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BOUNDEDNESS OF MULTILINEAR SINGULAR INTEGRAL OPERATORS ON THE HOMOGENEOUS MORREY – HERZ SPACES

ОБМЕЖЕНІСТЬ БАГАТОЛІНІЙНИХ СИНГУЛЯРНИХ ІНТЕГРАЛЬНИХ ОПЕРАТОРІВ НА ОДНОРІДНИХ ПРОСТОРАХ МОРРЕЯ – ГЕРЦА

A boundedness result is established for the multilinear singular integral operators on the homogeneous Morrey – Herz spaces. As applications, two corollaries about interesting cases of the boundedness of considered operators on the homogeneous Morrey – Herz spaces are obtained.

Встановлено обмеженість багатолінійних сингулярних інтегральних операторів на однорідних просторах Моррея – Герца. Як застосування, одержано два наслідки про цікаві випадки обмеженості розглядуваних операторів на однорідних просторах Моррея – Герца.

1. Introduction and main results. Let $\mathbb{R}^n$, $n \geq 1$, be the n-dimensional Euclidean space and let S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ and $\Omega \in L^r(S^{n-1})$ for some $r \in [1, \infty]$, and $\int_{S^{n-1}} \Omega(x')\, d\sigma(x') = 0$, where $x' = x|x|^{-1}$ for any $x \neq 0$. If $f \in L^q_\omega(\mathbb{R}^n)$, that is

$$\| f \|_{L^q_\omega(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} | f(x)|^q \, \omega(x)\, dx \right)^{1/q} < \infty .$$

The Calderón – Zygmund singular integral operator T is defined by

$$T f(x) = \text{p. v.} \int\limits_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)\, dy ,$$

and the truncated maximal operator T^* is defined by

$$T^* f(x) = \sup_{\varepsilon > 0} \left| \int\limits_{|x-y| > \varepsilon} \frac{\Omega(y')}{|y|^n} f(x-y)\, dy \right| ,$$

where $y' = y|y|^{-1} \in S^{n-1}$ and $f \in C_0^\infty(\mathbb{R}^n)$.

In 1971, Muckenhoupt and Wheeden [1] proved that if $\Omega(x') \in L^r(S^{n-1})$, $r > 1$, then the operators T and T^* are bounded on $L^q_{|x|^\beta}(\mathbb{R}^n)$, $1 < q < \infty$, provided that β be in the interval $\big(\max(-n, -1-(n-1)q/r'),\ \min(n(q-1), q-1+(n-1)q/r')\big)$. Here and in the following, r' denotes the dual exponent of r, i.e., $r' = r/(r-1)$, and $L^q_{|x|^\beta}(\mathbb{R}^n)$ denotes the weighted Lebesgue space. For general A_q weights, Duoandikoetxea [2] gave the weighted L^q, $1 < q < \infty$, boundedness of T and T^*.

The purpose of this paper is to consider the boundedness on homogeneous Morrey – Herz spaces for the multilinear singular integral operators $T_{A,m}$ and $T^*_{A,m}$ which are defined as follows,

$$T_{A,m}f(x) = \text{p. v.} \int_{\mathbb{R}^n} R_{m+1}(A;\, x,\, y)\frac{\Omega(x-y)}{|x-y|^{n+m}} f(y)\,dy,$$

$$T^*_{A,m}f(x) = \sup_{\varepsilon>0}\left| \int_{|x-y|>\varepsilon} R_{m+1}(A;\, x,\, y)\frac{\Omega(x-y)}{|x-y|^{n+m}} f(y)\,dy \right|,$$

where m is positive integer, A has derivatives of order m in $\mathrm{BMO}(\mathbb{R}^n)$, $R_{m+1}(A;\, x,\, y)$ denotes the $(m+1)$-th Taylor series remainder of A at x about y, that is

$$R_{m+1}(A;\, x,\, y) = A(x) - \sum_{|\mu|\le m} \frac{1}{\mu!} D^\mu A(y)(x-y)^\mu,$$

and $f \in C_0^\infty(\mathbb{R}^n)$.

Let $B_k = B(0, 2^k) = \left\{x \in \mathbb{R}^n : |x| \le 2^k\right\}$ and $C_k = B_k \backslash B_{k-1}$ for $k \in \mathbb{Z}$. Let $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$ be the characteristic function of the set C_k.

