UDC 517.9

T. S. Gadjiev, V. A. Mamedova (Inst. Math. and Mech. Nat. Acad. Sci. Azerbaijan, Baku)

ON REMOVABLE SETS OF SOLUTIONS OF SECOND-ORDER ELLIPTIC AND PARABOLIC EQUATIONS IN NONDIVERGENT FORM

ПРО УСУВНІ МНОЖИНИ РОЗВ'ЯЗКІВ ЕЛІПТИЧНИХ ТА ПАРАБОЛІЧНИХ РІВНЯНЬ ДРУГОГО ПОРЯДКУ У НЕДИВЕРГЕНТНІЙ ФОРМІ

We consider nondivergent elliptic and parabolic equations of the second order whose leading coefficients satisfy the uniform Lipschitz condition. We find the sufficient condition of removability of compact with respect to these equations in the space of Hölder functions.

Розглянуто недивергентні еліптичні та параболічні рівняння другого порядку, у яких коефіцієнти при старших членах задовольняють однорідну умову Ліпшиця. Знайдено достатню умову усувності компакту відносно цих рівнянь у просторі функцій Гельдера.

Introduction. The subject of this paper is finding the sufficient condition of removability of compact for nondivergent elliptic and parabolic equations in the space $C^{0,\lambda}\left(\overline{D}\right)$. This problem have been investigated by many researchers. For the Laplace equation the corresponding result was found by L. Carleson [1]. Concerning the second-order elliptic equations of divergent structure, we show in this direction the papers T. S. Gadjiev, V. A. Mamedova [2], E. I. Moiseev [3]. For a class of nondivergent elliptic equations of the second order with discontinuous coefficients of the removability condition was considered by I. T. Mamedov [4]. Note also the papers E. M. Landis [5], T. S. Gadjiev, V. A.Mamedova [6], in which the conditions of removability have been obtained for a compact in the space of continuous functions. In [7], T. Kilpelainen and X. Zhong have studied the divergent quasilinear equation without minor members proved the removability of compact. Removable sets for pointwise solutions of elliptic partial differential equations was found by J. Diederich [8]. Removable singularities of solutions of linear partial differential equations were considered in R. Harvey, J. Polking [9]. Exceptional sets at the boundary for subharmonic functions were investigated by B. Dahlberg [10].

The aim of our paper is to consider the removability question from the single point of view for nondivergent elliptic and parabolic equations. The paper consists of three parts: in the first part, we consider the Dirichlet problem for nondivergent elliptic equation of the second order; in the second part, we consider the Neumann problem for nondivergent parabolic equation of the second order; in the third part, we consider the mixed problem for nondivergent parabolic equation of the second order.

As opposed to previous works, in this paper, in terms of Hausdorff measures, more exact geometrical characteristics of removability are given. Note that in most cases in previous papers, characteristics of removability were basically presented for narrow class of equations in terms of capacities. The value of our paper is that for the first time in this work, we are considering the wide classes of the nondivergent elliptic and parabolic equations with minor members. Besides, the removability conditions of compact is obtained in terms of Hausdorff measure.

1. Let's consider Dirichlet problem for nondivergent elliptic equation of the second order. Let D be a bounded domain situated in n dimensional Euclidean space R^n of points $x = (x_1, \ldots, x_n)$, $n \geq 3$, ∂D be its boundary. Consider in D the elliptic equation

$$\mathcal{L}u = \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} + \sum_{i=1}^{n} b_i(x)u_i + c(x)u = 0, \tag{1}$$

in supposition that $\{a_{ij}(x)\}$ is a real symmetric matrix, moreover,

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \gamma^{-1}|\xi|^2, \quad \xi \in R^n, \quad x \in D, \tag{2}$$

$$a_{ij}(x) \in C^1\left(\overline{D}\right), \quad i,j = 1,\ldots,n, \tag{3}$$

$$|b_i(x)| \leq b_0, \quad -b_0 \leq c(x) \leq 0, \quad i = 1,\ldots,n, \quad x \in D. \tag{4}$$

Here, $u_i = \dfrac{\partial u}{\partial x_i}$, $u_{ij} = \dfrac{\partial^2 u}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, n$, $\gamma \in (0,1]$ and $b_0 \geq 0$ are constants. Besides we'll suppose that the lower coefficients of the operator $\mathcal{L}$ are measurable functions in D. Let $\lambda \in (0,1)$ be a number. Denote by $C^{0,\lambda}\left(\overline{D}\right)$ a Banach space of the functions $u(x)$ defined in D with the finite norm

$$\|u\|_{C^{0,\lambda}(D)} = \sup_{x \in D} |u(x)| + \sup_{\substack{x,y \in D \\ x \neq y}} \frac{|u(x) - u(y)|}{|x-y|^{\lambda}}.$$

The compact $E \subset \overline{D}$ is called exceptional with respect to the equation (1) in the space $C^{0,\lambda}\left(\overline{D}\right)$ if from

$$\mathcal{L}u = 0, \quad x \in D \backslash E, \quad u|_{\partial D \backslash E} = 0, \quad u(x) \in C^{0,\lambda}\left(\overline{D}\right) \tag{5}$$

it follows that $u(x) \equiv 0$ in D.

