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CONNECTIONS TO FIXED POINTS AND SIL'NIKOV SADDLE-FOCUS HOMOCLINIC ORBITS IN SINGULARLY PERTURBED SYSTEMS

ПОЄДНАННЯ НЕРУХОМИХ ТОЧОК ТА СІДЛОВІ ФОКУСНІ ГОМОКЛІНІЧНІ ОРБІТИ СІЛЬНІКОВА В СИНГУЛЯРНО ЗБУРЕНИХ СИСТЕМАХ

We consider a singularly perturbed system depending on two parameters with two (possibly the same) normally hyperbolic centre manifolds. We assume that the unperturbed system has an orbit connecting a hyperbolic fixed point on one centre manifold to a hyperbolic fixed point on the other. Then we prove some old and new results concerning the persistence of these connecting orbits and apply the results to find examples of systems in dimensions greater than three which possess Sil'nikov saddle-focus homoclinic orbits.

Розглянуто сингулярно збурену систему, що залежить від двох параметрів та має два (можливо, однакові) нормально гіперболічні центровані многовиди. При цьому припускається, що незбурена система має орбіту, яка поєднує гіперболічну нерухому точку на одному центрованому многовиді з гіперболічною нерухомою точкою на іншому. Доведено деякі відомі та нові результати щодо збереження цих орбіт та наведено приклади систем розмірності більше, ніж три, що мають сідлові фокусні гомоклінічні орбіти Сільнікова.

1. Introduction. In this paper, which continues [1], we consider a singularly perturbed system like:

$$\begin{aligned}\dot{x} &= \varepsilon f(x, y, \lambda, \varepsilon),\\ \dot{y} &= g(x, y, \lambda, \varepsilon)\end{aligned} \tag{1}$$

where $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, λ and ε are small real parameters and $f(x, y, \lambda, \varepsilon)$, $g(x, y, \lambda, \varepsilon)$ are C^r-functions in their arguments bounded with their derivatives, $r \geq 1$. We suppose that the following conditions hold:

(i) for any $x \in \mathbf{R}^m$, the equation

$$g(x, y, 0, 0) = 0$$

has C^r-solutions $y = v^{\pm}(x)$ (that may coincide) such that $v^{\pm}(x)$ and its derivatives are bounded on $\mathbf{R}$,

(ii) the infimums over $x \in \mathbf{R}^m$ of the moduli of the real parts of the eigenvalues of the Jacobian matrix $g_y(x, v^{\pm}(x), 0)$ are greater than a positive number δ_0. Moreover $g_y(x, v^+(x), 0)$ and $g_y(x, v^-(x), 0)$ have the same number of eigenvalues with positive (and hence also negative) real parts.

As explained in more detail in Section 2, conditions (i) and (ii) imply the existence of centre manifolds $y = v^+(x, \lambda, \varepsilon)$ and $y = v^-(x, \lambda, \varepsilon)$ for the perturbed system together

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with their associated centre stable and centre unstable manifolds. We also assume the following conditions:

(iii) there exists $\xi_0 \in \mathbf{R}^m$ such that the equation

$$\dot{y} = g(\xi_0, y, 0, 0)$$

has a solution $y_0(t)$ satisfying

$$y_0(t) \to v^-(\xi_0) \quad \text{as} \quad t \to -\infty, \qquad y_0(t) \to v^+(\xi_0) \quad \text{as} \quad t \to \infty,$$

(iv) $\dot{y}_0(t)$ is the unique bounded solution of the linear variational system:

$$\dot{y} = g_y(\xi_0, y_0(t), 0, 0)y \tag{2}$$

up to a scalar multiple.

Given these conditions, we would expect the perturbed system to have orbits connecting the two centre manifolds. However, in this paper, we are particularly interested in orbits which connect fixed points lying on the centre manifolds. So we need an additional condition:

(v) both equations on the centre manifolds

$$\dot{x} = F^{\pm}(x) = f(x, v^{\pm}(x), 0, 0)$$

have the same hyperbolic fixed point ξ_0 and the matrices $F_x^{\pm}(\xi_0)$ have the same number of eigenvalues, counted with mutiplicities, with positive (resp. negative) real parts and no eigenvalues with zero real part such that if Q_+ is the projection onto the stable subspace of $F_x^+(\xi_0)$ and Q_- is the projection onto the stable subspace of $F_x^-(\xi_0)$, then $\mathcal{R}Q_+ \cap \mathcal{N}Q_- = \{0\}$. Under this condition, the perturbed system

$$\dot{x} = F^{\pm}(x, \lambda, \varepsilon) = f(x, v^{\pm}(x, \lambda, \varepsilon), \lambda, \varepsilon) \tag{3}$$

has a hyperbolic fixed point $\xi_0^{\pm}(\lambda, \varepsilon)$ and

$$q^{\pm}(\lambda, \varepsilon) = \big(\xi_0^{\pm}(\lambda, \varepsilon), v^{\pm}(\xi_0^{\pm}(\lambda, \varepsilon), \lambda, \varepsilon\big)$$

is a hyperbolic fixed point of system (1). We make the additional assumption that for sufficiently small λ

$$\xi_0^+(\lambda, 0) = \xi_0^-(\lambda, 0) = \xi_0(\lambda).$$

Our objects in this paper are the following:

(a) to extend the result of [1] concerning the existence of solutions of (1) that connect a fixed point on one centre manifold to a fixed point on the other centre manifold to a more degenerate case;

(b) to give a general class of singularly perturbed systems in dimensions greater than three which possess Sil'nikov saddle-focus homoclinic orbits.

In [1] in the homoclinic case, following on from the work of Szmolyan [2] and Beyn – Stiefenhofer [3], we already gave a nondegeneracy condition (which corresponds to condition (vi) in Theorem 2 below) under which there is a curve $\lambda = \lambda(\varepsilon)$ in the parameter space along which system (1) has a connecting orbit. This condition says that the connecting orbit $y_0(t)$ in the system

$$\dot{y} = g(\xi_0(\lambda), y, \lambda, 0) \tag{4}$$

breaks as λ passes through 0. In this paper, in a slightly more general situation, we give a different and shorter proof of this theorem (see Theorem 2). We proceed in two steps, first we use the results of [4] to find the connecting orbits between the centre manifolds and then employ a further argument to pick out from these orbits the ones that connect the fixed points.

More importantly, we can use these techniques to treat degenerate cases. Such a degeneracy would arise if $y_0(t)$ does not break so that there is a one-parameter family $y(t, \lambda)$ of homoclinic orbits for (4). Then in Theorem 3 we need to add two additional conditions: one condition says that the centre stable and centre unstable manifolds intersect transversally along $(\xi_0, y_0(t))$ when $\lambda = \varepsilon = 0$ and the other is that a certain Melnikov function have a simple zero. Under these conditions we can again prove there is a curve $\lambda = \lambda(\varepsilon)$ in the parameter space along which system (1) has a connecting orbit. Note that there are other kinds of degeneracies that could arise such as what we called the Cherry and Duffing cases in [5] and [4] in which the centre stable and centre unstable manifolds do not intersect transversally along $(\xi_0, y_0(t))$ when $\lambda = \varepsilon = 0$. However, in this paper, we confine ourselves to what appears to be the simplest kind of degeneracy.

The theory of Sil'nikov saddle-focus homoclinic orbits is developed in [6] and [7]. Such orbits have been found in special systems (for example, see [8 – 10] and see [7] for others) but not many general classes of systems with such orbits have been found, apart from that of Rodriguez [11]. However, Rodriguez looked only at three-dimensional systems. In four dimensions two extra conditions must be verified. In [1] we exhibited a class of systems in 4 dimensions with saddle-focus homoclinic orbits. In this paper, with the help of [12], we show the conditions we gave in [1] can be weakened (in [1] we used too strong a condition to ensure Deng's condition (D_4) was satisfied) and we also give a result in n dimensions.

Now we summarize the contents of the paper. In Section 2, we recall the main result from [4] where we construct bifurcation equations, the zeros of which are initial values of solution of (1) which lie in the intersection of the global centre stable manifold corresponding to $y = v^+(x, \lambda, \varepsilon)$ and the global centre unstable manifold corresponding to $y = v^-(x, \lambda, \varepsilon)$. Then in Section 3 we prove Theorems 2 and 3 as described above and we give examples of the application of both. Then in Section 4, we prove the theorem concerning Sil'nikov saddle-focus homoclinic orbits and give an example of it as well.

2. Heteroclinic connections between the centre manifolds. In this section we recall the main result of [4], where under the conditions (i) – (iv) of the Introduction, we construct bifurcation equations, the zeros of which are initial values of solution of (1) which lie in the intersection of the global centre stable manifold corresponding to $y = v^+(x, \lambda, \varepsilon)$ and the global centre unstable manifold corresponding to $y = v^-(x, \lambda, \varepsilon)$.

First we observe that conditions (i) and (ii) of the Introduction imply the existence of $\varepsilon_0 > 0$, $\lambda_0 > 0$ and functions $v^\pm(x, \lambda, \varepsilon)$ which are defined for $x \in \mathbf{R}^m$, $|\lambda| \leq \lambda_0$ and $|\varepsilon| \leq \varepsilon_0$ such that $v^\pm(x, 0, 0) = v^\pm(x)$ and the manifolds $y = v^\pm(x, \lambda, \varepsilon)$ are invariant for the flow of (1) (see for example [13–15]). Moreover $v^{\pm}(x,\lambda,\varepsilon)$ are C^r and bounded with their derivatives. We will refer to $y = v^{\pm}(x,\lambda,\varepsilon)$ as global *centre manifolds* for system (1). We use the notation $x_c^{\pm}(t,\xi,\lambda,\varepsilon)$ for the solution of the initial value problem

$$\dot{x} = F^{\pm}(x,\lambda,\varepsilon) := f(x, v^{\pm}(x,\lambda,\varepsilon),\lambda,\varepsilon), \quad x(0) = \xi.$$

In this situation there also exist global centre stable and unstable manifolds. The *global centre stable manifold* $\mathcal{M}^{cs}$ consists of those solutions $(x(t), y(t))$ such that $|y(t) - v^{+}(x(t),\lambda,\varepsilon)| \to 0$ as $t \to \infty$ and the *global centre unstable manifold* $\mathcal{M}^{cu}$ consists of those solutions $(x(t), y(t))$ such that $|y(t) - v^{-}(x(t),\lambda,\varepsilon)| \to 0$ as $t \to -\infty$.

Next, according to condition (ii), for all $x \in \mathbf{R}^m$, the linear systems

$$\dot{y} = g_y(x, v^{\pm}(x), 0, 0)y$$

have exponential dichotomies on $\mathbf{R}$ with constant K, exponent δ_0 and projections, say, $P^0_{\pm}(x)$. Moreover $\operatorname{rank} P^0_{+}(x) = \operatorname{rank} P^0_{-}(x) = p$, p being the number of eigenvalues of $g_y(x, v^{\pm}(x), 0, 0)$ with negative real parts.

Then from (ii), (iii) and the roughness of exponential dichotomies, it follows that for any δ with $0 < \delta < \delta_0$ linear system (2) has an exponential dichotomy on $\mathbf{R}_+$ and $\mathbf{R}_-$ respectively with constants K and δ and respective projections $P_+(t) = Y(t)P_+Y^{-1}(t)$ and $P_-(t) = Y(t)P_-Y^{-1}(t)$, $Y(t)$ being the fundamental matrix with $Y(0) = \mathbf{I}$. Note that since from (iv) $\dot{y}_0(t)$ is, up to a scalar multiple, the unique bounded solution, it follows that $\mathcal{R}P_+ \cap \mathcal{N}P_-$ is the one-dimensional subspace spanned by $\dot{y}_0(0)$. Also $\mathcal{R}P_+ + \mathcal{N}P_-$ has codimension 1 and if we let ψ_0 be a unit vector orthogonal to $\mathcal{R}P_+ + \mathcal{N}P_-$, then the solution $\psi(t)$ of the adjoint equation

$$\dot{y} = -g^*_y(\xi_0, y_0(t), 0, 0)y \tag{5}$$

with $\psi(0) = \psi_0$ is, up to a scalar multiple, the unique bounded solution.

