicatively concave if and only if inequality* (1.2) *is reversed.*

The main purpose of this article is to discuss the Schur convexity and Schur multiplicative convexity of the symmetric function

$$F_n(x, r) = F_n(x_1, x_2, \ldots, x_n; r) = \sum_{1 \leq i_1 < i_2 \ldots i_r \leq n} \prod_{j=1}^{r} \frac{1 - x_{i_j}}{x_{i_j}} \tag{1.3}$$

for $x = (x_1, x_2, \ldots, x_n) \in (0, 1]^n$, $n \geq 2$, and $r = 1, 2, \ldots, n$, where $i_1, i_2, \ldots, i_n$ are positive integers.

Our main results are the Theorems 1.3 and 1.4.

Theorem 1.3. 1. $F_n(x, 1)$ *is Schur convex on* $(0, 1]^n$.

2. $F_n(x, r)$ *is Schur convex on* $\left(0, \frac{2n - r - 1}{2n - 2}\right)^n$ *for* $2 \leq r \leq n$.

3. $F_n(x, r)$ *is Schur concave on* $\left[\frac{2n - r - 1}{2n - 2}, 1\right]^n$ *for* $2 \leq r \leq n$.

Theorem 1.4. 1. $F_n(x, 1)$ *is Schur multiplicatively convex on* $(0, 1]^n$.

2. $F_n(x, n)$ *is Schur multiplicatively concave on* $(0, 1]^n$.

3. $F_n(x, r)$ *is Schur multiplicatively convex on* $\left(0, \frac{n - r}{n - 1}\right]^n$ *for* $n \geq 3$ *and* $2 \leq r \leq n - 1$.

4. $F_n(x, r)$ *is Schur multiplicatively concave on* $\left[\frac{n - r}{n - 1}, 1\right]^n$ *for* $n \geq 3$ *and* $2 \leq r \leq n - 1$.

As applications of Theorems 1.3 and 1.4, some inequalities are established by use of the theory of majorization in Section 4.

2. Lemmas. In this section, we establish and introduce several lemmas, which are used in the next sections.

For $t = (t_1, t_2, \ldots, t_n) \in (0, \infty)^n$ and $r \in \{0, 1, 2, \ldots, n\}$, $n \geq 2$, the r-th elementary symmetric function (see [3]) is defined as

$$E_n(t, r) = E_n(t_1, t_2, \ldots, t_n, r) = \begin{cases} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq n} \prod_{j=1}^{r} t_{i_j}, & r = 1, 2, \ldots, n, \\ 1, & r = 0, \end{cases}$$

where $i_1, i_2, \ldots, i_n$ are positive integers.

Lemma 2.1. *If* $1 \leq r \leq n - 1$, *then*

$$E_n^2(t, r) \geq E_n(t, r - 1) E_n(t, r + 1)$$

for $t = (t_1, t_2, \ldots, t_n) \in (0, \infty)^n$.

Proof. We use mathematical induction to prove Lemma 2.1.

(i) By simple computation, it is not difficult to verify that Lemma 2.1 is true for $n = 2$ and 3, and $n \ge 4$ and $r = 2$.

(ii) Assume that Lemma 2.1 is true for $3 \le n \le m - 1$ and $2 \le r \le n$. Then the definition of $E_n(t, r)$ yields that

$$\begin{aligned} E_m(t, r) &= E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-1)t_m + E_{m-1}(t_1, t_2, \dots, t_{m-1}; r), \\ E_m(t, r-1) &= E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-2)t_m + E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-1), \qquad (2.1) \\ E_m(t, r+1) &= E_{m-1}(t_1, t_2, \dots, t_{m-1}; r)t_m + E_{m-1}(t_1, t_2, \dots, t_{m-1}; r+1). \end{aligned}$$

Equation (2.1) leads to

$$E_m^2(t, r) - E_m(t, r-1)E_m(t, r+1) =$$

$$= [E_{m-1}^2(t_1, t_2, \dots, t_{m-1}; r-1) - E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-2) \times$$

$$\times E_{m-1}(t_1, t_2, \dots, t_{m-1}; r)]t_m^2 + [E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-1) \times$$

$$\times E_{m-1}(t_1, t_2, \dots, t_{m-1}; r) - E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-2) \times$$

$$\times E_{m-1}(t_1, t_2, \dots, t_{m-1}; r+1)]t_m + E_{m-1}^2(t_1, t_2, \dots, t_{m-1}; r) -$$

$$- E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-1)E_{m-1}(t_1, t_2, \dots, t_{m-1}; r+1). \qquad (2.2)$$