Denote by $B_R(z)$ and $S_R(z)$ the ball $\{x\colon |x-z| < R\}$ and the sphere $\{x : |x - z| = R\}$ of radius R with the center at the point $z \in R^n$ respectively. We'll need the following generalization of mean value theorem belonging to E. M. Landis and M. L. Gerver [11].

Let the domain G be considered between the spheres $S_R(0)$ and $S_{2R}(0)$ and let the part of the boundary of this domain, which is located strictly inside of lair $R < |x| < 2R$, be a smooth surface. If we specify it in this way, it shows $\partial G \cap \{x : R < |x| < 2R\}$ should not be ∂G. Further, let in $\overline{G}$ the uniformly positive definite matrix $\{a_{ij}(x)\}$, $i,j = 1,\ldots,n$, and the function $u(x) \in C^2(G) \cap C^1\left(\overline{G}\right)$ be given. Then there exists the piecewise smooth surface Σ dividing in G the spheres $S_R(0)$ and $S_{2R}(0)$ such that

$$\int\limits_{\Sigma} \left|\frac{\partial u}{\partial \nu}\right| ds \leq K \operatorname*{osc}_{G} u \frac{\operatorname{mes}_n G}{R^2}.$$

Here $K > 0$ is a constant, depending only on the matrix $\{a_{ij}(x)\}$ and n, and $\dfrac{\partial u}{\partial \nu}$ is a derivative by a conormal determined by the equality

$$\frac{\partial u(x)}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \cos(\bar{n}, x_j),$$

where $\cos(\bar{n}, x_j)$, $j = 1, \ldots, n$, are directing cosines of unit external normal vector to Σ.

***Remark* 1.** We say that Σ divides the spheres $S_R(0)$ and $S_{2R}(0)$ in G, if there exists such $\varepsilon > 0$ that each broken, laying in G and connecting the points belonging to ε neighbourhood of $S_R(0)$ and ε neighbourhood $S_{2R}(0)$ has not an empty intersection with ε.

Denote by $W_2^1(D)$ a Banach space of the functions $u(x)$ given in D with the finite norm

$$\|u\|_{W_2^1(D)} = \left(\int_D \left(u^2 + \sum_{i,j=1}^{n} u_i^2 \right) dx \right)^{1/2},$$

and let $\overset{\circ}{W}{}_2^1(D)$ be a completion $C_0^\infty(D)$ by the norm of the space $W_2^1(D)$.

By $m_H^s(A)$ we'll denote the Hausdorff measure of the set A of order $s > 0$. Further everywhere the notation $C(\ldots)$ means that the positive constant C depends only on content of brackets.

Theorem 1. *Let D be a bounded domain in R^n, $E \subset \overline{D}$ be a compact. If with respect to the coefficients of the operator $\mathcal{L}$ the conditions* (2) – (4) *are fulfilled, then for exceptionality of the compact E with respect to the equation* (1) *in the space $C^{0,\lambda}\left(\overline{D}\right)$ it sufficies that*

$$m_H^{n-2+\lambda}(E) = 0. \tag{6}$$

Proof. At first we show that without loss of generality we can suppose the condition $\partial D \in C^1$ to be fulfilled. Suppose that the condition (6) provides the exceptionability of the compact E for the domains, whose boundary is the surface of the class C^1, but $\partial D \notin C^1$ and when fulfilling (6) the compact E is not exceptional. Then the problem (5) has nontrivial solution $u(x)$, moreover $u|_E = f(x)$ and $f(x) \neq 0$. We always can suppose the lowest coefficients of the operator $\mathcal{L}$ to be infinitely differentiable in D. Moreover, without loss of generality, we'll suppose that the coefficients of the operator $\mathcal{L}$ are extended to a ball $B \supset \overline{D}$ with saving the conditions (2) – (4). Let $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \min\{f(x), 0\}$, and $u^\pm(x)$ be solutions of the boundary-value problems generalized by Wiener (see [12])

$$\mathcal{L}u^\pm = 0, \quad x \in D \backslash E, \qquad u^\pm\big|_{\partial D \backslash E} = 0, \qquad u^\pm\big|_E = f^\pm.$$

It is evident, that $u(x) = u^+(x) + u^-(x)$. Further, let D' be a domain such that $\partial D' \in C^1$, $\overline{D} \subset D'$, $\overline{D}' \subset B$, and $\vartheta^\pm(x)$ be solutions of the problems

$$\mathcal{L}\vartheta^\pm(x) = 0, \quad x \in D' \backslash E,$$

$$\vartheta^\pm\big|_{\partial D'} = 0, \qquad \vartheta^\pm\big|_E = f^\pm, \quad \vartheta^\pm(x) \in C^{0,\lambda}(D').$$