In Theorem 4 in [4] the following result, concerning the existence of heteroclinic connections between the two centre manifolds, has been proved. Such solutions lie in the intersection of the global centre stable and unstable manifolds.

Theorem 1. *Let f and g be bounded C^r functions, $r \geq 2$, with bounded derivatives, satisfying conditions* (i)–(iv) *of the Introduction. Then there exist ε_0, λ_0, α_0, C^{r-1} functions $\Delta(\xi,\lambda,\varepsilon)$ $Z(\xi,\lambda,\varepsilon)$ defined for $|\xi - \xi_0| \leq \alpha_0$, $|\lambda| \leq \lambda_0$, $|\varepsilon| \leq \varepsilon_0$ and a neighborhood O of $(\xi_0, y_0(0))$ such that if (ξ,η) satisfies*

$$\Delta(\xi,\lambda,\varepsilon) = 0, \qquad \eta = Z(\xi,\lambda,\varepsilon) \tag{6}$$

then (ξ,η) lies in the intersection of the global centre stable manifold corresponding to $y = v^{+}(x,\lambda,\varepsilon)$ and the global centre unstable manifold corresponding to $y = v^{-}(x,\lambda,\varepsilon)$ and, conversely, if (ξ,η) is in O and lies in this intersection and satisfies $\langle \eta - y_0(0), \dot{y}_0(0)\rangle = 0$, then (6) *holds. Moreover,*

$$\Delta(\xi_0, 0, 0) = 0, \qquad Z(\xi_0, 0, 0) = y_0(0),$$

$$\Delta_\xi(\xi_0, 0, 0) = -\int\limits_{-\infty}^{\infty} \psi^*(t) g_x(\xi_0, y_0(t), 0, 0)dt,$$

$$\Delta_\lambda(\xi_0,0,0) = -\int_{-\infty}^{\infty} \psi^*(t) g_\lambda(\xi_0, y_0(t), 0, 0)dt,$$

$$\Delta_\varepsilon(\xi_0,0,0) =$$

$$= -\int_{-\infty}^{\infty} \psi^*(t) \left\{ g_x(\xi_0, y_0(t), 0, 0) \int_0^t f(\xi_0, y_0(\tau), 0, 0)d\tau + g_\varepsilon(\xi_0, y_0(t), 0, 0) \right\} dt,$$

and there exist $\beta > 0$ $(\beta < \delta_0)$, $T > 0$, $\mu_1 > 0$, $\mu_2 > 0$ *and* $N_1 > 0$ *such that the solution* $(x(t), y(t)) = (\hat{x}(t,\xi,\lambda,\varepsilon), \hat{y}(t,\xi,\lambda,\varepsilon))$ *of* (1) *such that* $(x(0), y(0)) = (\xi, \eta)$ *satisfies*

$$\begin{gathered} e^{\beta|t\mp T|}\left|x(t) - x_c^\pm(\varepsilon(t\mp T), \hat{\xi}_\pm(\xi,\lambda,\varepsilon), \lambda, \varepsilon)\right| \le N_1|\varepsilon| \le \mu_1, \\ e^{\beta|t\mp T|}\left|y(t) - v^\pm(x(t),\lambda,\varepsilon)\right| \le \mu_2, \end{gathered} \tag{7}$$

for $t \ge T$ *and* $t \le -T$ *respectively, where the* $\hat{\xi}_\pm(\xi,\lambda,\varepsilon)$ *are* C^{r-1} *functions satisfying*

$$\begin{gathered} \hat{\xi}_\pm(\xi,\lambda,0) = \xi, \\ \hat{\xi}_{+,\varepsilon}(\xi,\lambda,0) = Tf(\xi, v^+(\xi,\lambda,0),\lambda,0) + \\ + \int_0^\infty f(\xi, \hat{y}(t,\xi,\lambda,0),\lambda,0) - f(\xi, v^+(\xi,\lambda,0),\lambda,0)\,dt, \\ \hat{\xi}_{-,\varepsilon}(\xi,\lambda,0) = -Tf(\xi, v^-(\xi,\lambda,0),\lambda,0) - \\ - \int_{-\infty}^0 f(\xi, \hat{y}(t,\xi,\lambda,0),\lambda,0) - f(\xi, v^-(\xi,\lambda,0),\lambda,0)\,dt. \end{gathered} \tag{8}$$

Moreover,

$$\hat{x}(t,\xi_0,0,0) = \xi_0, \qquad \hat{y}(t,\xi_0,0,0) = y_0(t), \tag{9}$$

$\hat{x}(t,\xi,\lambda,\varepsilon), \hat{y}(t,\xi,\lambda,\varepsilon)$ *are* C^{r-1} *in* $(\xi,\lambda,\varepsilon)$ *and if* σ *satisfies* $0 < r\sigma < \beta$, *the* k*-th order derivatives of* $\hat{x}(t,\xi,\lambda,\varepsilon) - x_c^\pm(\varepsilon(t\mp T), \hat{\xi}_\pm(\xi,\lambda,\varepsilon),\lambda,\varepsilon)$ *and* $\hat{y}(t,\xi,\lambda,\varepsilon) - - v^\pm(\hat{x}(t,\xi,\lambda,\varepsilon),\lambda,\varepsilon)$, *satisfy estimates similar to* (7) *with* $\beta - k\sigma$ *instead of* β *and possibly different constants* C_k *instead of* μ_1, μ_2.

Remarks. (i) Estimate (7) with $N_1|\varepsilon|$ and those concerning the derivatives of $\hat{x}(t,\xi,\lambda,\varepsilon) - x_c^\pm(\varepsilon(t\mp T), \hat{\xi}_\pm(\xi,\lambda,\varepsilon),\lambda,\varepsilon)$ and $\hat{y}(t,\xi,\lambda,\varepsilon) - v^\pm(\hat{x}(t,\xi,\lambda,\varepsilon),\lambda,\varepsilon)$ are not explicitly stated in [4], Theorem 4. However they follow from Eq. (63) and Theorems 1 and 2 in [4].

(ii) Eqns. (6) are what we call the *bifurcation equations*. Zeros (ξ,η) of these equations are initial values of solutions $(\hat{x}(t,\xi,\lambda,\varepsilon), \hat{y}(t,\xi,\lambda,\varepsilon))$ of (1) which lie in the intersection of the global centre stable manifold corresponding to $y = v^+(x,\lambda,\varepsilon)$ and the global centre unstable manifold corresponding to $y = v^-(x,\lambda,\varepsilon)$. The functions

$\hat{\xi}_\pm(\xi, \lambda, \varepsilon)$ tell us in which leaves of the stable (resp. unstable) foliation these solutions lie (see [4] for details).

3. Heteroclinic orbits connecting fixed points. In this section we assume that in addition to conditions (i) – (iv), condition (v) also holds. Our aim is to find a curve in the parameter space along which there exist heteroclinic orbits connecting the fixed point $q_-(\lambda, \varepsilon)$ to $q_+(\lambda, \varepsilon)$. First we construct the *slow* stable manifold of $q_+(\lambda, \varepsilon)$ and the *slow* unstable manifold of $q_-(\lambda, \varepsilon)$. By slow, we mean the stable and unstable manifolds lying inside the respective centre manifolds. Then we prove our two main theorems on the existence of heteroclinic orbits. Finally we give examples of the application of both theorems.

3.1. The slow stable and unstable manifolds. First we describe the local stable manifold of the fixed point $\xi_0^+(\lambda, \varepsilon)$ of the equation $\dot{x} = F^+(x, \lambda, \varepsilon)$ and the unstable manifold of the fixed points $\xi_0^-(\lambda, \varepsilon)$ of the equation $\dot{x} = F^-(x, \lambda, \varepsilon)$. As in [1], we can prove that the following holds: let ρ be a sufficiently small positive number. Then there exists ρ_1, $0 < \rho_1 < \rho$ such that if ε and λ are sufficiently small and if $\xi^+ \in \mathcal{R}Q_+$, $\xi^- \in \mathcal{N}Q_-$, with $|\xi^+| \le \rho_1$, $|\xi^-| \le \rho_1$, there exist unique solutions $x(t) = u^+(t, \xi^+, \lambda, \varepsilon)$ of equation $\dot{x} = F^+(x, \lambda, \varepsilon)$ and $x(t) = u^-(t, \xi^-, \lambda, \varepsilon)$ of equation $\dot{x} = F^-(x, \lambda, \varepsilon)$, that are defined for $t \ge 0$ and $t \le 0$ respectively, such that

$$\left|u^+(t, \xi^+, \lambda, \varepsilon) - \xi_0^+(\lambda, \varepsilon)\right| \le \rho \quad \text{for} \quad t \ge 0,$$

$$\left|u^-(t, \xi^-, \lambda, \varepsilon) - \xi_0^-(\lambda, \varepsilon)\right| \le \rho \quad \text{for} \quad t \le 0$$

and

$$\begin{aligned} &Q_+\left[u^+(0, \xi^+, \lambda, \varepsilon) - \xi_0^+(\lambda, \varepsilon)\right] = \xi^+, \\ &(\mathbf{I} - Q_-)\left[u^-(0, \xi^-, \lambda, \varepsilon) - \xi_0^-(\lambda, \varepsilon)\right] = \xi^-. \end{aligned} \tag{10}$$

Moreover, denoting with $\alpha > 0$ a positive number which strictly bounds from below the absolute values of the real parts of the eigenvalues of both matrices $F_x^\pm(\xi_0, 0, 0)$, then $u^\pm(t, \xi^\pm, \lambda, \varepsilon) - \xi_0^\pm(\lambda, \varepsilon)$ and their first derivatives with respect to $(\xi^\pm, \lambda, \varepsilon)$ satisfy exponential estimates like

$$\left|u(t)\right| \le L e^{-\alpha(t-s)} |u(s)| \tag{11}$$

where $t \ge s$ when $u(t) = u^+(t, \xi^+, \lambda, \varepsilon) - \xi_0^+(\lambda, \varepsilon)$ and $t \le s$ when $u(t) = u^-(t, \xi^-, \lambda, \varepsilon) - \xi_0^-(\lambda, \varepsilon)$. Also

$$u^+_{\xi^+}(0, 0, 0, 0) = Q_+ \quad \text{and} \quad u^-_{\xi^-}(0, 0, 0, 0) = \mathbf{I} - Q_-. \tag{12}$$

Next we note that, by uniqueness,

$$u^+(t, 0, \lambda, \varepsilon) = \xi_0^+(\lambda, \varepsilon), \qquad u^-(t, 0, \lambda, \varepsilon) = \xi_0^-(\lambda, \varepsilon) \tag{13}$$

and that

$$x_c^\pm(t, u^\pm(\tau, \xi^\pm, \lambda, \varepsilon), \lambda, \varepsilon) = u^\pm(t + \tau, \xi^\pm, \lambda, \varepsilon) \tag{14}$$

for any $\xi^+ \in \mathcal{R}Q_+$ and $\xi^- \in \mathcal{N}Q_-$ with $|\xi^\pm| < \rho_1$. Moreover, differentiating (13) we see that

$$u_\lambda^\pm(0,0,\lambda,0) = \frac{\partial \xi_0^\pm}{\partial \lambda}(\lambda,0) \tag{15}$$

for any $\lambda \in \mathbf{R}$.