By induction hypothesis we have

$$\frac{E_{m-1}(t_1, t_2, \dots, t_{m-1}; r)}{E_{m-1}(t_1, t_2, \dots, t_{m-1}; r+1)} \ge \frac{E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-1)}{E_{m-1}(t_1, t_2, \dots, t_{m-1}; r)} \ge$$

$$\ge \frac{E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-2)}{E_{m-1}(t_1, t_2, \dots, t_{m-1}; r-1)}. \qquad (2.3)$$

Now, equations (2.2) and (2.3) imply that

$$E_m^2(t, r) \ge E_m(t, r-1)E_m(t, r+1).$$

Therefore, Lemma 2.1 follows from (i) and (ii) together with the mathematical induction.

Lemma 2.2. *If* $n \ge 3$ *and* $1 \le r \le n-1$, *then the function*

$$\varphi_n(x_1, x_2, \dots, x_n; r) = \frac{F_n(x_1, x_2, \dots, x_n; r+1)}{F_n(x_1, x_2, \dots, x_n; r)}$$

is decreasing with respect to each x_i *in* $(0, 1)$, $i = 1, 2, \dots, n$.

Proof. Let $\psi_n(t_1, t_2, \dots, t_n; r) = \dfrac{E_n(t_1, t_2, \dots, t_n; r+1)}{E_n(t_1, t_2, \dots, t_n; r)}$ and $t_i = \dfrac{1 - x_i}{x_i}$, then from the symmetry of φ_n and ψ_n, and the monotonicity of $\dfrac{1-x}{x}$, we need only to prove that $\psi_n(t_1, t_2, \dots, t_n; r)$ is increasing with respect to t_1 in $(0, \infty)$. The proof is divided into three cases.

***Case* 1.** If $r = 1$, then

$$\psi_n(t_1, t_2, \ldots, t_n; 1) = \frac{t_1 \sum_{i=2}^n t_i + \sum_{2 \le i < j \le n} t_i t_j}{\sum_{i=1}^n t_i}$$

and

$$\frac{\partial \psi_n(t_1, t_2, \ldots, t_n; 1)}{\partial t_1} = \frac{\sum_{i=2}^n t_i^2 + \sum_{2 \le i < j \le n} t_i t_j}{\left(\sum_{i=1}^n t_i\right)^2} > 0.$$

***Case* 2.** If $r = n-1$, then

$$\psi_n(t_1, t_2, \ldots, t_n; n-1) = \frac{1}{\sum_{i=1}^n \frac{1}{t_i}},$$

we clearly see that $\psi_n(t_1, t_2, \ldots, t_n; n-1)$ is increasing with respect to t_1 in $(0, \infty)$.

***Case* 3.** If $n \ge 4$ and $2 \le r \le n-2$, then

$$\psi_n(t_1, t_2, \ldots, t_n; r) = \frac{t_1 E_{n-1}(t_2, t_3, \ldots, t_n; r) + E_{n-1}(t_2, t_3, \ldots, t_n; r+1)}{t_1 E_{n-1}(t_2, t_3, \ldots, t_n; r-1) + E_{n-1}(t_2, t_3, \ldots, t_n; r)}$$

and

$$\frac{\partial \psi_n(t_1, t_2, \ldots, t_n; r)}{\partial t_1} \ge 0.$$

From above Cases 1 – 3 we know that $\psi_n(t_1, t_2, \ldots, t_n; r)$ is increasing with respect to t_1 in $(0, \infty)$ for $n \ge 3$ and $1 \le r \le n-1$, and the proof of Lemma 2.2 is completed.