By the maximum principle for $x \in D$

$$0 \le u^+(x) \le \vartheta^+(x), \quad \vartheta^-(x) \le u^-(x) \le 0.$$

But according to our supposition $\vartheta^+(x) \equiv \vartheta^-(x) \equiv 0$. Hence, it follows that $u(x) \equiv 0$. So, we'll suppose that $\partial D \in C^1$. Now, let $u(x)$ be a solution of the problem (5), and the condition (6) be fulfilled. Give an arbitrary $\varepsilon > 0$. Then there exists a sufficiently small positive number δ and a system of the balls $\left\{B_{r_k}(x^k)\right\}$, $k = 1, 2, \ldots$, such that $r_k < \delta$, $E \subset \bigcup\limits_{k=1}^{\infty} B_{r_k}(x^k)$ and

$$\sum_{k=1}^{\infty} r_k^{n-2+\lambda} < \varepsilon. \tag{7}$$

Consider a system of the spheres $\left\{B_{2r_k}(x^k)\right\}$, and let $D_k = D \cap B_{2r_k}(x^k)$, $k = 1, 2, \ldots$. Without loss of generality, we can suppose that the cover $\left\{B_{2r_k}(x^k)\right\}$ has a finite multiplicity $a_0(n)$. By the Landis – Gerver theorem, for every k there exists a piecewise smooth surface Σ_k dividing in D_k the spheres $S_{r_k}(x^k)$ and $S_{2r_k}(x^k)$ such that

$$\int\limits_{\Sigma_k} \left|\frac{\partial u}{\partial \nu}\right| ds \le K \operatorname*{osc}_{D_k} u \frac{\mathrm{mes}_n D_k}{r_k^2}. \tag{8}$$

Since $u(x) \in C^{0,\lambda}\left(\overline{D}\right)$, there exists a constant $H_1 > 0$, depending only on the function $u(x)$, such that

$$\operatorname*{osc}_{D_k} u \le H_1 \left(2r_k\right)^{\lambda}. \tag{9}$$

Besides

$$\mathrm{mes}_n D_k \le \mathrm{mes}_n B_{2r_k}(x^k) = \Omega_n 2^n r_k^n, \quad k = 1, 2, \ldots, \tag{10}$$

where $\Omega_n = \mathrm{mes}_n B_1\left(0\right)$. Considering (9), (10) in (8), we get

$$\int\limits_{\Sigma_k} \left|\frac{\partial u}{\partial \nu}\right| ds \le C_1 r_k^{n-2+\lambda}, \quad k = 1, 2, \ldots, \tag{11}$$

where $C_1 = K H_1 2^{n+\lambda}$.

Let D_Σ be an open set, arranged in $D \backslash E$, whose boundary consists on unification of Σ and Γ, where $\Sigma = \bigcup\limits_{k=1}^{\infty} \Sigma_k$, $\Gamma = \partial D \backslash \bigcup\limits_{k=1}^{\infty} D_k^+$, D_k^+ be a part of D_k, remained after the partition of points, arranged between the Σ_k and $S_{2r_k}(x^k)$, $k = 1, 2, \ldots$. Denote by D'_Σ an arbitrary connected component of D_Σ, and by M-elliptic operator of a divergent structure

$$B = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

According to the Green formula for any functions $z(x)$ and $\omega(x)$ belonging to the intersection $C^2\left(D'_\Sigma\right) \cap C^1\left(\overline{D}'_\Sigma\right)$, we have

$$\int\limits_{D'_\Sigma} \left(zB\omega - \omega Bz\right) dx = \int\limits_{\partial D'_\Sigma} \left(z\frac{\partial \omega}{\partial \nu} - \omega \frac{\partial z}{\partial \nu}\right) ds. \tag{12}$$

Since $\partial D \in C^1$, we have $u(x) \in C^2\left(D'_\Sigma\right) \cap C^1\left(\overline{D}'_\Sigma\right)$ (see [13]). Supposing in (12) $z = 1$, $\omega = u^2$, we get

$$\int\limits_{D_\Sigma} B(u^2) dx = \int\limits_{\partial D'_\Sigma} u\frac{\partial u}{\partial \nu} ds.$$

But $|u(x)| \le M < \infty$ for $x \in \overline{D}$. Therefore allowing for (11) and (7) we conclude

$$\int\limits_{D'_\Sigma} B(u^2) dx \le 2Ma_0 \sum_{k=1}^{\infty} \int\limits_{\Sigma_k} \left|\frac{\partial u}{\partial \nu}\right| ds \le 2Ma_0 C_1 \sum_{k=1}^{\infty} r_k^{n-2+\lambda} < C_2 \varepsilon, \tag{13}$$

where $C_2 = 2Ma_0 C_1$.