3.2. Two conditions ensuring existence of homoclinic orbits to fixed points. In this subsection we prove two theorems on the existence of heteroclinic orbits. The first is essentially the same as the theorem proved in [1]. Our proof, though, here is different in that we proceed in two stages. Note that we do prove additional properties of the homoclinic solution in that paper. The second considers a more degenerate case which involves a Melnikov function. We hope this second theorem gives a guide as to how one might handle such degenerate cases.

We recall that $\alpha > 0$ is a positive number which bounds from below the absolute value of the eigenvalues of both matrices $F_x^\pm(\xi_0, 0, 0)$.

We first prove the following theorem.

Theorem 2. *Let f and g be C^r functions $(r \geq 2)$, bounded together with their derivatives and satisfying conditions* (i) – (v). *Suppose also that the condition*

(vi) $\displaystyle\int_{-\infty}^{\infty} \psi^*(t)\big[g_x(\xi_0, y_0(t), 0, 0)\xi_0'(0) + g_\lambda(\xi_0, y_0(t), 0, 0)\big]dt \neq 0$

holds. Then there exists a C^{r-1}-function $\lambda(\varepsilon)$ with $\lambda(0) = 0$ such that for ε sufficiently small and nonnegative, system (1) *with $\lambda = \lambda(\varepsilon)$ has a heteroclinic solution $p(t,\varepsilon) = (x(t,\varepsilon), y(t,\varepsilon))$, that is,*

$$p(t,\varepsilon) \neq q^\pm(\lambda(\varepsilon),\varepsilon)$$

but

$$p(t,\varepsilon) \to q^\pm(\lambda(\varepsilon),\varepsilon) \quad as \quad t \to \pm\infty. \tag{16}$$

Moreover,

$$p(t,0) = (\xi_0, y_0(t)), \tag{17}$$

and

$$\begin{aligned}
&\sup\nolimits_{t\geq 0} |x(t,\varepsilon) - \xi_0^+(\lambda(\varepsilon),\varepsilon)|e^{\varepsilon\alpha t} = O(\varepsilon),\\
&\sup\nolimits_{t\leq 0} |x(t,\varepsilon) - \xi_0^-(\lambda(\varepsilon),\varepsilon)|e^{-\varepsilon\alpha t} = O(\varepsilon),\\
&\sup\nolimits_{t\geq 0} |y(t,\varepsilon) - v^+(\xi_0^+(\lambda(\varepsilon),\lambda(\varepsilon),\varepsilon)|e^{\varepsilon\alpha t} = O(1),\\
&\sup\nolimits_{t\leq 0} |y(t,\varepsilon) - v^-(\xi_0^-(\lambda(\varepsilon),\lambda(\varepsilon),\varepsilon)|e^{-\varepsilon\alpha t} = O(1),\\
&\sup\nolimits_{t\in\mathbf{R}} |y(t,\varepsilon) - y_0(t)| = o(1)
\end{aligned} \tag{18}$$

as $\varepsilon \to 0$. Furthermore for $1 \leq k \leq r-1$, positive constants $\hat{C}_k$ exist such that the following hold for $t \geq 0$ (for v^+), $t \leq 0$ (for v^-):

$$\begin{aligned}
&\left|D_2^{(k)}[x(t,\varepsilon) - \xi_0^\pm(\lambda(\varepsilon),\varepsilon)]\right| e^{\varepsilon\alpha|t|} \leq \hat{C}_k,\\
&\left|D_2^{(k)}[y(t,\varepsilon) - v^\pm(\xi_0^\pm(\lambda(\varepsilon),\varepsilon),\lambda(\varepsilon),\varepsilon)]\right| e^{\varepsilon\alpha|t|} \leq \hat{C}_k,
\end{aligned} \tag{19}$$

where $D_2^{(k)}$ denotes the k-th derivative with respect to ε.

Proof. Our aim is to find a solution $(x(t), y(t))$ of (1) such that

$$\lim_{t\to\pm\infty} \left|x(t) - \xi_0^\pm(\lambda,\varepsilon)\right| = 0 \quad \text{and} \quad \lim_{t\to\pm\infty} \left|y(t) - v^\pm(\xi_0^\pm(\lambda,\varepsilon),\lambda,\varepsilon)\right| = 0. \tag{20}$$

According to Theorem 1, if $\Delta(\xi,\lambda,\varepsilon) = 0$ with $|\xi - \xi_0|$, $|\lambda|$, and $|\varepsilon|$ sufficiently small, we can find a solution $(x(t), y(t)) = \big(\hat{x}(t,\xi,\lambda,\varepsilon), \hat{y}(t,\xi,\lambda,\varepsilon)\big)$ of (1) that satisfies

$$\begin{aligned} &\sup\nolimits_{t\geq 0} e^{\beta t}\left|x(t+T) - x_c(\varepsilon t, \hat{\xi}_+(\xi,\lambda,\varepsilon),\lambda,\varepsilon)\right| \leq \mu_1, \\ &\sup\nolimits_{t\geq 0} e^{\beta t}\left|y(t+T) - v_+(x(t+T),\lambda,\varepsilon)\right| \leq \mu_2 \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\sup\nolimits_{t\leq 0} e^{-\beta t}\left|x(t-T) - x_c(\varepsilon t, \hat{\xi}_-(\xi,\lambda,\varepsilon),\lambda,\varepsilon)\right| \leq \mu_1, \\ &\sup\nolimits_{t\leq 0} e^{-\beta t}\left|y(t-T) - v_-(x(t-T),\lambda,\varepsilon)\right| \leq \mu_2 \end{aligned} \tag{22}$$

with $\mu_1 = N_1|\varepsilon|$ (see (7)) and the other statements of Theorem 1 hold.

Now, if we can find $\xi^+ \in \mathcal{R}Q_+$ and $\xi^- \in \mathcal{N}Q_-$ such that

$$\hat{\xi}_+(\xi,\lambda,\varepsilon) = u^+(0,\xi^+,\lambda,\varepsilon) \quad \text{and} \quad \hat{\xi}_-(\xi,\lambda,\varepsilon) = u^-(0,\xi^-,\lambda,\varepsilon), \tag{23}$$

then

$$x_c(\varepsilon t, \hat{\xi}_+(\xi,\lambda,\varepsilon),\lambda,\varepsilon) = u^+(\varepsilon t,\xi^+,\lambda,\varepsilon),$$

and

$$\begin{aligned} x(t+T) - \xi_0^+(\lambda,\varepsilon) &= \left[x(t+T) - x_c(\varepsilon t, \hat{\xi}_+(\xi,\lambda,\varepsilon),\lambda,\varepsilon)\right] + \\ &\quad + \left[u^+(\varepsilon t,\xi^+,\lambda,\varepsilon) - \xi_0^+(\lambda,\varepsilon)\right], \\ y(t+T) - v_+(\xi_0^+(\lambda,\varepsilon),\lambda,\varepsilon) &= \left[y(t+T) - v_+(x(t+T),\lambda,\varepsilon)\right] + \\ &\quad + \left[v_+(x(t+T),\lambda,\varepsilon) - v_+(\xi_0^\pm(\lambda,\varepsilon),\lambda,\varepsilon)\right]. \end{aligned} \tag{24}$$

Thus, for $t \to \infty$, (20) follows from (7) and (11). A similar argument applies when $t \to -\infty$.

So assume that

$$\begin{aligned} &\Delta(\xi,\lambda,\varepsilon) = 0, \\ &u^+(0,\xi^+,\lambda,\varepsilon) - \hat{\xi}_+(\xi,\lambda,\varepsilon) = 0, \\ &u^-(0,\xi^-,\lambda,\varepsilon) - \hat{\xi}_-(\xi,\lambda,\varepsilon) = 0 \end{aligned} \tag{25}$$

has a C^{r-1} solution $(\xi,\xi^+,\xi^-,\lambda) = \big(\xi(\varepsilon),\xi^+(\varepsilon),\xi^-(\varepsilon),\lambda(\varepsilon)\big) \in \mathbf{R}^m \times \mathcal{R}Q_+ \times \mathcal{N}Q_- \times \times \mathbf{R}$, such that

$$\big(\xi(0),\xi^+(0),\xi^-(0),\lambda(0)\big) = (\xi_0,0,0,0). \tag{26}$$

Then the C^{r-1}-function

$$x(t,\varepsilon) = \hat{x}(t,\xi(\varepsilon),\lambda(\varepsilon),\varepsilon), \qquad y(t,\varepsilon) = \hat{y}(t,\xi(\varepsilon),\lambda(\varepsilon),\varepsilon)$$

is, for $0 < \varepsilon < \varepsilon_0$, a solution of (1) with $\lambda = \lambda(\varepsilon)$, such that

$$\lim_{t\to\pm\infty} x(t,\varepsilon) = \xi_0^{\pm}(\lambda(\varepsilon),\varepsilon), \qquad \lim_{t\to\pm\infty} y(t,\varepsilon) = v^{\pm}(\xi_0^{\pm}(\lambda(\varepsilon),\varepsilon),\lambda(\varepsilon),\varepsilon)).$$

Next from (24) and (11), for $t \geq T$

$$\bigl|x(t,\varepsilon) - \xi_0(\lambda(\varepsilon),\varepsilon)\bigr|e^{\varepsilon\alpha t} \leq N_1|\varepsilon| + L\bigl|\hat{\xi}^{+}(\xi(\varepsilon),\lambda(\varepsilon),\varepsilon) - \xi_0(\lambda(\varepsilon),\varepsilon)\bigr| = O(\varepsilon).$$

Moreover, from (9) we obtain

$$x(t,0) = \hat{x}(t,\xi_0,0,0) = \xi_0, \qquad y(t,0) = \hat{y}(t,\xi_0,0,0) = y_0(t)$$

and hence the estimate

$$\sup_{0\leq t\leq T} \bigl|x(t,\varepsilon) - \xi_0^{+}(\lambda(\varepsilon),\varepsilon)\bigr|e^{\varepsilon\alpha t} = O(\varepsilon)$$

follows from the continuity and $\xi_0^{+}(0,0) = \xi_0$. A similar argument applies when $t \leq 0$. This proves the first two estimates in (18). The third and the fourth estimates in (18) follow from a similar argument, using again (24) when $|t| \geq T$ and the continuous dependence on the data when $|t| \leq T$. Note that in this case Theorem 1 implies only that the difference $\bigl|y(t,\varepsilon) - v^{\pm}(x(t,\varepsilon),\lambda(\varepsilon),\varepsilon)\bigr|e^{\varepsilon\alpha t}$ is bounded.

Next, using (24), (11) and Theorem 1, we see that (19) holds.