Lemma 2.3. *Let* $x = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$ *and* $\sum_{i=1}^n x_i = s$. *If* $\lambda \le 1$, *then*

$$\frac{s - \lambda x}{n - \lambda} = \left(\frac{s - \lambda x_1}{n - \lambda}, \frac{s - \lambda x_2}{n - \lambda}, \ldots, \frac{s - \lambda x_n}{n - \lambda} \right) \prec (x_1, x_2, \ldots, x_n) = x.$$

Proof. For any $x = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$, we clearly see that

$$\left(\frac{1}{n-1} \sum_{i \ne 1} x_i, \frac{1}{n-1} \sum_{i \ne 2} x_i, \ldots, \frac{1}{n-1} \sum_{i \ne n} x_i \right) \prec (x_1, x_2, \ldots, x_n) = x,$$

multiply by $n-1$, add $(1-\lambda)x$ to both sides and divided by $n - \lambda$, we get

$$\frac{s - \lambda x}{n - \lambda} = \left(\frac{s - \lambda x_1}{n - \lambda}, \frac{s - \lambda x_2}{n - \lambda}, \ldots, \frac{s - \lambda x_n}{n - \lambda} \right) \prec (x_1, x_2, \ldots, x_n) = x.$$

***Remark* 2.1.** Lemma 2.3 was prove by S. H. Wu [30] in the case of $0 \le \lambda \le 1$.

3. Proof of Theorems 1.3 and 1.4. ***Proof of Theorem*** **1.3.** 1. If $r = 1$ and $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$, then (1.3) leads to

$$F_n(x, 1) = F_n(x_1, x_2, \ldots, x_n, 1) = \sum_{i=1}^{n} \frac{1 - x_i}{x_i} \tag{3.1}$$

and

$$(x_1 - x_2)\left(\frac{\partial F_n(x,1)}{\partial x_1} - \frac{\partial F_n(x,1)}{\partial x_2}\right) = \frac{(x_1 - x_2)^2 (x_1 + x_2)}{x_1^2 x_2^2} \geq 0. \tag{3.2}$$

Therefore, Theorem 1.3(1) follows from (3.2) and Theorem 1.1 together with Remark 1.1.

2. If $2 \leq r \leq n$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{2n - r - 1}{2n - 2}\right)^n$, then the proof is divided into six cases.

***Case* 2.1.** If $n = 2$, $r = 2$ and $x = (x_1, x_2) \in \left(0, \frac{1}{2}\right)^n$, then

$$F_2(x, 2) = F_2(x_1, x_2; 2) = \frac{(1 - x_1)(1 - x_2)}{x_1 x_2} \tag{3.3}$$

and

$$(x_1 - x_2)\left(\frac{\partial F_2(x,2)}{\partial x_1} - \frac{\partial F_2(x,2)}{\partial x_2}\right) = \frac{(x_1 - x_2)^2 (1 - x_1 - x_2)}{x_1^2 x_2^2} \geq 0.$$

***Case* 2.2.** If $n \geq 3$, $r = n$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{1}{2}\right)^n$, then

$$F_n(x, n) = F_n(x_1, x_2, \ldots, x_n; n) = \prod_{i=1}^{n} \frac{1 - x_i}{x_i} \tag{3.4}$$

and

$$(x_1 - x_2)\left(\frac{\partial F_n(x,n)}{\partial x_1} - \frac{\partial F_n(x,n)}{\partial x_2}\right) = \frac{(x_1 - x_2)^2 F_n(x,n)}{x_1 x_2 (1 - x_1)(1 - x_2)}(1 - x_1 - x_2) \geq 0.$$

***Case* 2.3.** If $n = 3$, $r = 2$ and $x = (x_1, x_2, x_3) \in \left(0, \frac{3}{4}\right)^n$, then

$$F_3(x, 2) = F_3(x_1, x_2, x_3; 2) =$$

$$= \frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + \frac{(1 - x_1)(1 - x_3)}{x_1 x_3} + \frac{(1 - x_2)(1 - x_3)}{x_2 x_3} \tag{3.5}$$

and

$$(x_1 - x_2)\left(\frac{\partial F_3(x, 2)}{\partial x_1} - \frac{\partial F_3(x, 2)}{\partial x_2} \right) =$$

$$= \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} \left[(1 - x_1 - x_2) + (x_1 + x_2)\left(\frac{1}{x_3} - 1 \right) \right] \geq$$

$$\geq \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} \left[1 - \frac{2}{3}(x_1 + x_2) \right] \geq 0.$$