On the other hand

$$B(u^2) = 2uBu + 2\sum_{i,j=1}^{n} a_{ij}(x)\, u_i u_j,$$

and besides

$$B(x) = \mathcal{L}u + \sum_{i=1}^{n} d_i(x) u_i - c(x)u,$$

where $d_i = \sum_{i=1}^{n} \dfrac{\partial a_{ij}(x)}{\partial x_j} - b_i(x)$, $i = 1, \ldots, n$. It is clear that by virtue of conditions (3), (4) $\left|d_i(x)\right| \le d_0 < \infty$, $i = 1, \ldots, n$. Thus from (13) we obtain

$$2\int\limits_{D'_\Sigma} u \sum_{i=1}^{n} d_i(x) u_{x_i} dx - 2\int\limits_{D'_\Sigma} u^2 c(x) dx + 2\int\limits_{D'_\Sigma} \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i} u_{x_j} dx < C_2 \varepsilon.$$

Hence, it follows that for any $\alpha > 0$

$$\begin{aligned} 2\gamma \int\limits_{D'_\Sigma} |\nabla u|^2 dx &< 2d_0 \int\limits_{D'_\Sigma} |u|\,|u_i|\, dx + C_2\varepsilon \le \\ &\le d_0\lambda \int\limits_{D'_\Sigma} |\nabla u|^2 dx + \frac{d_0 n}{\lambda} \int\limits_{D'_\Sigma} u^2 dx + C_2\varepsilon \le \\ &\le d_0\lambda \int\limits_{D'_\Sigma} |\nabla u|^2 dx + \frac{d_0 n M^2 \mathrm{mes}_n D}{\lambda} + C_2\varepsilon. \end{aligned} \tag{14}$$

Supposing $\lambda = \dfrac{\gamma}{d_0}$ from (14) we conclude

$$\int\limits_{D'_\Sigma} |\nabla u|^2 dx \le C_3,$$

where $C_3 = \dfrac{d_0 n M^2 \mathrm{mes}_n D}{\lambda} + \dfrac{C_2}{\gamma}$ (without loss of generality, we suppose that $\varepsilon \le 1$).

Hence, it follows that

$$\int\limits_{D} |\nabla u|^2 dx \le C_4 (C_3, E, D).$$

Thus $u(x) \in W_2^1(D)$. From the boundary condition and $\mathrm{mes}_{n-1}(\partial D \cap E) = 0$ we get $u(x) \in \mathring{W}_2^1(D)$. Now, let $\sigma \ge 2$ be a number, which will be chosen later, $D_\Sigma^+ = \{x\colon x \in D'_\Sigma, u(x) > 0\}$. Without loss of generality, we suppose that the set D_Σ^+ isn't empty. Supposing in (12) $z = 1$, $\omega = u^\sigma$ we get

$$\int\limits_{D_\Sigma^+} M(u^\sigma)\, dx = \sigma \int\limits_{\partial D_\Sigma^+} u^{\sigma-1} \left|\frac{\partial u}{\partial \nu}\right| ds < C_5(a_0, M, \sigma, C_1)\,\varepsilon.$$

But, on the other hand

$$M\left(u^{\sigma}\right)=\sigma u^{\sigma-1}Mu+\sigma(\sigma-1)u^{\sigma-2}\sum_{i,j=1}^{n}a_{ij}(x)u_iu_j=$$

$$=\sigma u^{\sigma-1}\sum_{i=1}^{n}d_i(x)u_i-\sigma u^{\sigma}c(x)+\sigma(\sigma-1)u^{\sigma-2}\sum_{i,j=1}^{n}a_{ij}(x)u_iu_j.$$

Hence, we conclude

$$\sigma(\sigma-1)\int\limits_{D_{\Sigma}^{+}}u^{\sigma-2}\sum_{i,j=1}^{n}a_{ij}(x)u_iu_jdx+\sigma\int\limits_{D_{\Sigma}^{+}}u^{\sigma-1}\sum_{i=1}^{n}d_i(x)u_idx<C_5\varepsilon. \tag{15}$$

Let $D^{+}=\{x\colon x\in D, u(x)>0\}$, D_1^{+} be an arbitrary connected component of D_1^{+}. Subject to the arbitrariness of ε from (13) we get

$$(\sigma-1)\gamma\int\limits_{D_1^{+}}u^{\sigma-2}\left|\nabla u\right|^2dx\le d_0\int\limits_{D_1^{+}}u^{\sigma-1}\sum_{i=1}^{n}\left|u_i\right|dx.$$

Thus, for any $\mu>0$

$$(\sigma-1)\gamma\int\limits_{D_1^{+}}u^{\sigma-2}\left|\nabla u\right|^2dx\le\frac{d_0\mu}{2}\int\limits_{D_1^{+}}u^{\sigma-2}\left(\sum_{i=1}^{n}\left|u_i\right|\right)^2dx+\frac{d_0}{2\mu}\int\limits_{D_1^{+}}u^{\sigma}dx. \tag{16}$$