Definition 1 [3]. *Let* $\alpha \in \mathbb{R}$, $0 < p \le \infty$, $0 < q < \infty$ *and* $\lambda \ge 0$. *The homogeneous Morrey – Herz spaces* $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ *are defined by*

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n) = \left\{ f \in L^q_{\mathrm{loc}}\big(\mathbb{R}^n\backslash\{0\}\big) \colon \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)} \right)^{1/p}$$

with the usual modifications made when $p = \infty$.

Now, let us state the main results of this paper.

Theorem 1. *Let* $R_{m+1}(A;\, x,\, y)$ *and* A *be defined as above*, $r \ge 1$, $m \in \mathbb{N}$, $\lambda \ge 0$, $0 < p \le \infty$, $1 < q < \infty$, $\max\big(-n/q + \lambda,\ -1/q - (n-1)/r' + \lambda\big) < \alpha < \min\big(n(1-1/q) + \lambda,\ 1 - 1/q + (n-1)/r' + \lambda\big)$. *And let* $\tilde{\Omega}$ *be a homogeneous function of degree zero on* $\mathbb{R}^n$ *with* $\tilde{\Omega} \in L^r(\ln L)(S^{n-1})$, *that is*

$$\int_{S^{n-1}} \left|\tilde{\Omega}(x')\right|^r \ln\left(2 + \left|\tilde{\Omega}(x')\right|\right) d\sigma(x') \;<\; \infty .$$

If a sublinear operator $\tilde{T}_{b,m}$ *is bounded on* $L^q(\mathbb{R}^n)$ *and there is a constant* C *independent of* f *such that*

$$\left|\tilde{T}_{b,m} f(x)\right| \;\le\; C \int_{\mathbb{R}^n} \frac{\left|\tilde{\Omega}(x-y)\right|}{|x-y|^{n+m}} \left|R_{m+1}(A;\, x,\, y)\right| \left|f(y)\right| dy$$

for any $f \in L^1(\mathbb{R}^n)$ *with compact support and* $x \notin \operatorname{supp} f$, *then* $\tilde{T}_{b,m}$ *is also bounded on* $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose value may differ from line to line.

2. Proof of Theorem 1. Let $\bar{\Omega}$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and $\bar{\Omega} \in L^r(S^{n-1})$ for some $r \in [1, \infty]$. The truncated operator $S_{t;b,\bar{\Omega}}$ is defined by

$$S_{t;A,\bar{\Omega}} f(x) \;=\; t^{-n} \int_{|x-y|<t} \frac{\left|\bar{\Omega}(x-y)\right|}{|x-y|^m} \left|R_{m+1}(A;\, x,\, y)\right| \left|f(y)\right| dy .$$

In this section we give two lemmas which are the key to the proof of Theorem 1.

Lemma 1 [4]. *Let* $A(x)$ *be a function on* $\mathbb{R}^n$ *with derivatives of order* m *in* $L^r(\mathbb{R}^n)$ *for some* $r \in (n, \infty]$. *Then*

$$\left|R_m(A;\, x,\, y)\right| \;\le\; C_{m,n} |x-y|^m \sum_{|\mu|=m} \left(\frac{1}{\left|\bar{\Omega}(x,\, y)\right|} \int_{\bar{\Omega}(x,\, y)} \left|D^{\mu} A(z)\right|^r dz \right)^{1/r},$$

where $\bar{\Omega}(x, y)$ *is the cube centered at* x *with sides parallel to the coordinate axes and whose side length is* $5\sqrt{n}\,|x-y|$.

Lemma 2 [5]. *Let* $R_{m+1}(A;\, x,\, y)$ *and* $A(x)$ *be defined as above. Let* $\bar{\Omega} \in L^{\infty}(S^{n-1})$ *be a homogeneous function of degree zero on* $\mathbb{R}^n$, $m \in \mathbb{N}$, $r \ge 1$, $|\mu| = m$, $1 < q < \infty$ *and* $D^{\mu} A \in \mathrm{BMO}(\mathbb{R}^n)$. *Set*

$$\xi_{\bar{\Omega}} \;=\; \inf\left\{ \xi > 0 : \frac{\|\bar{\Omega}\|_r}{\xi} \ln\left(2 + \frac{\|\bar{\Omega}\|_{\infty}}{\xi}\right) \le 1 \right\}.$$

If $\max\left(-n, -1-(n-1)q/r'\right) < \beta < \min\left(n(q-1), q-1+(n-1)q/r'\right)$, *then the truncated operator* $S_{t;A,\bar{\Omega}}$ *is bounded on* $L^q_{|x|^{\beta}}(\mathbb{R}^n)$ *with bound* $C\sum_{|\mu|=m} \left\| D^{\mu} A \right\|_{\mathrm{BMO}(\mathbb{R}^n)} \xi_{\bar{\Omega}}$.