***Case* 2.4.** If $n \geq 4$, $r = 2$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{2n-3}{2n-2} \right)^n$, then

$$F_n(x, 2) = F_n(x_1, x_2, \ldots, x_n; 2) =$$

$$= \frac{(1 - x_1)(1 - x_2)}{x_1 x_2} + \left(\frac{1 - x_1}{x_1} + \frac{1 - x_2}{x_2} \right) \sum_{i=3}^{n} \frac{1 - x_i}{x_i} + \sum_{3 \leq i < j \leq n} \frac{(1 - x_i)(1 - x_j)}{x_i x_j} \tag{3.6}$$

and

$$(x_1 - x_2)\left(\frac{\partial F_n(x, 2)}{\partial x_1} - \frac{\partial F_n(x, 2)}{\partial x_2} \right) =$$

$$= \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} \left[(1 - x_1 - x_2) + (x_1 + x_2) \sum_{i=3}^{n} \frac{1 - x_i}{x_i} \right] \geq$$

$$\geq \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} \left[1 - \frac{n-1}{2n-3}(x_1 + x_2) \right] \geq 0.$$

***Case* 2.5.** If $n \geq 4$, $r = n - 1$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{n}{2n-2} \right)^n$, then

$$F_n(x, n-1) = F_n(x_1, x_2, \ldots, x_n; n-1) =$$

$$= \left[\frac{(1 - x_1)(1 - x_2)}{x_1 x_2} \sum_{i=3}^{n} \frac{x_i}{1 - x_i} + \frac{1 - x_1}{x_2} + \frac{1 - x_2}{x_2} \right] \prod_{i=3}^{n} \frac{1 - x_i}{x_i} \tag{3.7}$$

and

$$(x_1 - x_2)\left(\frac{\partial F_n(x, n-1)}{\partial x_1} - \frac{\partial F_n(x, n-1)}{\partial x_2} \right) =$$

$$= \frac{(x_1 - x_2)^2}{x_1^2 x_2^2} \left(1 - x_1 - x_2 + \frac{x_1 + x_2}{\sum_{i=3}^{n} \frac{x_i}{1 - x_i}} \right) \left(\sum_{i=3}^{n} \frac{x_i}{1 - x_i} \right) \prod_{i=3}^{n} \frac{1 - x_i}{x_i} \geq$$

$$\geq \frac{(x_1-x_2)^2}{x_1^2x_2^2}\left[1-\frac{n-1}{n}(x_1+x_2)\right]\left(\sum_{i=3}^{n}\frac{x_i}{1-x_i}\right)\prod_{i=3}^{n}\frac{1-x_i}{x_i} \geq 0.$$

***Case* 2.6.** If $n \geq 5$, $3 \leq r \leq n-2$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{2n-r-1}{2n-2}\right)^n$, then (1.3) and Lemma 2.2 yield that

$$F_n(x, r) = F_n(x_1, x_2, \ldots, x_n; r) = \frac{1-x_1}{x_1}\frac{1-x_2}{x_2}F_{n-2}(x_3, x_4, \ldots, x_n; r-2) +$$

$$+\left(\frac{1-x_1}{x_1}+\frac{1-x_2}{x_2}\right)F_{n-2}(x_3, x_4, \ldots, x_n; r-1) + F_{n-2}(x_3, x_4, \ldots, x_n; r) \tag{3.8}$$

and

$$(x_1-x_2)\left(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\right) = \frac{(x_1-x_2)^2}{x_1^2x_2^2}F_{n-2}(x_3, x_4, \ldots, x_n; r-2) \times$$

$$\times\left[(1-x_1-x_2)+\frac{F_{n-2}(x_3, x_4, \ldots, x_n; r-1)}{F_{n-2}(x_3, x_4, \ldots, x_n; r-2)}(x_1+x_2)\right] \geq$$

$$\geq \frac{(x_1-x_2)^2}{x_1^2x_2^2}F_{n-2}(x_3, x_4, \ldots, x_n; r-2) \times$$

$$\times\left[(1-x_1-x_2)+\frac{\dfrac{(n-2)!}{(r-1)!(n-r-1)!}}{\dfrac{(n-2)!}{(r-2)!(n-r)!}}\frac{r-1}{2n-r-1}(x_1+x_2)\right] =$$

$$= \frac{(x_1-x_2)^2}{x_1^2x_2^2}F_{n-2}(x_3, x_4, \ldots, x_n; r-2)\left[1-\frac{n-1}{2n-r-1}(x_1+x_2)\right] \geq 0.$$

Therefore, Theorem 1.3(2) follows from Cases 2.1 – 2. 6 and Theorem 1.1 together with Remark 1.1.

3. If $2 \leq r \leq n$ and $x = (x_1, x_2, \ldots, x_n) \in \left(\frac{2n-r-1}{2n-2}, 1\right)^n$, then the similar proofs as in Theorem 1.3(2) show that $F_n(x, r)$ is Schur concave on $\left[\frac{2n-r-1}{2n-2}, 1\right]^n$.