But, on the other hand

$$I=-\sigma\sum_{i=1}^{n}\int\limits_{D_1^{+}}x_iu^{\sigma-1}u_idx=-\sum_{i=1}^{n}\int\limits_{D_1^{+}}x_i\left(u^{\sigma}\right)_idx=n\int\limits_{D_1^{+}}u^{\sigma}dx,$$

and besides, for any $\beta>0$

$$I=\frac{\sigma\beta}{2}\int\limits_{D_1^{+}}r^2u^{\sigma}dx+\frac{\sigma}{2\beta}\int\limits_{D_1^{+}}u^{\sigma-2}\left(\sum_{i=1}^{n}\frac{x_i}{r}u_i\right)^2dx\le$$

$$\le\frac{\sigma\beta}{2}\int\limits_{D_1^{+}}r^2u^{\sigma}dx+\frac{\sigma}{2\beta}\int\limits_{D_1^{+}}u^{\sigma-2}\left|\nabla u\right|^2dx,$$

where $r=|x|$. Denote by $\omega(D)$ the quantity $\sup\limits_{x\in D}|x|$. Without loss of generality, we'll suppose that $\omega(D)=1$. Then

$$I\le\frac{\sigma}{2\beta}\int\limits_{D_1^{+}}u^{\sigma}dx+\frac{\sigma}{2\beta}\int\limits_{D_1^{+}}u^{\sigma-2}\left|\nabla u\right|^2dx.$$

Thus

$$\left(n-\frac{\sigma\beta}{2}\right)\int\limits_{D_1^{+}}u^{\sigma}dx\le\frac{\sigma}{2\beta}\int\limits_{D_1^{+}}u^{\sigma-2}\left|\nabla u\right|^2dx.$$

Now, choosing $\beta = \dfrac{n}{\sigma}$, we get finally

$$\int\limits_{D_1^+} u^\sigma dx \le \frac{\sigma^2}{n^2} \int\limits_{D_1^+} u^{\sigma-2} |\nabla u|^2 dx. \tag{17}$$

Subject to (17) in (16) we conclude

$$(\sigma-1)\gamma \int\limits_{D_1^+} u^{\sigma-2} |\nabla u|^2 dx \le \left(\frac{d_0 \mu n}{2} + \frac{d_0 \sigma^2}{2\mu n^2}\right) \int\limits_{D_1^+} u^{\sigma-2} |\nabla u|^2 dx. \tag{18}$$

Now, choose μ such that

$$(\sigma-1)\gamma > \frac{d_0 \mu n}{2} + \frac{d_0 \sigma^2}{2\mu n^2}. \tag{19}$$

Then from (17)–(19) it will follow that $u(x) \equiv 0$ in D_1^+, and thus $u(x) \equiv 0$ in D. Suppose that $\mu = \dfrac{(\sigma-1)\gamma}{d_0 n}$. Then (19) is equivalent to the condition

$$n > \left(\frac{\sigma}{\sigma-1}\right)^2 \left(\frac{d_0}{\gamma}\right)^2. \tag{20}$$

At first, suppose, that

$$n > \left(\frac{d_0}{\gamma}\right)^2. \tag{21}$$

Let's choose and fix such big $\sigma \ge 2$, that by fulfilling (21) the inequality (20) was true. Thus the theorem is proved, if with respect to n the condition (21) is fulfilled. Show that it is true for any n. For this, at first, note, that if $n \ge 3$, then condition (21) will take the form

$$n > \left(\frac{d_0 \omega(D)}{\gamma}\right)^2.$$

Besides, the assertion of the theorem remains valid if in the problem (5) we replace the condition $u|_{\partial D \setminus E} = 0$ by the conditions $u|_{\Gamma_1} = 0$ and $\left.\dfrac{\partial u}{\partial v}\right|_{\Gamma_2} = 0$, where $\Gamma_1 \cup \Gamma_2 = \partial D \setminus E$.

Now, let the condition (21) be not fulfilled. Denote by k the least natural number, for which

$$n + k > \left(\frac{d_0}{\gamma}\right)^2. \tag{22}$$

Consider $(n+k)$-dimensional semi-cylinder $D' = D \times (-\delta_0, \delta_0) \times \ldots \times (-\delta_0, \delta)$, where the number $\delta_0 > 0$ will be chosen later. Since $\omega(D) = 1$, we have $\omega(D') \le 1 + \delta_0 \sqrt{k}$. Let's choose and fix δ_0 such small that, along with the condition (22), the condition

$$n + k > \left(\frac{d_0 \omega(D')}{\gamma}\right)^2 \tag{23}$$

is fulfilled too.

Let

$$y = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+k}), \qquad E' = \underbrace{E \times [-\delta_0, \delta_0] \times \dots \times [-\delta_0, \delta_0]}_{k \text{ times}}.$$

Consider on the domain D' the equation

$$\mathcal{L}'_\vartheta = \sum_{i,j=1}^{n} a_{ij}(x)\vartheta_{ij} + \sum_{i=1}^{k} \frac{\partial^2 \vartheta}{\partial x_{n+i}^2} + \sum_{i=1}^{n} b_i(x)\vartheta_i + c(x)\vartheta = 0. \tag{24}$$

It is easy to see that the function $\vartheta\,(y) = u\,(x)$ is a solution of the equation (24) in $D' \backslash E'$. Besides, $m_H^{n+k-2+\lambda}\,(E') = (2\delta_0)^k m_H^{n-2+\lambda}(E) = 0$, the function $\vartheta\,(y)$ vanishes on $\left(\partial D \times \underbrace{[-\delta_0, \delta_0] \times \dots \times [-\delta_0, \delta_0]}_{k \text{ times}}\right) \backslash E'$ and $\dfrac{\partial \vartheta}{\partial \nu'} = 0$ at $x_{n+i} = \pm\delta_0$, $i = 1, \dots, k$, where $\dfrac{\partial}{\partial \nu'}$ is a derivative by the conormal, generated by the operator $\mathcal{L}'$. Noting that $\gamma\,(\mathcal{L}') = \gamma\,(\mathcal{L})\,, d_0\,(\mathcal{L}') = d_0\,(\mathcal{L})$ and subject to the condition (23), from the proved above we conclude that $\vartheta\,(y) \equiv 0$, i.e., D'.