Now we prove the last estimate in (18). We have

$$\bigl|y(t,\varepsilon) - y_0(t)\bigr| \leq \bigl|y(t,\varepsilon) - v^{\pm}(x(t,\varepsilon),\lambda(\varepsilon),\varepsilon)\bigr| +$$

$$+\bigl|v^{\pm}(x(t,\varepsilon),\lambda(\varepsilon),\varepsilon) - v^{\pm}(\xi_0^{\pm}(\varepsilon),\lambda(\varepsilon),\varepsilon)\bigr| +$$

$$+\bigl|v^{\pm}(\xi_0^{\pm}(\varepsilon),\lambda(\varepsilon),\varepsilon) - v^{\pm}(\xi_0,0,0)\bigr| + \bigl|y_0(t) - v^{\pm}(\xi_0,0,0)\bigr|,$$

where now we write $\xi_0^{\pm}(\varepsilon) = \xi_0^{\pm}(\lambda(\varepsilon),\varepsilon)$. Then, from (21) and the first inequality in (18), we see that there exists a constant M_1 such that for $t \geq T$ we have

$$\bigl|y(t,\varepsilon) - v^{+}(x(t,\varepsilon),\lambda(\varepsilon),\varepsilon)\bigr| \leq \mu_2 e^{-\beta(t-T)},$$

$$\bigl|v^{+}(x(t,\varepsilon),\lambda(\varepsilon),\varepsilon) - v^{+}(\xi_0^{+}(\varepsilon),\lambda(\varepsilon),\varepsilon)\bigr| \leq M_1\varepsilon e^{-\varepsilon\alpha(t-T)}.$$

Also, from the smoothness of the three functions $v^{+}(x,\lambda,\varepsilon)$, $\xi_0^{+}(\varepsilon)$, $\lambda(\varepsilon)$ and from (26), there exists a constant M_2

$$\bigl|v^{+}(\xi_0^{+}(\varepsilon),\lambda(\varepsilon),\varepsilon) - v^{+}(\xi_0,0,0)\bigr| \leq M_2\varepsilon.$$

So, given $\rho > 0$, there exists $\bar{T}_\rho = T + \beta^{-1}\log(\mu_2\rho^{-1}) > 0$ such that, for $t \geq \bar{T}_\rho$ we have

$$\bigl|y(t,\varepsilon) - y_0(t)\bigr| \leq \rho + (M_1 + M_2)\varepsilon + \bigl|y_0(t) - v^{+}(\xi_0,0,0)\bigr|.$$

Since $\lim\limits_{|t|\to\infty} y_0(t) = v^{+}(\xi_0,0,0)$ there exists $T_\rho > \bar{T}_\rho$ such that for $t \geq T_\rho$, $\bigl|y_0(t) - v^{+}(\xi_0,0,0)\bigr| < \rho$. So we have proved that, given $\rho > 0$, there exists $T_\rho > 0$ such that for $t \geq T_\rho$, we have

$$\big|y(t,\varepsilon) - y_0(t)\big| \leq \rho + (M_1 + M_2)\varepsilon.$$

By similar arguments, we can show this inequality holds for $t \leq -T_\rho$ also. Next since $y(t,0) = y_0(t)$, from continuity we see that

$$\lim_{\varepsilon\to 0} \sup_{|t|\leq T_\rho} \big|y(t,\varepsilon) - y_0(t)\big| = 0.$$

Then for any $\rho > 0$ there exists $\bar\varepsilon > 0$ such that for $0 \leq \varepsilon \leq \bar\varepsilon$ we have $\sup\limits_{t\in\mathbf{R}} \big|y(t,\varepsilon) - - y_0(t)\big| < \rho$. Hence the last equality in (18) follows.

So all we need to prove is that system (25) has a C^{r-1} solution $(\xi, \xi^+, \xi^-, \lambda) = = (\xi(\varepsilon), \xi^+(\varepsilon), \xi^-(\varepsilon), \lambda(\varepsilon))$ that satisfies (26).

Now, from $\hat\xi_\pm(\xi,\lambda,0) = \xi$, we see that, when $\varepsilon = 0$, system (25) reads:

$$\begin{aligned} &\Delta(\xi,\lambda,0) = 0,\\ &u^+(0,\xi^+,\lambda,0) - \xi = 0,\\ &u^-(0,\xi^-,\lambda,0) - \xi = 0 \end{aligned} \tag{27}$$

and has the solution $(\xi, \xi^+, \xi^-, \lambda) = (\xi_0, 0, 0, 0)$ (see (13) and Theorem 1). Then the linear part of the left-hand side of (27) at the point $(\xi_0, 0, 0, 0)$ is given by, according to (13), (12):

$$L\colon (\xi, \xi^+, \xi^-, \lambda) \mapsto \begin{pmatrix} \Delta_\xi(\xi_0,0,0)\xi + \Delta_\lambda(\xi_0,0,0)\lambda \\ \xi^+ + \xi_0'(0)\lambda - \xi \\ \xi^- + \xi_0'(0)\lambda - \xi \end{pmatrix}$$

hence if $(\xi, \xi^+, \xi^-, \lambda)$ belongs to its kernel we must have

$$\begin{aligned} &\Delta_\xi(\xi_0,0,0)\xi + \Delta_\lambda(\xi_0,0,0)\lambda = 0,\\ &\xi^+ + \xi_0'(0)\lambda - \xi = 0,\\ &\xi^- + \xi_0'(0)\lambda - \xi = 0. \end{aligned} \tag{28}$$

Subtracting the second equation from the third we see that $\xi^+ = \xi^-$ and hence $\xi^+ = \xi^- = 0$ since $\xi^+ \in \mathcal{R}Q_+$, $\xi^- \in \mathcal{N}Q_-$ and $\mathcal{R}Q_+ \cap \mathcal{N}Q_- = \{0\}$. Thus $(\xi, \xi^+, \xi^-, \lambda) \in \mathcal{N}L$ if and only if

$$\begin{aligned} &\Delta_\xi(\xi_0,0,0)\xi + \Delta_\lambda(\xi_0,0,0)\lambda = 0,\\ &\xi = \xi_0'(0)\lambda. \end{aligned}$$

Plugging the second equation into the first we see that λ has to satisfy

$$\big[\Delta_\xi(\xi_0,0,0)\xi_0'(0)+\Delta_\lambda(\xi_0,0,0)\big]\lambda=0.$$

From Theorem 1 and assumption (vi) we see that the above equation has only the solution $\lambda=0$ and hence (28) has the unique solution $(\xi,\xi^+,\xi^-,\lambda)=(0,0,0,0)$. Thus L is invertible and then, for $0<\varepsilon<\varepsilon_0$, sufficiently small, (25) has a unique C^r-solution

$$(\xi,\xi^+,\xi^+,\lambda)=\big(\xi(\varepsilon),\xi^+(\varepsilon),\xi^-(\varepsilon),\lambda(\varepsilon)\big)$$

such that

$$\big(\xi(0),\xi^+(0),\xi^-(0),\lambda(0)\big)=(\xi_0,0,0,0).$$

This concludes the proof of the theorem.

Now we prove the second theorem in this subsection.

Theorem 3. *Let f and g be C^{r+2} functions* ($r\geq 2$), *bounded together with their derivatives and satisfying conditions* (i) – (v). *Suppose also that the three conditions*,

(vii) $\displaystyle\int_{-\infty}^{\infty}\psi^*(t)g_x(\xi_0,y_0(t),0,0)dt\neq 0$;

(viii) *the stable and unstable manifolds respectively of the hyperbolic equilibria $v^+(\xi_0(\lambda),\lambda,0)$ and $v^-(\xi_0(\lambda),\lambda,0)$ of*

$$\dot y=g(\xi_0(\lambda),y,\lambda,0)$$

intersect near $y_0(0)$ so that there is a solution $y_0(t,\lambda)\to v^\pm(\xi_0(\lambda),\lambda,0)$ as $t\to\pm\infty$ with $y(0,\lambda)$ depending continuously on λ and $y_0(0,0)=y_0(0)$;

(ix) *if we let $Q_\pm(\lambda)$ be the projection on to the stable subspace of the linear system*

$$\dot x=F_x^\pm(\xi_0(\lambda),\lambda,0)x \tag{29}$$

along the unstable subspace, $Q(\lambda)$ the projection on to $\mathcal{R}Q_+(\lambda)$ along $\mathcal{N}Q_-(\lambda)$ and $\psi(t,\lambda)$ the unique (*up to a multiplicative constant*), *bounded solution of the adjoint linear system*

$$\dot y+g_y^*(\xi_0(\lambda),y_0(t,\lambda),\lambda,0)y=0,$$

then the Melnikov function

$$\mathcal{M}(\lambda)=\int\limits_{-\infty}^{\infty}\psi(t,\lambda)^*\Bigg\{g_\varepsilon(\xi_0(\lambda),y_0(t,\lambda),\lambda,0)+g_x(\xi_0(\lambda),y_0(t,\lambda),\lambda,0)\times$$

$$\times\left[Q(\lambda)\left(\int\limits_{-\infty}^{t}f(\xi_0(\lambda),y_0(\tau,\lambda),\lambda,0)d\tau+\frac{\partial\xi_0^-}{\partial\varepsilon}(\lambda,0)\right)-\right.$$

$$\left.-\ (\mathbf{I}-Q(\lambda))\left(\int\limits_{t}^{\infty}f(\xi_0(\lambda),y_0(\tau,\lambda),\lambda,0)d\tau-\frac{\partial\xi_0^+}{\partial\varepsilon}(\lambda,0)\right)\right]\Bigg\}dt$$

has a simple zero at $\lambda=0$

hold. Then we get the same conclusions as in Theorem 2.

Proof. As in the proof of Theorem 2, all we need to prove is that system (25) has a C^{r-1} solution $(\xi, \xi^+, \xi^-, \lambda) = \big(\xi(\varepsilon), \xi^+(\varepsilon), \xi^-(\varepsilon), \lambda(\varepsilon)\big)$ that satisfies (26).

First note by solving the equation

$$\langle y_0(t, \lambda) - y_0(0), \dot{y}_0(0) \rangle = 0$$

for a continuous function $t = t(\lambda)$ with $t(0) = 0$ and then replacing $y_0(t, \lambda)$ by $y_0(t + t(\lambda), \lambda)$, we can assume without loss of generality that

$$\big\langle y_0(0, \lambda) - y_0(0), \dot{y}_0(0) \big\rangle = 0.$$

Now, since for λ sufficiently small the point $(\xi_0(\lambda), y_0(0, \lambda))$ belongs to the neighborhood whose existence is stated in Theorem 1 and $(\xi_0(\lambda), y_0(0, \lambda))$ also belongs to the intersection of the centre stable and the centre unstable manifolds and satisfies $\big\langle y_0(0, \lambda) - y_0(0), \dot{y}_0(0) \big\rangle = 0$, it follows from Theorem 1 that

$$\Delta(\xi_0(\lambda), \lambda, 0) = 0, \qquad Z(\xi_0(\lambda), \lambda, 0) = y_0(0, \lambda). \tag{30}$$

Hence

$$y_0(t, \lambda) = \hat{y}(t, \xi_0(\lambda), \lambda, 0) \tag{31}$$

because both functions satisfy the same equation and have the same value at $t = 0$. Next, from (7), (8) we see that $\hat{x}(t, \xi_0(\lambda), \lambda, 0) = \hat{\xi}_\pm(\xi_0(\lambda), \lambda, 0) = \xi_0(\lambda)$ and hence, using (31) and the properties of $\hat{y}(t, \xi, \lambda, \varepsilon)$, we get

$$\left| \frac{\partial^k}{\partial \lambda^k} [y_0(t, \lambda) - v^\pm(\xi_0(\lambda), \lambda, 0)] \right| \leq C_k e^{-(\beta - k\sigma)|t|}$$

for $k = 0, \ldots, r-1$ and $t \geq 0$ in the case of v^+ and $t \leq 0$ in the case of v^-. Thus $y_0(0, \lambda)$ is C^{r+1} and its derivatives with respect to λ are bounded. Moreover, using the inequality with $k = 0$ we see that $y_0(t, \lambda) \to v^\pm(\xi_0(\lambda), \lambda, 0)$ as $t \to \pm\infty$ exponentially and uniformly with respect to λ.