***Proof of Theorem* 1.4.** 1. If $r = 1$ and $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$, then (3.1) yields that

$$(x_1-x_2)\left(x_1\frac{\partial F_n(x,1)}{\partial x_1}-x_2\frac{\partial F_n(x,1)}{\partial x_2}\right) = \frac{(x_1-x_2)^2}{x_1x_2} \geq 0. \tag{3.9}$$

Therefore, Theorem 1.4(1) follows from (3.9) and Theorem 1.2.

2. If $r = n$ and $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$, then (3.3) and (3.4) lead to that

$$(x_1 - x_2)\left(x_1 \frac{\partial F_n(x,n)}{\partial x_1} - x_2 \frac{\partial F_n(x,n)}{\partial x_2} \right) = -\frac{(x_1 - x_2)^2}{(1-x_1)(1-x_2)} F_n(x,n) \le 0. \quad (3.10)$$

Therefore, Theorem 1.4(2) follows from (3.10) and Theorem 1.2.

3. If $n \ge 3$, $2 \le r \le n-1$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{n-r}{n-1}\right)^n$, then the proof is divided into three cases.

***Case* 3.1.** If $n \ge 3$, $r = 2$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{n-2}{n-1}\right)^n$, then (3.5) and (3.6) yield that

$$(x_1 - x_2)\left(x_1 \frac{\partial F_n(x,2)}{\partial x_1} - x_2 \frac{\partial F_n(x,2)}{\partial x_2} \right) = \frac{(x_1 - x_2)^2}{x_1 x_2}\left(-1 + \sum_{i=3}^{n} \frac{1-x_i}{x_i} \right) \ge 0.$$

***Case* 3.2.** If $n \ge 4$, $r = n-1$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{1}{n-1}\right)^n$, then (3.7) implies that

$$(x_1 - x_2)\left(x_1 \frac{\partial F_n(x,n-1)}{\partial x_1} - x_2 \frac{\partial F_n(x,n-1)}{\partial x_2} \right) =$$

$$= \frac{(x_1 - x_2)^2}{x_1 x_2}\left(1 - \sum_{i=3}^{n} \frac{x_i}{1-x_i} \right) \prod_{i=3}^{n} \frac{1-x_i}{x_i} \ge 0.$$

***Case* 3.3.** If $n \ge 5$, $3 \le r \le n-2$ and $x = (x_1, x_2, \ldots, x_n) \in \left(0, \frac{n-r}{n-1}\right)^n$, then from (3.8) and Lemma 2.2 together with (1.3) we see that

$$(x_1 - x_2)\left(x_1 \frac{\partial F_n(x,r)}{\partial x_1} - x_2 \frac{\partial F_n(x,r)}{\partial x_2} \right) =$$

$$= \frac{(x_1 - x_2)^2}{x_1 x_2} F_{n-2}(x_3, x_4, \ldots, x_n; r-2)\left[\frac{F_{n-2}(x_3, x_4, \ldots, x_n; r-1)}{F_{n-2}(x_3, x_4, \ldots, x_n; r-2)} - 1 \right] \ge$$

$$\ge \frac{(x_1 - x_2)^2}{x_1 x_2} F_{n-2}(x_3, x_4, \ldots, x_n; r-2)\left[\frac{\dfrac{(n-2)!}{(r-1)!(n-r-1)!}}{\dfrac{(n-2)!}{(r-2)!(n-r)!}} \frac{r-1}{n-r} - 1 \right] = 0.$$

Therefore, Theorem 1.4(3) follows from Cases 3.1 – 3.3 and Theorem 1.2.

4. The proofs is completely parallel to that in Theorem 1.4(3).

4. Applications. In this section, we establish some inequalities by use of Theorems 1.3, 1.4 and the theory of majorization.

Theorem 4.1. *If* $n \geq 2$, $x = (x_1, x_2, \ldots, x_n) \in (0, 1]^n$ *and* $s = \sum_{i=1}^n x_i$, *then*

(1) $\displaystyle\sum_{i=1}^{n} \frac{1-x_i}{x_i} \geq \sum_{i=3}^{n} \frac{(n-s)-\lambda(1-x_i)}{s-\lambda x_i}$ *for* $\lambda \leq 1$;

(2) $F_n(x, r) \geq F_n\left(\dfrac{s-\lambda x}{n-\lambda}; r\right)$ *for* $2 \leq r \leq n$, $x \in \left(0, \dfrac{2n-r-1}{2n-2}\right]^n$ *and* $\lambda \leq 1$;

(3) $F_n(x, r) \leq F_n\left(\dfrac{s-\lambda x}{n-\lambda}; r\right)$ *for* $2 \leq r \leq n$, $x \in \left[\dfrac{2n-r-1}{2n-2}, 1\right]^n$ *and* $\lambda \leq 1$.