Here and in what follows, let $\|f\|_p$ denote $\|f\|_{L^p(S^{n-1})}$. And without loss of generality, we may assume that $\sum_{|\mu|=m} \left\| D^{\mu} A \right\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$.

***Proof of Theorem* 1.** We choose $\alpha_1, \alpha_2 \in \mathbb{R}$, such that $\max\left(-n/q, -1/q-(n-1)/r'\right) < \alpha_1/q < \alpha-\lambda < \alpha_2/q < \min\left(n(1-1/q), 1-1/q+(n-1)/r'\right)$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x).$$

Then, we have

$$\left\|\tilde{T}_{A,m}(f)\right\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left\|\chi_k\tilde{T}_{A,m}(f)\right\|^p_{L^q(\mathbb{R}^n)}\right)^{1/p} \le$$

$$\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}\left\|\chi_k\tilde{T}_{A,m}(f_j)\right\|_{L^q(\mathbb{R}^n)}\right)^p\right)^{1/p} +$$

$$+ C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\left\|\chi_k\tilde{T}_{A,m}(f_j)\right\|_{L^q(\mathbb{R}^n)}\right)^p\right)^{1/p} +$$

$$+ C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k+2}^{\infty}\left\|\chi_k\tilde{T}_{A,m}(f_j)\right\|_{L^q(\mathbb{R}^n)}\right)^p\right)^{1/p} \equiv$$

$$\equiv E_1 + E_2 + E_3.$$

For E_2, by the $L^q(\mathbb{R}^n)$ boundedness of $\tilde{T}_{A,m}$, we have

$$E_2 \le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left(\sum_{j=k-1}^{k+1}\left\|\chi_k f_j\right\|_{L^q(\mathbb{R}^n)}\right)^p\right)^{1/p} \le$$

$$\le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\left\|f\chi_k\right\|^p_{L^q(\mathbb{R}^n)}\right)^{1/p} =$$

$$= C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

For E_1, note that when $x\in C_k$, $j\le k-2$, and $y\in C_j$, then $2|y|\le|x|$. Therefore, for $x\in C_k$,

$$\left|\tilde{T}_{A,m}f_j(x)\right| \le C\int_{\mathbb{R}^n}\frac{\left|\tilde{\Omega}(x-y)\right|}{|x-y|^{n+m}}\left|R_{m+1}(A;\,x,\,y)\right|\left|f_j(y)\right|dy \le$$

$$\le \frac{C}{|x|^n}\int_{|x-y|\le\frac{3}{2}|x|}\frac{\left|\tilde{\Omega}(x-y)\right|}{|x-y|^m}\left|R_{m+1}(A;\,x,\,y)\right|\left|f_j(y)\right|dy \le CS_{2^{k+1};A,\tilde{\Omega}}f_j(x).$$

Let $F_0=\left\{x\in S^{n-1}:\left|\tilde{\Omega}(x)\right|\le 2\right\}$ and $F_d=\left\{x\in S^{n-1}:2^d<\left|\tilde{\Omega}(x)\right|\le 2^{d+1}\right\}$ for positive integer d. Denote by $\tilde{\Omega}_d$ the restriction of $\tilde{\Omega}$ on F_d. Then

$$S_{2^{k+1};A,\tilde{\Omega}}f_j(x) = \sum_{d=0}^{\infty} S_{2^{k+1};A,\tilde{\Omega}_d}f_j(x).$$

Thus, by Lemma 2 and the conditions in Theorem 1, we have

$$\left\| S_{2^{k+1};A,\tilde\Omega} f_j \right\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)} \le \sum_{d=0}^{\infty} \left\| S_{2^{k+1};A,\tilde\Omega_d} f_j \right\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)} \le$$

$$\le C \sum_{d=0}^{\infty} \xi_{\bar\Omega_d} \left\| f_j \right\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)} \le C \left\| f_j \right\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)}. \tag{1}$$