Now from assumption (ii), it follows that

$$\dot{y} = g_y(\xi_0(\lambda), y_0(t, \lambda), \lambda, 0) y$$

has an exponential dichotomy on both $\mathbf{R}_+$ and $\mathbf{R}_-$ with projections $P_+(\lambda)$ and $P_-(\lambda)$ respectively that can be assumed to be C^{r+1} (see, for example, [16–18]). Also we can take $P_+(0) = P_+$ and $P_-(0) = P_-$ (see Section 2). Moreover

$$\mathcal{R}P_+(\lambda) \cap \mathcal{N}P_-(\lambda) = \text{span}\,\big\{\dot{y}_0(0, \lambda)\big\},$$

since this intersection certainly contains the subspace on the right and has dimension at most 1 by continuity. It follows that $\mathcal{N}P_+^*(\lambda) \cap \mathcal{R}P_-^*(\lambda)$ is the subspace of initial values of bounded solutions of the adjoint system

$$\dot{y} = -g_y^*\big(\xi_0(\lambda), y_0(t, \lambda), \lambda, 0\big) y. \tag{32}$$

Then if we define $\psi(t, \lambda)$ to be the solution of the adjoint system with

$$\psi(0,\lambda) = \psi_0(\lambda) \in \mathcal{N}P_+^*(\lambda) \cap \mathcal{R}P_-^*(\lambda),$$

where $\psi_0(\lambda)$ is C^{r+1} with

$$\psi(0,0) = \psi_0,$$

then, since $y_0(t,\lambda)$ is C^{r+1} and is bounded together with its derivatives, it follows from [19] that $\psi(t,\lambda)$ can be assumed to be C^{r+1} and it and its derivatives tend to zero exponentially and uniformly with respect to λ as $|t| \to \infty$. Thus, in particular, $\mathcal{M}(\lambda)$ is C^{r-1}. (We pause here to observe that if $\psi(t,\lambda)$ is a bounded solution of the adjoint system (32), then $\psi(t - t(\lambda),\lambda)$ is a bounded solution of

$$\dot{y} = -g_y^*(\xi_0(\lambda), y_0(t - t(\lambda),\lambda),\lambda,0)y$$

and hence $\mathcal{M}(\lambda)$ does not change if we replace $y_0(t,\lambda)$ by $y_0(t - t(\lambda),\lambda)$, which was the original $y_0(t,\lambda)$, and $\psi(t,\lambda)$ by $\psi(t - t(\lambda),\lambda)$.)

This being said we go back to the problem of solving equation (25). Well, in this case, from (8), (13) and (30) we see that equation (27) has the solution

$$\xi = \xi_0(\lambda), \qquad \xi^+ = 0, \qquad \xi^- = 0, \quad \lambda = 0,$$

and the kernel of the linear map obtained by differentiating the left-hand side of (27) with respect to $(\xi^+,\xi^-,\xi,\lambda)$ at the point $(\xi^+,\xi^-,\xi,\lambda) = (0,0,\xi_0(\lambda),\lambda)$ consists of those $(\xi^+,\xi^-,\xi,\lambda)$ such that

$$\Delta_\xi(\xi_0(\lambda),\lambda,0)\xi + \Delta_\lambda(\xi_0(\lambda),\lambda,0)\lambda = 0,$$
$$u_{\xi^+}^+(0,0,\lambda,0)\xi^+ + u_\lambda^+(0,0,\lambda,0)\lambda - \xi = 0,$$
$$u_{\xi^-}^-(0,0,\lambda,0)\xi^- + u_\lambda^-(0,0,\lambda,0)\lambda - \xi = 0$$

that is, using (15),

$$\begin{aligned} &\Delta_\xi(\xi_0(\lambda),\lambda,0)\xi + \Delta_\lambda(\xi_0(\lambda),\lambda,0)\lambda = 0, \\ &u_{\xi^+}^+(0,0,\lambda,0)\xi^+ + \xi_0'(\lambda)\lambda - \xi = 0, \\ &u_{\xi^-}^-(0,0,\lambda,0)\xi^- + \xi_0'(\lambda)\lambda - \xi = 0. \end{aligned} \tag{33}$$

Subtracting the third equation from the second we obtain

$$u_{\xi^+}^+(0,0,\lambda,0)\xi^+ = u_{\xi^-}^-(0,0,\lambda,0)\xi^-.$$

When $\lambda = 0$ the above equation is equivalent to (see (12))

$$\xi^+ = \xi^- = 0$$

since $\xi^+ \in \mathcal{R}Q_+$ and $\xi^- \in \mathcal{N}Q_-$. Thus we obtain the same conclusion for small $\lambda \neq 0$, because of continuity. Then $(\xi^+,\xi^-,\xi,\lambda)$ satisfies (33) if and only if $\xi^+ = \xi^- = 0$ and the kernel of the linear map consists of those $(\xi^+,\xi^-,\xi,\lambda)$ such that $\xi^+ = \xi^- = 0$ and

$$\Delta_\xi(\xi_0(\lambda),\lambda,0)\xi + \Delta_\lambda(\xi_0(\lambda),\lambda,0)\lambda = 0,$$

$$\xi = \xi_0'(\lambda)\lambda.$$

Plugging the second equation in the first and using $\Delta(\xi_0(\lambda),\lambda,0) = 0$ we conclude that the solution of system (33) is the one dimensional space spanned by $(\xi^+,\xi^-,\xi,\lambda) = (0,0,\xi_0'(\lambda),1)$.

We use the Crandall – Rabinowitz theorem as given in [20] (Theorem 4.1), a suitable version of which can be stated as follows:

Proposition 1. *Let* $\mathcal{F}\colon E\times\mathbf{R}\mapsto G$, $(z,\varepsilon)\mapsto\mathcal{F}(z,\varepsilon)$, *be a* C^r *mapping* $(r\geq 2)$, *where* E *and* G *are Banach spaces. Suppose that there exists a* C^r *function* $\phi(\lambda)$ *defined on an interval* I *such that* $\phi'(\lambda)\neq 0$ *and such that*

$$\mathcal{F}(\phi(\lambda),0) = 0$$

and $L(\lambda) = \mathcal{F}_z(\phi(\lambda),0)$ *is Fredholm of index zero with nullspace spanned by* $\phi'(\lambda)$. *Define*

$$d(\lambda) = \psi(\lambda)\big(\mathcal{F}_\lambda(\phi(\lambda),0)\big),$$

where $\psi(\lambda)\in G^*$ *is a* C^r *function such that* $\mathcal{N}(\psi(\lambda)) = \mathcal{R}(L(\lambda))$. *Then if* $d(\lambda)$ *has a simple zero at* λ_0, *the equation*

$$\mathcal{F}(z,\varepsilon) = 0$$

has a solution $z(\varepsilon)$ *for* ε *sufficiently small. Moreover,* $z(0) = \phi(\lambda_0)$, $z(\varepsilon)$ *is a* C^{r-1} *function and* $\mathcal{F}_z(z(\varepsilon),\varepsilon)$ *is invertible if* $\varepsilon\neq 0$.

Let $\mathcal{F}(\xi^+,\xi^-,\xi,\lambda,\varepsilon)\colon \mathcal{R}Q_+\times\mathcal{N}Q_-\times\mathbf{R}^m\times\mathbf{R}\times\mathbf{R}\to\mathbf{R}\times\mathbf{R}^{2m}$ be the map defined by the left-hand side of (25). We have seen that

$$\mathcal{F}(0,0,\xi_0(\lambda),\lambda,0) = 0$$

and that its linearization at $(\xi^+,\xi^-,\xi,\lambda) = (0,0,\xi_0(\lambda),\lambda)$ with $\varepsilon = 0$: $L(\lambda) = \mathcal{F}_{(\xi^+,\xi^-,\xi,\lambda)}(0,0,\xi_0(\lambda),\lambda,0)$ has the one dimensional kernel

$$\mathcal{N}L(\lambda) = \operatorname{span}\big\{(0,0,\xi_0'(\lambda),1)\big\}.$$

To apply the Crandall – Rabinowitz theorem, we need to determine a vector $\Psi(\lambda)$ such that $\mathcal{R}L(\lambda) = \{\Psi(\lambda)\}^\perp$. Now, $\mathcal{R}L(\lambda)$ consists of those $(\sigma,u_1,u_2)\in\mathbf{R}\times\mathbf{R}^m\times\mathbf{R}^m$ for which $(\xi^+,\xi^-,\xi,\mu)\in\mathcal{R}Q_+\times\mathcal{N}Q_-\times\mathbf{R}^m\times\mathbf{R}$ exists such that

$$\begin{aligned}
&\Delta_\xi(\xi_0(\lambda),\lambda,0)\xi + \Delta_\lambda(\xi_0(\lambda),\lambda,0)\lambda = \sigma,\\
&u^+_{\xi^+}(0,0,\lambda,0)\xi^+ + \xi_0'(\lambda)\lambda - \xi = u_1,\\
&u^-_{\xi^-}(0,0,\lambda,0)\xi^- + \xi_0'(\lambda)\lambda - \xi = u_2.
\end{aligned}\tag{34}$$

Subtracting the third equation from the second we see that (ξ^+,ξ^-) has to be a solution of

$$u^+_{\xi^+}(0,0,\lambda,0)\xi^+ - u^-_{\xi^-}(0,0,\lambda,0)\xi^- = u_1 - u_2. \tag{35}$$

Now $Q_+(\lambda)\colon \mathbf{R}^m \to \mathbf{R}^m$ is the projection on to the stable space of the linear system

$$\dot{u} = F^+_x(\xi_0(\lambda),\lambda,0)u \tag{36}$$

along the unstable space. Moreover since $u^+(t,\xi^+,\lambda,0)$ satisfies the equation $\dot{x} = F^+(x,\lambda,0)$ together with the first equality in (10), we see that $u^+_{\xi^+}(t,0,\lambda,0)$ is a solution of (36) which is bounded on $\mathbf{R}_+$ (because of (11)) and satisfies

$$Q_+ u^+_{\xi^+}(0,0,\lambda,0) = Q_+.$$

Hence $u^+_{\xi^+}(0,0,\lambda,0)$ in an isomorphism from $\mathcal{R}Q_+$ onto $\mathcal{R}Q_+(\lambda)$ with Q_+ as its inverse. Similarly $u^-_{\xi^-}(0,0,\lambda,0)$ in an isomorphism from $\mathcal{N}Q_-$ onto $\mathcal{N}Q_-(\lambda)$ with $\mathbf{I} - Q_-$ as its inverse. So, using (35) and the fact that $\mathcal{R}Q(\lambda) = \mathcal{R}Q_+(\lambda)$, $\mathcal{N}Q(\lambda) = \mathcal{N}Q_-(\lambda)$, we get

$$u^+_{\xi^+}(0,0,\lambda,0)\xi^+ = Q(\lambda)u^+_{\xi^+}(0,0,\lambda,0)\xi^+ = Q(\lambda)(u_1 - u_2)$$

and

$$u^-_{\xi^-}(0,0,\lambda,0)\xi^- = (\mathbf{I} - Q(\lambda))u^-_{\xi^-}(0,0,\lambda,0)\xi^- = (\mathbf{I} - Q(\lambda))(u_2 - u_1).$$

Then, from the last two equations in (34) we obtain

$$\xi = \xi_0'(\lambda)\lambda - \big[(\mathbf{I} - Q(\lambda))u_1 + Q(\lambda)u_2\big].$$

Plugging this equality into the first equation in (34) and using again the first equality in (30) we obtain

$$\sigma = -\Delta_\xi(\xi_0(\lambda),\lambda,0)\big[(\mathbf{I} - Q(\lambda))u_1 + Q(\lambda)u_2\big].$$