Proof. Theorem 4.1(1) follows from Theorem 1.3(1), Lemma 2.3 and (1.3); Theorem 4.1(2) follows from Theorem 1.3(2) and Lemma 2.3; and Theorem 4.1(3) follows from Theorem 1.3(3) and Lemma 2.3.

If we take $s = 1$ in Theorem 4.1(1), then we get the following corollary.

***Corollary* 4.1.** *If* $n \geq 2$, $x = (x_1, x_2, \ldots, x_n) \in (0, 1)^n$ *with* $\sum_{i=1}^n x_i = 1$, *and* $\lambda \leq 1$, *then*

$$\sum_{i=1}^{n} \frac{1}{x_i} \geq (n-\lambda)\sum_{i=1}^{n} \frac{1}{1-\lambda x_i}.$$

If we take $r = n$ in Theorem 4.1(2) and (3), respectively, then we have the following corollary.

***Corollary* 4.2.** *If* $n \geq 2$, $s = \sum_{i=1}^n x_i$ *and* $\lambda \leq 1$, *then*

(1) $\displaystyle\prod_{i=1}^{n}\left(\frac{1}{x_i}-1\right) \geq \prod_{i=1}^{n}\left(\frac{n-\lambda}{s-\lambda x_i}-1\right)$ *for* $(x_1, x_2, \ldots, x_n) \in \left(0, \dfrac{1}{2}\right]^n$;

(2) $\displaystyle\prod_{i=1}^{n}\left(\frac{1}{x_i}-1\right) \leq \prod_{i=1}^{n}\left(\frac{n-\lambda}{s-\lambda x_i}-1\right)$ *for* $(x_1, x_2, \ldots, x_n) \in \left[\dfrac{1}{2}, 1\right]^n$.

Theorem 4.2. *If* $n \geq 2$, $x = (x_1, x_2, \ldots, x_n) \in (0, 1]^n$, $A_n(x) = \dfrac{\sum_{i=1}^n x_i}{n}$, $G_n(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ *and* $H_n(x) = \dfrac{n}{\sum_{i=1}^n \frac{1}{x_i}}$, *then*

(1) $A_n(x) \geq H_n(x)$;

(2) $\displaystyle\sum_{1\leq i_1<i_2<\ldots<i_r\leq n} \prod_{j=1}^{r}\left(\frac{1}{x_{i_j}}-1\right) \geq \frac{n!}{r!(n-r)!}\left[\frac{A_n(1-x)}{A_n(x)}\right]^r$ *for* $2 \leq r \leq n$ *and* $x \in \left(0, \dfrac{2n-r-1}{2n-2}\right]^n$;

(3) $\sum_{1\le i_1<i_2<\ldots<i_r\le n}\prod_{j=1}^{r}\left(\frac{1}{x_{i_j}}-1\right)\le\frac{n!}{r!(n-r)!}\left[\frac{A_n(1-x)}{A_n(x)}\right]^r$ *for* $2\le r\le n$ *and* $x\in\left[\frac{2n-r-1}{2n-2},1\right]^n$;

(4) $G_n(x)\ \ge\ H_n(x)$;

(5) $G_n(x)+G_n(1-x)\ \le\ 1$;

(6) $\sum_{1\le i_1<i_2<\ldots<i_r\le n}\prod_{j=1}^{r}\left(\frac{1}{x_{i_j}}-1\right)\ge\frac{n!}{r!(n-r)!}\left[\frac{G_n(x)-1}{G_n(x)}\right]^r$ *for* $n\ge 3,\ 2\le r\le n-1$ *and* $x\in\left(0,\frac{n-r}{n-1}\right]^n$;

(7) $\sum_{1\le i_1<i_2<\ldots<i_r\le n}\prod_{j=1}^{r}\left(\frac{1}{x_{i_j}}-1\right)\le\frac{n!}{r!(n-r)!}\left[\frac{G_n(x)-1}{G_n(x)}\right]^r$ *for* $n\ge 3,\ 2\le r\le n-1$ *and* $x\in\left[\frac{n-r}{n-1},1\right]^n$.