The theorem is proved.

***Remark* 2.** As is seen from the proof, the assertion of the theorem remains valid if, instead of the condition (3), it is required that the coefficients $a_{ij}(x)$, $i, j = 1, \dots, n$, have to satisfy in domain D the uniform Lipschitz condition.

2. Let's consider Neumann problem for nondivergent parabolic equation of the second order. In the case of Laplace operator, the question on removability sets relative to the Neumann problem was studied in the papers [2] and [3]. The questions of removability for solutions of the first boundary-value problem for elliptic and parabolic equations were considered in the papers [5] and [14]. In the paper [15] the analogous questions of boundary-value problems are considered for linear and quasilinear elliptic equations.

Let's consider cylindrical domain $Q_T = \Omega \times (0, T)\,,\ 0 < T < \infty$, in $(n+1)$-dimensional Euclidean space of the points $(x_1, \dots, x_n, t)$ in R^{n+1}, $n \geq 2$, where $\Omega \subset R^n$ is a bounded domain, $\partial\Omega$ is its boundary. Let E_0 be some compact set lying on $\partial\Omega$, $E = E_0 \times (0, T)\,,\ Q_0 = \big\{(x,t)\colon x \in \Omega,\ t = 0\big\}$. $\Gamma\,(Q_T) = Q_0 \cup (\partial\Omega \times (0,T))$ be a parabolic boundary Q_T. Let's consider the following boundary-value problem in Q_T:

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x,t) u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t) u_{x_i} + c(x,t)u - u_t = 0 \quad \text{in} \quad Q_T, \tag{25}$$

$$\left.\frac{\partial u}{\partial \nu}\right|_{\Gamma(Q_T)\backslash E} = 0, \tag{26}$$

where $\dfrac{\partial u}{\partial \nu}$ is a derivative by conormal. $\partial\Omega$ is a sufficiently smooth surface.

Let's call the set E removable relative to the second boundary-value problem (25), (26) in $C^{0,\lambda}\left(\overline{Q}_T\right)$, $0 < \lambda < 1$, if from

$$Lu = 0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma(Q_T)\backslash E} = 0, \quad u(x,t) \in C^{0,\lambda}\left(\overline{Q}_T\right), \tag{27}$$

it follows that $u(x,t) \equiv 0$ in Q_T, i.e., problem (25), (26) has only trivial solution.

Relative to the coefficients we assume the fulfilment of the following conditions:

$$\gamma|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \gamma^{-1}|\xi|^2, \quad \xi \in E_n, \tag{28}$$

$$\left|a_{ij}(x,t) - a_{ij}(y,t)\right| \leq k_1 |x-y|, \tag{29}$$

$$\left|b_i(x,t)\right| \leq b_0, \quad -b_0 \leq c(x,t) \leq 0. \tag{30}$$

Here $\gamma \in (0,1]$, $i,j = \overline{1,n}$, $b_0 > 0$, k_1 are constants. Besides, the lower coefficients are the functions measurable in Q_T.

Denote by $B_R(z)$ and $S_R(z)$ the ball $\{x\colon |x-z| < R\}$ and the sphere $\{x\colon |x-z| = R\}$ of radius R with the center at the point $z \in R^n$.

We assume that $u(x,t)$ is a solution of the first boundary-value problem for heat conduction equation and consider the function $z(x) = \int_0^t u^2(x,t)dt$. Let's fix an arbitrary t^0, $0 < t^0 < T$. At above mentioned conditions on coefficients, for an arbitrary $\varepsilon > 0$ we can find the surfaces Σ_i, isolating the ball of radius r_i from the ball of radius $2r_i$ in the cylinder Q_T and isolating the singular points $\Gamma(Q_T)$ so that

$$\int_{\Sigma_i} \left|\frac{\partial u}{\partial \nu}\right| ds \leq C_1 \operatorname*{osc}_{r_i < r < 2r_i} u r_i^{n-2}. \tag{31}$$

The existence of such surfaces follows from [11].

Let $\mathcal{D}_\Sigma$ be an open set, situated in $Q_T \backslash E$, whose boundary consists of unification of Σ and Γ, where $\Sigma = \bigcup_{k=1}^{\infty} \Sigma_k$, $D_k = D \cap B_{2r_k}(x^k)$, $k = 1, 2$, $\Gamma = \partial D \backslash \bigcup_{k=1}^{\infty} D_k$, D_k^+ be a part of D_k remained after elimination of points, arranged between the Σ_k and $S_{2r_k}(x^k)$, $k = 1, 2, \ldots$. Denote by $\mathcal{D}'_\Sigma$ an arbitrary connected component $\mathcal{D}_\Sigma$.