Thus the range of $L(\lambda)$ is the nullspace of the linear functional $\Psi(\lambda)$, defined by

$$\Psi(\lambda)\left(\begin{bmatrix}\sigma\\ u_1\\ u_2\end{bmatrix}\right) = \sigma + \Delta_\xi(\xi_0(\lambda),\lambda,0)\big[(\mathbf{I} - Q(\lambda))u_1 + Q(\lambda)u_2\big].$$

Finally, we evaluate $\mathcal{F}_\varepsilon(0,0,\xi_0(\lambda),\lambda,0)$. Using (13) we see that

$$\mathcal{F}_\varepsilon(0,0,\xi_0(\lambda),\lambda,0) = \begin{pmatrix} \Delta_\varepsilon(\xi_0(\lambda),\lambda,0) \\ \dfrac{\partial\xi_0^+}{\partial\varepsilon}(\lambda,0) - \dfrac{\partial\hat{\xi}_+}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) \\ \dfrac{\partial\xi_0^-}{\partial\varepsilon}(\lambda,0) - \dfrac{\partial\hat{\xi}_-}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) \end{pmatrix}.$$

Thus our Melnikov function is

$$d(\lambda) = \Psi(\lambda)\mathcal{F}_\varepsilon(0,0,\xi_0(\lambda),\lambda,0) =$$

$$= \Delta_\varepsilon(\xi_0(\lambda),\lambda,0) - \Delta_\xi(\xi_0(\lambda),\lambda,0)\left\{(\mathbf{I}-Q(\lambda))\left[\frac{\partial\hat{\xi}_+}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) - \frac{\partial\xi_0^+}{\partial\varepsilon}(\lambda,0)\right]+\right.$$

$$\left.+Q(\lambda)\left[\frac{\partial\hat{\xi}_-}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) - \frac{\partial\xi_0^-}{\partial\varepsilon}(\lambda,0)\right]\right\}.$$

This turns out to be difficult to calculate. We replace it by

$$\widetilde{d}(\lambda) = \left[\psi_0^*(\lambda)\psi_0\right] d(\lambda).$$

Since $\psi_0^*(\lambda)\psi_0 = 1$ when $\lambda = 0$, we see that $\widetilde{d}(\lambda)$ has a simple zero at $\lambda = 0$ if and only if $d(\lambda)$ has. Then, setting

$$\widetilde{\Delta}(\xi,\lambda,\varepsilon) = \left[\psi_0^*(\lambda)\psi_0\right]\Delta(\xi,\lambda,\varepsilon),$$

we see that

$$\widetilde{\Delta}_\xi(\xi,\lambda,\varepsilon) = \left[\psi_0^*(\lambda)\psi_0\right]\Delta_\xi(\xi,\lambda,\varepsilon), \qquad \widetilde{\Delta}_\varepsilon(\xi,\lambda,\varepsilon) = \left[\psi_0^*(\lambda)\psi_0\right]\Delta_\varepsilon(\xi,\lambda,\varepsilon)$$

and

$$\widetilde{d}(\lambda) = \widetilde{\Delta}_\varepsilon(\xi_0(\lambda),\lambda,0) -$$

$$-\widetilde{\Delta}_\xi(\xi_0(\lambda),\lambda,0)\left\{\big(\mathbf{I}-Q(\lambda)\big)\left[\frac{\partial\hat{\xi}_+}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) - \frac{\partial\xi_0^+}{\partial\varepsilon}(\lambda,0)\right]+\right.$$

$$\left.+Q(\lambda)\left[\frac{\partial\hat{\xi}_-}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) - \frac{\partial\xi_0^-}{\partial\varepsilon}(\lambda,0)\right]\right\}.$$

To calculate further we observe that from the equality

$$f(\xi_0(\lambda), v^\pm(\xi_0(\lambda),\lambda,0),\lambda,0) = 0$$

(see assumption (v)), and (8), (31) it follows that

$$\frac{\partial\hat{\xi}_+}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) = \int_0^\infty f(\xi_0(\lambda), y_0(t,\lambda),\lambda,0)dt,$$

$$\frac{\partial\hat{\xi}_-}{\partial\varepsilon}(\xi_0(\lambda),\lambda,0) = -\int_{-\infty}^0 f(\xi_0(\lambda), y_0(t,\lambda),\lambda,0)dt.$$

Also, as in the proof of Theorem 1 in [4], we can derive the formulae

$$\widetilde{\Delta}_\xi(\xi_0(\lambda),\lambda,0) = -\int_{-\infty}^\infty \psi^*(t,\lambda)g_x(\xi_0(\lambda),y_0(t,\lambda),\lambda,0)dt$$

and

$$\widetilde{\Delta}_\varepsilon(\xi_0(\lambda),\lambda,0) = -\int\limits_{-\infty}^{\infty} \psi^*(t,\lambda)\Bigg\{ g_\varepsilon(\xi_0(\lambda), y_0(t,\lambda),\lambda,0) +$$

$$+ g_x(\xi_0(\lambda), y_0(t,\lambda),\lambda,0)\int\limits_0^t f(\xi_0(\lambda), y_0(\tau,\lambda),\lambda,0)d\tau\Bigg\} dt.$$

Then it follows that

$$\widetilde{d}(\lambda) = -\mathcal{M}(\lambda).$$

Thus, if $\mathcal{M}(\lambda)$ has a simple zero at $\lambda = 0$, from Proposition 1 we obtain the existence of a C^{r-1}–solution $(\xi^+,\xi^-,\xi,\lambda) = (\xi^+(\varepsilon),\xi^-(\varepsilon),\xi(\varepsilon),\lambda(\varepsilon))$ of (25) that satisfies (26). The proof is complete.

Remark. We can also show the solution $p(t,\varepsilon)$ found in Theorems 2 and 3 has the following properties:

(a) *$\dot{p}(t,\varepsilon)$ is not in the tangent space to the stable fibre through $p(t,\varepsilon)$, provided that*

$$Q\left[-\frac{\partial \xi_0^+}{\partial\varepsilon}(0,0) + \frac{\partial \xi_0^-}{\partial\varepsilon}(0,0) + \int\limits_{-\infty}^{\infty} f(\xi_0, y_0(t),0,0)dt\right] \neq 0, \tag{37}$$

where Q is the projection with the same range as Q_+ and the same nullspace as Q_-, and where, according to Theorem 3 *in* [4], *vectors in the tangent space to the stable fibre at $p(t,\varepsilon)$ are the initial values of the solutions of the variational system along $p(t,\varepsilon)$ which approach zero as $t \to \infty$ at the exponential rate $\beta - \sigma$.*

To prove the statement we first note from (7) and the equation after (23) that for $t \geq T$

$$\left|x(t,\varepsilon) - u^+(\varepsilon(t-T),\xi^+(\varepsilon),\lambda(\varepsilon),\varepsilon)\right| \leq \mu_1 e^{-\beta(t-T)},$$

$$\left|y(t,\varepsilon) - v^+(u^+(\varepsilon(t-T),\xi^+(\varepsilon),\lambda(\varepsilon),\varepsilon),\lambda(\varepsilon),\varepsilon)\right| \leq \mu_2 e^{-\beta(t-T)}.$$

We next prove that $u^+(0,\xi^+(\varepsilon),\lambda(\varepsilon),\varepsilon) \neq \xi_0^+(\lambda(\varepsilon),\varepsilon) = u^+(0,0,\lambda(\varepsilon),\varepsilon)$ (see (13)). Since $u^+(0,\xi^+(0),\lambda(0),0) = u^+(0,0,0,0)$ we compute

$$\frac{d}{d\varepsilon}\left[u^+(0,\xi^+(\varepsilon),\lambda(\varepsilon),\varepsilon) - u^+(0,0,\lambda(\varepsilon),\varepsilon)\right]\Big|_{\varepsilon=0} = Q_+\frac{d\xi^+}{d\varepsilon}(0)$$

(see (12)). To calculate $\dfrac{d\xi^+}{d\varepsilon}(0)$, we differentiate the last two equations in (25) with $\xi^\pm = \xi^\pm(\varepsilon)$, $\lambda = \lambda(\varepsilon)$, $\xi = \xi(\varepsilon)$ with respect to ε at $\varepsilon = 0$ to get

$$\frac{d\xi^+}{d\varepsilon}(0) + \xi_0'(0)\lambda'(0) + \frac{\partial\xi_0^+}{\partial\varepsilon}(0,0) = \xi'(0) + \hat{\xi}_{+,\varepsilon}(\xi_0,0,0),$$

$$\frac{d\xi^-}{d\varepsilon}(0) + \xi_0'(0)\lambda'(0) + \frac{\partial\xi_0^-}{\partial\varepsilon}(0,0) = \xi'(0) + \hat{\xi}_{-,\varepsilon}(\xi_0,0,0).$$

Subtracting these equations, we get

$$\frac{d\xi^+}{d\varepsilon}(0) - \frac{d\xi^-}{d\varepsilon}(0) = -\frac{\partial\xi_0^+}{\partial\varepsilon}(0,0) + \frac{\partial\xi_0^-}{\partial\varepsilon}(0,0) + \int\limits_{-\infty}^{\infty} f(\xi_0, y_0(t), 0, 0)dt$$

from which it follows that

$$\frac{d\xi^+}{d\varepsilon}(0) = Q\left[-\frac{\partial\xi_0^+}{\partial\varepsilon}(0,0) + \frac{\partial\xi_0^-}{\partial\varepsilon}(0,0) + \int\limits_{-\infty}^{\infty} f(\xi_0, y_0(t), 0, 0)dt\right]$$

and hence that

$$Q_+\frac{d\xi^+}{d\varepsilon}(0) = Q\left[-\frac{\partial\xi_0^+}{\partial\varepsilon}(0,0) + \frac{\partial\xi_0^-}{\partial\varepsilon}(0,0) + \int\limits_{-\infty}^{\infty} f(\xi_0, y_0(t), 0, 0)dt\right] \neq 0$$

since Q and Q_+ have the same range. Thus $u^+(0, \xi^+(\varepsilon), \lambda(\varepsilon), \varepsilon) \neq \xi_0^+(\lambda(\varepsilon), \varepsilon)$ if $\varepsilon > 0$ is sufficiently small.

Now write $x_c(t)$ for $u^+(t + \varepsilon T, \xi^+(\varepsilon), \lambda(\varepsilon), \varepsilon)$. Note that $\dot{x}_c(0) \neq 0$ since $x_c(t)$ is not an equilibrium. Then for $t \geq 0$

$$\big|x(t+T,\varepsilon) - x_c(\varepsilon t)\big| \leq \mu_1 e^{-\beta t},$$

$$\big|y(t+T,\varepsilon) - v_+(x(t+T,\varepsilon), \lambda(\varepsilon), \varepsilon)\big| \leq \mu_2 e^{-\beta t}.$$

Note that from these inequalities we also obtain for $t \geq 0$

$$\big|y(t+T,\varepsilon) - v_+(x_c(\varepsilon t), \lambda(\varepsilon), \varepsilon)\big| \leq (\hat{N}\mu_1 + \mu_2)e^{-\beta t}$$

where $\hat{N}$ is an upper bound for the norm of $v_x(x, \lambda, \varepsilon)$. Then since

$$\dot{x}(t,\varepsilon) = \varepsilon f(x(t,\varepsilon), y(t,\varepsilon), \lambda(\varepsilon), \varepsilon),$$

$$\varepsilon\dot{x}_c(\varepsilon t) = \varepsilon f\big(x_c\big(\varepsilon t), v_+(x_c(\varepsilon t), \lambda(\varepsilon), \varepsilon), \lambda(\varepsilon), \varepsilon\big),$$

we see that for $t \geq 0$

$$\big|\dot{x}(t+T,\varepsilon) - \varepsilon\dot{x}_c(\varepsilon t)\big| \leq N|\varepsilon|\big[(\hat{N}+1)\mu_1 + \mu_2\big]e^{-\beta t}.$$

Now $\dot{x}_c(t)$ is a solution of

$$\dot{x} = F_x^+(x_c(t), \lambda(\varepsilon), \varepsilon)x$$

and so for all t

$$\big|\dot{x}_c(t)\big| \geq |\dot{x}_c(0)|e^{-N|t|},$$

where N is a bound on $|F_x^+(x, \lambda, \varepsilon)|$. It follows that for $t \geq 0$

$$\big|\dot{p}(t+T,\varepsilon)\big| \geq \big|\dot{x}(t+T,\varepsilon)\big| \geq \big|\varepsilon\dot{x}_c(\varepsilon t, \varepsilon)\big| - N|\varepsilon|\Big[\big(\hat{N}+1\big)\mu_1 + \mu_2\Big]e^{-\beta t} \geq$$

$$\geq \big|\varepsilon\dot{x}_c(0)\big|e^{-N\varepsilon t} - N|\varepsilon|\big[(\hat{N}+1)\mu_1 + \mu_2\big]e^{-\beta t}.$$

If $N\varepsilon < \beta$, it follows that $e^{N\varepsilon t}|\dot{p}(t,\varepsilon)|$ does not tend to 0 as $t \to \infty$. So if $N\varepsilon < \beta - \sigma$, it follows that $\dot{p}(t,\varepsilon)$ cannot be in the tangent space to the stable fibre at $p(t,\varepsilon)$.