Proof. We clearly see that

$$\big(A_n(x),A_n(x),\ldots,A_n(x)\big)\prec(x_1,x_2,\ldots,x_n) \tag{4.1}$$

and

$$\log\big(G_n(x),G_n(x),\ldots,G_n(x)\big)\prec\log(x_1,x_2,\ldots,x_n). \tag{4.2}$$

Therefore, Theorem 4.2(1) follows from Theorem 1.3(1), (4.1) and (1.3). Theorem 4.2(2) and (3) follow from (4.1), (1.3) and Theorem 1.3(2) and (3), respectively. Theorem 4.2(4) follows from Theorem 1.4(1), (4.2) and (1.3). Theorem 4.2(5) follows from Theorem 1.4(2), (4.2) and (1.3). Theorem 4.2(6) and (7) follow from (4.2), (1.3) and Theorem 1.4(3)and (4), respectively.

If we take $r=n$ in Theorem 4.2(2), then we get the following corollary.

***Corollary* 4.3.** *If* $n\ge 2$ *and* $x=(x_1,x_2,\ldots,x_n)\in\left(0,\frac{1}{2}\right]^n$, *then*

$$\frac{G_n(1-x)}{G_n(x)}\ge\frac{A_n(1-x)}{A_n(x)}.$$

***Remark* 4.1.** The inequality in Corollary 4.3 is known as Ky Fan's inequality [20, p. 363; 2, p. 5]. There are already at least ten proofs of this result, see, for example, [1, 17, 18] and references cited therein.

Theorem 4.3. *Let* $\mathcal{A}=A_1A_2\ldots A_{n+1}$ *be a* n*-dimensional simplex in* R^n, $n\ge 2$, *and* P *be an arbitrary point in the interior of* $\mathcal{A}$. *If* B_i *is the intersection point of straight line* A_iP *and the hyperplane* $\sum_i=A_1A_2\ldots A_{i-1}A_{i+1}\ldots A_{n+1}$, $i=1,2,\ldots\ldots,n+1$, *then*

(1) $\sum_{i=1}^{n+1} \frac{PA_i}{PB_i} \geq n(n+1)$;

(2) $\sum_{i=1}^{n+1} \frac{PB_i}{PA_i} \geq \frac{n+1}{n}$.

Proof. One can easily see that $\sum_{i=1}^{n+1} \frac{PB_i}{A_iB_i} = 1$ and $\sum_{i=1}^{n+1} \frac{PA_i}{A_iB_i} = n$. Therefore, Theorem 4.3 follows from Theorem 1.3(1) and (1.3) together with the fact that

$$\left(\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1} \right) \prec \left(\frac{PB_1}{A_1B_1}, \frac{PB_2}{A_2B_2}, \ldots, \frac{PB_{n+1}}{A_{n+1}B_{n+1}} \right)$$

and

$$\left(\frac{n}{n+1}, \frac{n}{n+1}, \ldots, \frac{n}{n+1} \right) \prec \left(\frac{PA_1}{A_1B_1}, \frac{PA_2}{A_2B_2}, \ldots, \frac{PA_{n+1}}{A_{n+1}B_{n+1}} \right).$$