In obtaining the last inequality, we use the fact that $\sum_{d=0}^{\infty} \xi_{\bar\Omega_d} \le C$. In fact, our hypothesis on $\tilde\Omega$ now says that $\sum_{d>0} d \left\| \tilde\Omega_d \right\|_r < \infty$. Set $\xi_d = d \left\| \tilde\Omega_d \right\|_r + 2^{-d}$. It is obvious that

$$\frac{\left\| \tilde\Omega_d \right\|_r}{\xi_d} \ln\left(2 + \frac{\left\| \tilde\Omega_d \right\|_\infty}{\xi_d}\right) \le \frac{\left\| \tilde\Omega_d \right\|_r}{d \left\| \tilde\Omega_d \right\|_r} \ln\left(2^{2d+1}\right) \le C.$$

This in turn implies that $\xi_{\bar\Omega_d} \le C\left(d \left\| \tilde\Omega_d \right\|_r + 2^{-d}\right)$. Therefore,

$$\sum_{d \ge 0} \xi_{\bar\Omega_d} = \xi_{\tilde\Omega_0} + \sum_{d>0} \xi_{\bar\Omega_d} \le C + C \sum_{d>0} d \left\| \tilde\Omega_d \right\|_r + C \sum_{d>0} 2^{-d} \le C.$$

Thus, by (1), it follows that

$$E_1 \le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k(\alpha - \alpha_2/q)p} \left(\sum_{j=-\infty}^{k-2} \left\| \chi_k \tilde T_{A,m}(f_j) \right\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)} \right)^p \right)^{1/p} \le$$

$$\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k(\alpha - \alpha_2/q)p} \left(\sum_{j=-\infty}^{k-2} \left\| S_{2^{k+1};A,\tilde\Omega}(f_j) \right\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)} \right)^p \right)^{1/p} \le$$

$$\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k(\alpha - \alpha_2/q)p} \left(\sum_{j=-\infty}^{k-2} \left\| f_j \right\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)} \right)^p \right)^{1/p} \le$$

$$\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(\lambda - \alpha + \alpha_2/q) - j\lambda} \times \right.\right.$$

$$\left.\left. \times \left(\sum_{l=-\infty}^{j} 2^{l\alpha p} \left\| f_l \right\|^p_{L^q(\mathbb{R}^n)} \right)^{1/p} \right)^p \right)^{1/p} \le$$

$$\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(\lambda - \alpha + \alpha_2/q)} \left\| f \right\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \right)^p \right)^{1/p} \le$$

$$\le C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p} \right)^{1/p} \left\| f \right\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \le$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} 2^{k_0\lambda} \| f \|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = C \| f \|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

Now, let us turn to estimate for E_3. Note that when $x \in C_k$, $j \geq k+2$, and $y \in C_j$, then $2|x| \leq |y|$ and

$$\left| \tilde{T}_{A,m} f_j(x) \right| \leq C \int_{\mathbb{R}^n} \frac{\left| \tilde{\Omega}(x-y) \right|}{|x-y|^{n+m}} \left| R_{m+1}(A;x,y) \right| \left| f_j(y) \right| dy \leq$$

$$\leq C 2^{-jn} \int_{|x-y| \leq 2^{j+1}} \frac{\left| \tilde{\Omega}(x-y) \right|}{|x-y|^{m}} \left| R_{m+1}(A;x,y) \right| \left| f_j(y) \right| dy \leq C S_{2^{j+1};A,\tilde{\Omega}} f_j(x).$$

Thus, by (1), similar to the proof of E_1, we have

$$E_3 \leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k(\alpha-\alpha_1/q)p} \left(\sum_{j=k+2}^{\infty} \left\| \chi_k \tilde{T}_{A,m}(f_j) \right\|_{L^q_{|x|^{\alpha_1}}(\mathbb{R}^n)} \right)^p \right)^{1/p} \leq$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k(\alpha-\alpha_1/q)p} \left(\sum_{j=k+2}^{\infty} \left\| S_{2^{j+1};A,\tilde{\Omega}}(f_j) \right\|_{L^q_{|x|^{\alpha_1}}(\mathbb{R}^n)} \right)^p \right)^{1/p} \leq$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k(\alpha-\alpha_1/q)p} \left(\sum_{j=k+2}^{\infty} \left\| (f_j) \right\|_{L^q_{|x|^{\alpha_1}}(\mathbb{R}^n)} \right)^p \right)^{1/p} \leq$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda-\alpha_1/q)-j\lambda} \times \right.\right.$$