(b) *Similarly we can prove that* $\dot{p}(t,\varepsilon)$ *is not in the tangent space to the unstable fibre through* $p(t,\varepsilon)$, *provided that*

$$(\mathbf{I} - Q)\left[-\frac{\partial \xi_0^+}{\partial \varepsilon}(0,0) + \frac{\partial \xi_0^-}{\partial \varepsilon}(0,0) + \int_{-\infty}^{\infty} f(\xi_0, y_0(t), 0, 0)dt\right] \neq 0,$$

where, according to Theorem 3 *in* [4], *vectors in the tangent space to the unstable fibre at* $p(t,\varepsilon)$ *are the initial values of the solutions of the variational system along* $p(t,\varepsilon)$ *which approach zero as* $t \to -\infty$ *at the exponential rate* $\beta - \sigma$ (*see Theorem* 1).

(c) *It is in general position, that is, the tangent spaces to the stable manifold* $\mathcal{W}^s$ *of the hyperbolic equilibrium* $q^+(\lambda(\varepsilon),\varepsilon)$ *and to the unstable manifold* $\mathcal{W}^u$ *of the hyperbolic equilibrium* $q^-(\lambda(\varepsilon),\varepsilon)$ *of the system*

$$\dot{x} = \varepsilon f(x, y, \varepsilon, \lambda(\varepsilon)),$$
$$\dot{y} = g(x, y, \varepsilon, \lambda(\varepsilon))$$

at $p(t,\varepsilon)$ *intersect in the one-dimensional subspace spanned by* $\dot{p}(t,\varepsilon)$.

The proof of this fact goes as in [1, p. 46], so we will not repeat it here.

(d) *From Section* 3.1 *in* [5] *it follows that, if*

$$\int_{-\infty}^{\infty} \psi^*(t) g_x(\xi_0, y_0(t), 0, 0)dt \neq 0,$$

then $T_{p(0,\varepsilon)}\mathcal{M}^{cs}$ *and* $T_{p(0,\varepsilon)}\mathcal{M}^{cu}$ *intersect transversely.*

3.3. Examples. Here we give examples of the application of Theorems 2 and 3. First, for an example of Theorem 2, we consider the following system, with $x, y \in \mathbf{R}$:

$$\dot{x} = \varepsilon f(x, y, \dot{y}, \lambda, \varepsilon),$$
$$\ddot{y} = -g(y) + \lambda h_1(x, \lambda, \varepsilon)\dot{y} + \varepsilon h(x, y, \dot{y}, \lambda, \varepsilon)$$

where the functions involved are sufficiently smooth. Setting $y_1 = y$, $y_2 = \dot{y}$ we obtain the system

$$\begin{aligned} \dot{x} &= \varepsilon f(x, y_1, y_2, \lambda, \varepsilon), \\ \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -g(y_1) + \lambda h_1(x, \lambda, \varepsilon) y_2 + \varepsilon h(x, y_1, y_2, \lambda, \varepsilon). \end{aligned} \tag{38}$$

We assume the following conditions hold: there exists a point $\xi_0 \in \mathbf{R}$ such that

$$
\begin{aligned}
&f(\xi_0,0,0,0,0)=0, \quad g(0)=0,\\
&f_x(\xi_0,0,0,0,0)<0, \quad g'(0)<0, \qquad (39)\\
&h_1(\xi_0,0,0)\neq 0
\end{aligned}
$$

and the second order equation $\ddot{y}+g(y)=0$ has the solution $p(t)\neq 0$ which is homoclinic to $y=0$, that is:

$$\lim_{|t|\to\infty} p(t)=\lim_{|t|\to\infty}\dot{p}(t)=0.$$

Note that, taking $G(y)=\int_0^y g(u)du$ then any solution of $\ddot{y}+g(y)=0$ satisfies $\frac{1}{2}\dot{y}^2+G(y)=\text{const}$. In particular $\frac{1}{2}\dot{p}^2(t)+G(p(t))=0$. Hence we see that $p(t)\neq 0$ for all $t\in\mathbf{R}$, since otherwise, if $p(t^*)=0$ for some t^*, then we also have $\dot{p}(t^*)=0$ and hence $p(t)=0$ since both $y=p(t)$ and $y=0$ are solutions of the Cauchy problem:

$$
\begin{aligned}
&\ddot{y}+g(y)=0,\\
&y(t^*)=\dot{y}(t^*)=0.
\end{aligned}
$$

As a consequence, either $p(t)<0$ or $p(t)>0$ for all $t\in\mathbf{R}$.

When $\lambda=\varepsilon=0$, the equations are decoupled and we have the single normally hyperbolic centre manifold $(y_1,y_2)=v(x)=0$, that persists for $\lambda\neq 0$ also, that is, $v(x,\lambda,0)=0$. The equation $\dot{x}=f(x,0,0,0,0)$ on the centre manifold has the exponentially stable fixed point $x=\xi_0$, and the y-equation has for all x the homoclinic orbit

$$y_0(t)=\begin{bmatrix} p(t)\\ \dot{p}(t)\end{bmatrix}.$$

Finally, the bounded solution of the adjoint system is

$$\psi(t)=\begin{bmatrix} \ddot{p}(t)\\ -\dot{p}(t)\end{bmatrix}.$$

Hence we see that conditions (i) – (v) are satisfied and condition (vi) reads:

$$-\int_{-\infty}^{\infty}\dot{p}(t)h_1(\xi_0,0,0)\dot{p}(t)dt=-h_1(\xi_0,0,0)\int_{-\infty}^{\infty}\dot{p}(t)^2dt\neq 0.$$

Thus we conclude that, if (39) holds, then for some $\lambda=\lambda(\varepsilon)$, equation (38) has a solution $p(t,\varepsilon)=\big(x(t,\varepsilon),y(t,\varepsilon)\big)$ homoclinic to a fixed point of (38) with $\lambda=\lambda(\varepsilon)$, close to $(\xi_0,0)$. Moreover, by the Remarks after Theorem 3, $p(t,\varepsilon)$ is in general position, and $\dot{p}(t,\varepsilon)$ is not in the tangent space to the stable fibre at $p(t,\varepsilon)$ if

$$\int_{-\infty}^{\infty} f(\xi_0, p(t), \dot{p}(t), 0, 0)dt \neq 0$$

(note that (39) implies that ξ_0 is a stable fixed point for the equation $\dot{x} = f(x, 0, 0, 0, 0)$ and hence $Q = \mathbf{I}$). For example if $f(x, y, \dot{y}, \lambda, \varepsilon) = y - x$ we have $\xi_0 = 0$ and

$$\int_{-\infty}^{\infty} f(\xi_0, p(t), \dot{p}(t), 0, 0)dt = \int_{-\infty}^{\infty} p(t)dt \neq 0$$

since either $p(t) < 0$ or $p(t) > 0$ for all $t \in \mathbf{R}$.

As an example for Theorem 3, we consider the following system:

$$\begin{aligned} \dot{x} &= \varepsilon(f(x) + \lambda h_0(y_1) + \varepsilon h(x, y)), \\ \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -(1+\lambda)^2 \widetilde{g}(y_1) + k(x) y_2 \end{aligned} \tag{40}$$

where x, y_1, y_2 are in $\mathbf{R}$, the functions are sufficiently smooth and we assume that

$$\begin{aligned} &f(0) = \widetilde{g}(0) = h(0, 0) = k(0) = h_0(0) = 0, \\ &f'(0) < 0, \qquad \widetilde{g}'(0) < 0, \quad k'(0) \neq 0, \end{aligned}$$

and also that the equation $\ddot{y} + \widetilde{g}(y) = 0$ has a homoclinic orbit $y_0(t) = (p(t), \dot{p}(t))$ associated with the saddle point $(0, 0)$. Without loss of generality we can assume that $\dot{p}(0) = 0$ and hence $p(t) = p(-t)$ since both solve the Cauchy problem

$$\begin{aligned} &\ddot{y} + \widetilde{g}(y) = 0, \\ &y(0) = p(0), \qquad \dot{y}(0) = 0. \end{aligned}$$

Then we assume that

$$\int_0^{\infty} h_0(p(\tau))d\tau \neq 0.$$

Now system (40) has the single normally hyperbolic centre manifold $y = v(x, \lambda, \varepsilon) = 0$ and, when $\lambda = \varepsilon = 0$, the system $\dot{x} = f(x)$ on the centre manifold has the exponentially stable fixed point $\xi_0 = 0$. For small λ and ε, the equation on the centre manifold is $\dot{x} = f(x) + \varepsilon h(x, 0)$ with no λ and so the equilibrium near 0 is $\xi_0(\lambda, \varepsilon) = \xi_0(\varepsilon)$. Then differentiating the equation $f(\xi_0(\varepsilon)) + \varepsilon h(\xi_0(\varepsilon), 0) = 0$, we see that

$$\xi_0'(0) = -\frac{h(0, 0)}{f'(0)} = 0.$$

Next observe that the y-equations in (40) with $x = \xi_0(\lambda, 0) = 0$ and $\varepsilon = 0$ have the homoclinic orbit

$$y_0(t,\lambda) = \begin{bmatrix} p((1+\lambda)t) \\ (1+\lambda)\dot{p}((1+\lambda)t) \end{bmatrix}$$

and we have

$$\psi(t,\lambda) = \begin{bmatrix} (1+\lambda)^2\ddot{p}((1+\lambda)t) \\ -(1+\lambda)\dot{p}((1+\lambda)t) \end{bmatrix}.$$

As an expected consequence

$$\int_{-\infty}^{\infty} \psi^*(t,\lambda)g_\lambda(x,y_0(t,\lambda),0,0)dt = 2(1+\lambda)^2 \int_{-\infty}^{\infty} \dot{p}((1+\lambda)t)\ddot{p}((1+\lambda)t)dt =$$

$$= 2(1+\lambda)\int_{-\infty}^{\infty} \dot{p}(t)\ddot{p}(t)dt = 0.$$

Next we have

$$\int_{-\infty}^{\infty} \psi^*(t,0)g_x(\xi_0,y_0(t,0),0,0)dt = -k'(0)\int_{-\infty}^{\infty} \dot{p}(t)^2 dt \neq 0.$$