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1. *Alzer H.* A short proof of Ky Fan’s inequality // Arch. Math. (Brno). – 1991. – **27B**. – P. 199 – 200.
2. *Beckenbach E. F., Bellman R.* Inequalities. – Berlin: Springer-Verlag, 1961.
3. *Bullen P. S.* Handbook of means and their inequalities. – Dordrecht: Kluwer Acad. Publ., 2003.
4. *Chu Y. M., Zhang X. M.* Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave // J. Math. Kyoto Univ. – 2008. – **48**, № 1. – P. 229 – 238.
5. *Chu Y. M., Zhang X. M., Wang G. D.* The Schur geometrical convexity of the extended mean values // J. Convex Anal. – 2008. – **15**, № 4. – P. 707 – 718.
6. *Constantine G. M.* Schur-convex functions on the spectra of graphs // Discrete Math. – 1985. – **45**, № 2 – 3. – P. 181 – 188.
7. *Guan K. Z.* The Hamy symmetric function and its generalization // Math. Inequal. and Appl. – 2006. – **9**, № 4. – P. 797 – 805.
8. *Guan K. Z.* Schur-convexity of the complete symmetric function // Ibid. – P. 567 – 576.
9. *Guan K. Z.* Some properties of a class of symmetric functions // J. Math. Anal. and Appl. – 2007. – **336**, № 1. – P. 70 – 80.
10. *Guan K. Z.* A class of symmetric functions for multiplicatively convex function // Math. Inequal. and Appl. – 2007. – **10**, № 4. – P. 745 – 753.
11. *Guan K. Z., Shen J. H.* Schur-convexity for a class of symmetric function and its applications // Ibid. – 2006. – **9**, № 2. – P. 199 – 210.
12. *Hardy G. H., Littlewood J. E., Pólya G.* Some simple inequalities satisfied by convex functions // Messenger Math. – 1929. – **58**. – P. 145 – 152.
13. *Hwang F. K., Rothblum U. G.* Partition-optimization with Schur convex sum objective functions // SIAM J. Discrete Math. – 2004/2005. – **18**, № 3. – P. 512 – 524.
14. *Hwang F. K., Rothblum U. G., Shepp L.* Monotone optimal multipartitions using Schur convexity with respect to partial orders // Ibid. – 1993. – **6**, № 4. – P. 533 – 574.
15. *Jiang W. D.* Some properties of dual form of the Hamy's symmetric function // J. Math. Inequal. – 2007. – **1**, № 1. – P. 117 – 125.
16. *Marshall A. W., Olkin I.* Inequalities: theory of majorization and its applications. – New York: Acad. Press, 1979.
17. *McGregor M. T.* On some inequalities of Ky Fan and Wang-Wang // J. Math. Anal. and Appl. – 1993. – **180**, № 1. – P. 182 – 188.
18. *Mercer A. McD.* A short proof of Ky Fan’s arithmetic-geometric inequality // Ibid. – 1996. – **204**, № 3. – P. 940 – 942.
19. *Merkle M.* Convexity, Schur-convexity and bounds for the gamma function involving the digamma function // Rocky Mountain J. Math. – 1998. – **28**, № 3. – P. 1053 – 1066.
20. *Mitrinovic D. S.* Analytic inequalities. – New York: Springer-Verlag, 1970.

21. *Niculescu C. P.* Convexity according to the geometric mean // Math. Inequal. and Appl. – 2000. – **3**, № 2. – P. 155 – 167.
22. *Pečarić J., Proschan F., Tong Y. L.* Conves functions, partial orderings, and statistical applications. – New York: Acad. Press, 1992.
23. *Qi F.* A note on Schur-convexity of extended mean values // Rocky Mountain J. Math. – 2005. – **35**, № 5. – P. 1787 – 1793.
24. *Qi F., Sándor J., Dragomir S. S., Sofo A.* Note on the Schur-convexity of the extended mean values // Taiwan. J. Math. – 2005. – **9**, № 3. – P. 411 – 420.
25. *Shaked M., Shanthikumar J. G., Tong Y. L.* Parametric Schur convexity and arrangement monotonicity properties of partial sums // J. Multivar. Anal. – 1995. – **53**, № 2. – P. 293 – 310.
26. *Shi H. N.* Schur-convex functions related to Hadamard-type inequalities // J. Math. Inequal. – 2007. – **1**, № 1. – P. 127 – 136.
27. *Shi H. N., Wu S. H., Qi F.* An alternative note on the Schur-convexity of the extended mean values // Math. Inequal. and Appl. – 2006. – **9**, № 2. – P. 219 – 224.
28. *Stepniak C.* An effective characterization of Schur-convex function with applications // J. Convex Anal. – 2007. – **14**, № 1. – P. 103 – 108.
29. *Stepniak C.* Stochastic ordering and Schur-convex functions in comparison of linear experiments // Metrika. – 1989. – **36**, № 5. – P. 291 – 298.
30. *Wu S. H.* Generalization and sharpness of the power means inequality and their applications // J. Math. Anal. and Appl. – 2005. – **312**, № 2. – P. 637 – 652.
31. *Zhang X. M.* Optimization of Schur-convex functions // Math. Inequal. and Appl. – 1998. – **1**, № 3. – P. 319 – 330.
32. *Zhang X. M.* Schur-convex functions and isperimetric inequalities // Proc. Amer. Math. Soc. – 1998. – **126**, № 2. – P. 461 – 470.

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