Further,

$$\frac{\partial z}{\partial \nu} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i}\nu_i = \sum_{i=1}^{n} 2\nu_i \int_0^t u u_{x_i} dt = \int_0^t 2u \sum_{i=1}^{n} \nu_i u_{x_i} dt = 2\int_0^t u \frac{\partial u}{\partial \nu} dt,$$

where ν_i are directive cosines. Here, we are to take into account that by virtue of cylinder property of Q_T ν_i remain fixed at any t. By Green formula

$$\int_{\mathcal{D}_\Sigma} \left(2\int_0^t |\nabla_x u|^2 dt \right) dx + \int_{\mathcal{D}_\Sigma} \left(2\int_0^t u u_t dt \right) dx = \sum_{j=1}^{m} \int_{\Sigma_j} \frac{\partial z}{\partial \nu} ds$$

or

$$2\int_0^t |\nabla_x u|^2 dt dx + \int_{\mathcal{D}_\Sigma} \left[u^2(x,t) - u^2(x,0)\right] dx \leq \sum_{j=1}^{m} \int_{\Sigma_j} \left|\frac{\partial z}{\partial \nu}\right| ds$$

and allowing for $u\big|_{Q_0} = 0$ we have

$$\int_{\mathcal{D}_\Sigma} \int_0^t |\nabla_x u|^2 dt dx \leq \frac{C_2}{2} \sum_{j=1}^{m} \operatorname*{osc}_{r_j \leq r \leq 2r_j} z r_j^{n-2}.$$

Since $\left|u(x_1,t)-u(x_2,t)\right| \le C_3\left|x_1-x_2\right|^{\lambda}$ we have

$$\left|z(x_1)-z(x_2)\right| \le \int\limits_0^t \left|u(x_1,t)+u(x_2,t)\right|\left|u(x_2,t)-u(x_1,t)\right|dt \le$$

$$\le 2C_3\sup_{Q_T}|u|\left|x_1-x_2\right|^{\lambda}|t| \le C_4\left|x_1-x_2\right|^{\lambda},$$

and so

$$\int\limits_{\mathcal{D}_\Sigma}\int\limits_0^t \left|\nabla_x u\right|^2 dt\,dx \le \frac{C_5}{2}4^{\lambda}\sum_{j=1}^m r_j^{n-2+\lambda} \le C_6\varepsilon.$$

Hence, by virtue of arbitrariness of ε we obtain that

$$\int\limits_{\mathcal{D}_\Sigma}\int\limits_0^t \left|\nabla_x u\right|^2 dtdx = 0,$$

or $\left|\nabla_x u(x,t)\right| = 0$. Hence, allowing for $u_t = \Delta u = 0$, we have $u(x,t) \equiv \mathrm{const}$. But $u|_{Q_0} = 0$, therefore $u(x,t) \equiv 0$.

Now let $u(x,t)$ be a solution of problem (25), (26). Taking the function $z(x)$ and treating as in the work [15], allowing for the above mentioned estimations $z(x)$ we'll obtain $u\left(x,t\right) \equiv 0$.

So, the following theorem is proved.

Theorem 2. *Let $Q_T = \Omega \times (0,T)$ be a cylindrical domain in R^{n+1}, $n \ge 2$, $E \subset \overline{Q}_T$ be some compact, and let conditions* (28)–(30) *be fulfilled relative to the coefficients. Then for removability of the compact E relative to problem* (25), (26) *in the space $C^{0,\lambda}\left(\overline{Q}_T\right)$, it suffices that*

$$m_H^{n-2+\lambda}(E) = 0.$$

3. Let's consider the mixed boundary-value problem for the second order nondivergent parabolic equation. Let Γ_1 and Γ_2 be such two sets that $\Gamma\left(Q_T\right)\backslash E = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \varnothing$.

Let's consider the following mixed problem:

$$\begin{gathered}\mathcal{L}u = \sum_{i,j=1}^n a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^n b_i(x,t)u_{x_i} + c(x,t)u - u_t = 0 \quad \text{in} \quad Q_T, \\ u|_{\Gamma_1} = 0, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma_2} = 0.\end{gathered} \tag{32}$$

We find solution of problem (32) from the class $C^{2,1}\left(Q_T\right) \cap C^0\left(\overline{Q}_T \backslash E\right)$.

Theorem 3. *Let $Q_T \subset R^{n+1}$, $n \ge 2$, be a cylinder, $E \subset \overline{Q}_T$ be a compact, and let conditions* (28)–(30) *be fulfilled relative to the coefficients. Then for removability of the compact E relative to problem* (32) *in the space $C^{0,\lambda}\left(\overline{Q}_T\right)$ it suffices that*

$$m_H^{n-2+\lambda}(E) = 0.$$

The Theorem 3 is proved by the same ideas that in Theorem 1.