$$\left.\left. \times \left(\sum_{l=-\infty}^{j} 2^{l\alpha p} \| f_l \|^p_{L^q(\mathbb{R}^n)} \right)^{1/p} \right)^p \right)^{1/p} \leq$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p} \left(\sum_{j=k+2}^{\infty} 2^{(k-j)(\alpha-\lambda-\alpha_1/q)} \| f \|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \right)^p \right)^{1/p} \leq$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left(\sum_{k=-\infty}^{k_0} 2^{k\lambda p} \right)^{1/p} \| f \|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} \leq$$

$$\leq C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} 2^{k_0\lambda} \| f \|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = C \| f \|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

Therefore, Theorem 1 is proved.

3. Corollary. The above proof of Theorem 1 also indicates that the boundedness of operator $\tilde{T}_{A,m}$ on Lebesgue spaces with power weights implies its boundedness on homogeneous Morrey – Herz spaces. Similar result is proved by Lu and Xu in [3].

The following lemma for the boundedness of operators $T_{A,m}$ and $T^*_{A,m}$ on the Lebesgue spaces with power weights can be found in [5].

Lemma 3 [5]. *Let* $R_{m+1}(A;\, x,\, y)$ *and* A *be defined as above,* $m \in \mathbb{N}$, $r > 1$, *and* $\Omega \in L^r(\ln L)(S^{n-1})$. *If* $\max\big(-n,\, -1-(n-1)\,q/r'\big) < \beta < \min\big(n(q-1),\, q-1+(n-1)\,q/r'\big)$ *and* $1 < q < \infty$, *then*

$$\big\| T_{A,m}\, f \big\|_{L^q_{|x|^\beta}(\mathbb{R}^n)} \;\le\; C\,\| f \|_{L^q_{|x|^\beta}(\mathbb{R}^n)}$$

and

$$\big\| T^*_{A,m}\, f \big\|_{L^q_{|x|^\beta}(\mathbb{R}^n)} \;\le\; C\,\| f \|_{L^q_{|x|^\beta}(\mathbb{R}^n)}.$$

As a simple corollary of Theorem 1 and Lemma 3, when $r > 1$, we have the following result.

***Corollary* 1.** *Let* $R_{m+1}(A;\, x,\, y)$ *and* A *be defined as above,* $m \in \mathbb{N}$, $r > 1$, $1 < q < \infty$, $0 < p \le \infty$, *and* $\Omega \in L^r(\ln L)(S^{n-1})$. *If* $\max\big(-n/q + \lambda,\, -1/q - (n-1)\,r' + \lambda\big) < \alpha < \min\big(n(1-1/q) + \lambda,\, 1 - 1/q + (n-1)\,r' + \lambda\big)$, *then* $T_{A,m}$ *and* $T^*_{A,m}$ *are bounded on* $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.

Lemma 4 [5]. *Let* $R_{m+1}(A;\, x,\, y)$ *and* A *be defined as above,* $m \in \mathbb{N}$, $r = 1$, *and* $\Omega \in L(\ln L)(S^{n-1})$. *If* $-1 < \beta < q - 1$ *and* $1 < q < \infty$, *then*

$$\big\| T_{A,m}\, f \big\|_{L^q_{|x|^\beta}(\mathbb{R}^n)} \;\le\; C\,\| f \|_{L^q_{|x|^\beta}(\mathbb{R}^n)}$$

and

$$\big\| T^*_{A,m}\, f \big\|_{L^q_{|x|^\beta}(\mathbb{R}^n)} \;\le\; C\,\| f \|_{L^q_{|x|^\beta}(\mathbb{R}^n)}.$$

As a simple corollary of Theorem 1 and Lemma 4, when $r = 1$, we have the following result.

***Corollary* 2.** *Let* $R_{m+1}(A;\, x,\, y)$ *and* A *be defined as above,* $m \in \mathbb{N}$, $1 < q < \infty$, $0 < p \le \infty$, *and* $\Omega \in L(\ln L)(S^{n-1})$. *If* $-1 < \beta < q - 1$, *then* $T_{A,m}$ *and* $T^*_{A,m}$ *are bounded on* $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.

According to [3] (Theorem 2.1), Corollaries 1 and 2 are proved easily.

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