Let's consider the following equation in Q_T:

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) +$$

$$+\sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u + b\left(x,t,u,\nabla u\right) - u_t = 0. \tag{33}$$

Assume that $a_{ij}(x,t)$ are bounded, measurable functions satisfying condition (28), $b_i(x,t), c(x,t)$ satisfy condition (30) and

$$\left|b\left(x,t,u,\nabla u\right)\right| \le g(u)\cdot\left|\nabla u\right|^2, \qquad \int\limits_0^k g(u)du < \infty, \quad k < \infty. \tag{34}$$

For equation (33) we consider the problem

$$Lu = 0 \quad \text{in} \quad Q_T\backslash E, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\Gamma(Q_T)\backslash E} = 0. \tag{35}$$

We try to find a solution of this problem in the class

$$\left\{W_2^1\left(Q_T\right)\cap C^{0,\lambda}\left(\overline{Q}_T\right), \ \ 0 \le u(x,t) \le k\right\}.$$

Theorem 4. *Let Q_T be a cylindrical domain in R^{n+1}, $n\ge 2$, $E\subset \overline{Q}_T$ be some compact, and let relative to the coefficients of equation* (33) *conditions* (28), (30), (34) *be fulfilled. Then for removability of the compact E relative to problem* (35) *it suffices that*

$$m_H^{n-2+\lambda}(E) = 0.$$

Before we pass to the proof, let's note that if the solutions are sought in the class $\left\{W_2^1(Q_T)\cap C^0\left(\overline{Q}_T\right), 0\le u(x,t)\le k\right\}$, then the set E is removable if

$$m_H^{n-2}(E) < \infty.$$

***Proof of Theorem* 3.** The function $\vartheta(x,t) = \displaystyle\int_0^{u(x,t)} \exp\left(\frac{1}{\lambda_1}\int_0^t g\left(g(\tau)d\tau\right)\right)dt$ is a subsolution of the linear operator

$$L_1 = \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}\right) - \frac{\partial}{\partial t}.$$

Further, analogously to the proof of Theorem 1, we obtain that $\vartheta(x,t)\equiv 0$, which proves the theorem.

1. *Carleson L.* Selected problems on exceptional sets. – Toronto etc.: D. Van. Nostrand Comp., 1967. – 126 p.
2. *Gadjiev T. S., Mamedova V. A.* On removable sets of solutions of boundary-value problems for quasilinear elliptic equations // Trudi Voron. Zimney Mat. Shkoly. – 2006. – P. 12 – 17.

3. *Moiseev E. I.* On exceptional and unexceptional boundary sets of Neuman problem // Differents. Uravneniya. – 1973. – **9**, № 5. – P. 901 – 911 (in Russian).
4. *Mamedov I. T.* On exceptional sets of solutions of Gilbarg – Serrin equation in the space of Hölder functions // Izv. AN Azerb. Ser. phys.-tech., math. Issue math. and mech. – 1998. – **18**, № 2. – P. 46 – 51 (in Russian).
5. *Landis E. M.* To question on uniqueness of solution of the first boundary-value problem for elliptic and parabolic equations of the second order // Uspekhi Mat. Nauk. – 1978. – **33**, № 3. – P. 151 (in Russian).
6. *Gadjiev T. S., Mamedova V. A.* Removable sets of solutions of the second order boundary-value problem for second order parabolic equations // Trans. NAS Azerb. – 2006. – **25**, № 1. – P. 59 – 64.
7. *Kilpelainen T., Zhong X.* Removable sets for continuous solutions of quasilinear elliptic equations // Proc. Amer. Math. Soc. – 2002. – **130**, № 6. – P. 1681 – 1688 (electronic).
8. *Diederich J.* Removable sets for pointwise solutions of elliptic partial differential equations // Ibid. – 1972. – **165**. – P. 333 – 352.
9. *Harvey R., Polking J.* Removable singularities of solutions of linear partial differential equations // Acta Math. – 1970. – **125**. – P. 39 – 56.
10. *Dahlberg B. E. J.* On exceptional sets at the boundary for subharmonic functions // Ark. mat. – 1977. – **15**, № 2. – P. 305 – 312.
11. *Gerver M. L., Landis E. M.* One generalization of theorem on value for multivariable functions // Dokl. AN SSSR. – 1962. – **146**, № 4. – P. 761 – 764 (in Russian).
12. *Landis E. M.* Second order equations of elliptic and parabolic types (in Russian). – M.: Nauka, 1971. – 288 p.
13. *Gilbarg D., Trudinger N. S.* Elliptic partial differential equations of second order. – Berlin etc.: Springer, 1977. – 401 p.
14. *Kayzer V., Muller B.* Removable sets for head conduction // Vestnik Mosc. Univ. – 1973. – № 5. – P. 26 – 32 (in Russian).
15. *Mamedov I. T., Mamedova V. A.* On exceptional sets of solutions of the second order elliptic equations in nondivergent form // Proc. Inst. Math. and Mech. NAS Azerb. – 2002. – **17**, № 25. – P. 115 – 121.

Received 07.04.09