Then, the linear system (29) in this case reads: $\dot{x} = f'(0)x$ and has the exponentially stable equilibrium $x = 0$. Hence $Q(\lambda) = Q_\pm(\lambda) = \mathbf{I}$ and $\mathcal{M}(\lambda)$ reads:

$$\mathcal{M}(\lambda) = -k'(0)\int_{-\infty}^{\infty} (1+\lambda)^2\dot{p}^2((1+\lambda)t)\int_{-\infty}^{t} \lambda h_0(p((1+\lambda)\tau)d\tau dt =$$

$$= -\lambda k'(0)\int_{-\infty}^{\infty} \dot{p}^2(t)\int_{-\infty}^{t} h_0(p(\tau))d\tau dt =$$

$$= -\lambda k'(0)\int_{-\infty}^{\infty} \dot{p}^2(t)dt\int_{-\infty}^{0} h_0(p(t))dt =$$

$$= -\lambda k'(0)\int_{-\infty}^{\infty} \dot{p}^2(t)dt\int_{0}^{\infty} h_0(p(t))dt$$

where we have used the fact that $\dot{p}(t)$ and $\int_0^t h(p(\tau))d\tau$ are odd functions. Thus $\mathcal{M}(\lambda)$ has a simple zero at $\lambda = 0$ and hence we conclude that, for some $\lambda = \lambda(\varepsilon)$, system (40) has a homoclinic solution $p(t,\varepsilon) = (x(t,\varepsilon), y(t,\varepsilon))$ such that

$$\lim_{|t|\to\infty} x(t,\varepsilon) = \xi_0(\varepsilon), \qquad \lim_{|t|\to\infty} y(t,\varepsilon) = 0.$$

Moreover, by the Remarks after Theorem 3, $p(t,\varepsilon)$ is in general position and the centre stable and centre unstable manifolds intersect transversely along $p(t,\varepsilon)$.

Note that Kokubu et al. [21] consider systems such as

$$\dot{x} = \varepsilon(1-x^2),$$

$$\dot{y}_1 = y_2,$$

$$\dot{y}_2 = xy_2 - y_1(y_1 - a - \lambda).$$

Such a system satisfies our conditions (i) to (viii) if $a \neq 0$. However the Melnikov function is identically zero and so our theorem cannot be applied.

4. Sil'nikov saddle-focus homoclinic orbits. Saddle-focus homoclinic orbits were first studied by Sil'nikov [6]. In this section we review the definition of such orbits and describe the equivalent definition developed in [12]. Then we state a theorem which gives a general class of singularly perturbed systems in dimension $n \geq 4$ which have saddle-focus homoclinic orbits.

We consider an autonomous system

$$\dot{x} = F(x) \tag{41}$$

in $\mathbf{R}^n$, where $n \geq 3$ and F is C^1, with a hyperbolic equilibrium. We denote the stable manifold by $\mathcal{W}^s$ and the unstable manifold by $\mathcal{W}^u$. Also we denote by $\phi(t,\xi)$ the solution $x(t)$ of (1) with $x(0) = \xi$.

Here we give the definition of saddle-focus homoclinic orbit as given in Deng [7]. The first two conditions are:

(D_1) the eigenvalues of $F'(q)$ having the smallest positive real part are $\mu \pm i\omega$ with $\omega > 0$ and

$$0 < \mu < -\mathrm{Re}(\lambda)$$

for all eigenvalues λ with negative real parts;

(D_2) there is a homoclinic orbit $p(t)$ to q, that is, $p(t) \neq q$ and $p(t) \in \mathcal{W}^s \cap \mathcal{W}^u$, such that

$$\dim T_{p(t)}\mathcal{W}^s \cap T_{p(t)}\mathcal{W}^u = 1.$$

These are the only conditions needed in 3 dimensions, although note that in 3 dimensions the second part of (D_2) is automatically satisfied. In higher dimensions, two additional conditions are needed. We denote by $\mathcal{W}^{uu}$ the strong unstable manifold of the equilibrium q. This is a locally invariant manifold containing q whose tangent space at q consists of the sum of the generalized eigenspaces of $F'(q)$ corresponding to the eigenvalues with real part greater than μ. Solutions of (1) starting in this manifold approach q as $t \to -\infty$ at an exponential rate faster than μ. The two conditions are:

(D_3) as $t \to -\infty$, $p(t)$ is asymptotically tangent to the linear span of the eigenvectors of $\mu \pm i\omega$;

(D$_4$) there is a submanifold $\mathcal{M}_0$ of $\mathcal{W}^u$ containing $p(0)$ with $\dim \mathcal{M}_0 = \dim \mathcal{W}^{uu}$ such that

$$\lim_{t\to\infty} T_{p(t)}\mathcal{M}_t = T_q\mathcal{W}^{uu},$$

where $\mathcal{M}_t = \phi(t, \mathcal{M}_0)$.

If there is such a homoclinic orbit, Silnikov and Deng show the presence of chaotic dynamics near it.

Now we discuss conditions (D$_3$) and (D$_4$). Note that Deng's condition (D$_4$), which he refers to as the "strong inclination" condition, corresponds to Sil'nikov's condition (D) in [6]. In [12], we show that these two conditions can be formulated in terms of certain subspaces related to the variational system

$$\dot{x} = F'(p(t))x \tag{42}$$

of which $\phi_x(t, p(0))$ is the fundamental matrix with $\phi_x(0, p(0)) = \mathbf{I}$. Consider a homoclinic orbit $p(t) = \phi(t, p(0))$ satisfying Deng's condition (D$_1$). Then choose ν so that

$$\mu < \nu < \mathrm{Re}(\lambda)$$

for all eigenvalues λ of $F'(q)$ with real part greater than μ. We define the *centre stable subspace* as

$$W^{cs} = \left\{\xi\colon \sup_{t\geq 0} e^{-\nu t}\big|\phi_x(t, p(0))\xi\big| < \infty\right\}.$$

We show in [12] that this definition is independent of the choice of ν and that

$$\dim W^{cs} = \dim \mathcal{W}^s + 2 = n + 2 - \dim \mathcal{W}^u. \tag{43}$$

Next we define the *strong unstable subspace* as

$$W^{uu} = \left\{\xi\colon \sup_{t\leq 0} e^{-\nu t}\big|\phi_x(t, p(0))\xi\big| < \infty\right\}.$$

We show in [12] that this definition is independent of the choice of ν and that

$$\dim W^{uu} = \dim \mathcal{W}^u - 2.$$

In [12], we show the following.

Proposition 2. *Suppose* (41) *has a hyperbolic equilibrium q satisfying* (D$_1$). *Let $p(t)$ be an associated homoclinic orbit. Then*

(i) (D$_3$) *holds if and only if $p'(0)$ is not in the strong unstable subspace W^{uu}*;

(ii) *Deng's condition* (D$_4$) *and Sil'nikov's condition* (D) *are equivalent to the condition*

$$\dim(T_{p(0)}\mathcal{W}^u \cap W^{cs}) = 2 \qquad \textit{or} \qquad T_{p(0)}\mathcal{W}^u + W^{cs} = \mathbf{R}^n.$$

We now prove the following theorem, which gives a general class of systems with Sil'nikov saddle-focus homoclinic orbits.

Theorem 4. *Consider system* (1) *with* $m = 2$ *such that conditions* (i)–(iv) *hold with* $v^+(x) = v^-(x) = v(x)$ *and condition (v) holds with* $\xi_0^{\pm}(\lambda, \varepsilon) = \xi_0(\lambda, \varepsilon)$ *such that the derivative with respect to* x *of* $f(x, v(x), 0, 0)$ *at* ξ_0 *has eigenvalues* $\mu \pm i\omega$ *with* $\mu > 0$ *and* $\omega > 0$. *Further suppose either* (vi) *in Theorem* 2 *holds or* (vii), (viii) *and* (ix) *in Theorem* 3 *hold. Then we deduce the existence of a function* $\lambda(\varepsilon)$ *and a homoclinic orbit* $p(t, \varepsilon) = (x(t, \varepsilon), y(t, \varepsilon))$ *for system* (1) *with* $\lambda = \lambda(\varepsilon)$ *as in Theorem* 2 *and if we assume in addition that*

$$\int_{-\infty}^{\infty} f(\xi_0, y_0(t), 0, 0)dt \neq 0, \qquad \int_{-\infty}^{\infty} \psi^*(t) g_x(\xi_0, y_0(t), 0, 0)dt \neq 0,$$

then $p(t, \varepsilon)$ *is a Sil'nikov saddle-focus homoclinic orbit.*

Proof. Now under condition (vi) or conditions (vii), (viii) and (ix) we get the existence of $\lambda(\varepsilon)$ and the solution $p(t, \varepsilon)$ as described in Theorem 2.

Now we verify Deng's conditions for system (1) with $\lambda = \lambda(\varepsilon)$. First note that

$$q(\lambda(\varepsilon), \varepsilon) = \big(\xi_0(\lambda(\varepsilon), \varepsilon), v(\xi_0(\lambda(\varepsilon), \varepsilon), \lambda(\varepsilon), \varepsilon)\big)$$

is an equilibrium for system (1) with $\lambda = \lambda(\varepsilon)$. The corresponding variational matrix is

$$A = \begin{bmatrix} \varepsilon f_x & \varepsilon f_y \\ g_x & g_y \end{bmatrix},$$

where $f_x = f_x(\xi_0(\lambda(\varepsilon), \varepsilon), v(\xi_0(\lambda(\varepsilon), \varepsilon), \lambda(\varepsilon), \varepsilon), \lambda, \varepsilon)$, etc. If we take

$$T = \begin{bmatrix} \mathbf{I}_m & 0 \\ v_x & \mathbf{I}_n \end{bmatrix},$$

where $v_x = v_x(\xi_0(\lambda(\varepsilon), \varepsilon), \lambda(\varepsilon), \varepsilon)$, then we see that

$$T^{-1}AT = \begin{bmatrix} \varepsilon(f_x + f_y v_x) & \varepsilon f_y \\ 0 & g_y - \varepsilon f_y v_x \end{bmatrix}.$$

Now when $\varepsilon = 0$, the eigenvalues of $f_x + f_y v_x$ are $\mu \pm i\omega$ and the eigenvalues of g_y have real parts with absolute value greater than δ_0. So if ε is small enough, A has a pair of eigenvalues $\varepsilon(\mu \pm i\omega + O(\varepsilon))$ and the other eigenvalues have real parts with absolute value greater than or equal to δ_0. Hence, if ε is sufficiently small, (D_1) is satisfied by the equilibrium $q(\lambda(\varepsilon), \varepsilon)$.

Next we note that (D_2) follows from Theorem 2 and (a) in the Remark after Theorem 3.

In regard to (D_3), it follows from (b) in the Remark after Theorem 3, where we note that here $Q = 0$, that $\dot{p}(0, \varepsilon)$ is not in the tangent space to the unstable fibre at $p(0, \varepsilon)$. However according to Theorem 3 in [4], vectors in the tangent space to the unstable fibre at $p(0, \varepsilon)$ are the initial values of the solutions of the variational system along $p(t, \varepsilon)$ which approach zero as $t \to -\infty$ at the exponential rate $\beta - \sigma$. Now here we choose ν so that $\varepsilon\mu < \nu < \delta_0$ and we have $\beta < \delta_0$ (see Theorem 1). So we can assume

$\nu > \beta - \sigma$. It follows that W^{uu} is a subspace of the tangent space to the unstable fibre at $p(0,\varepsilon)$. Hence $\dot{p}(0,\varepsilon)$ is not in W^{uu}.

Now we show that the last condition in Theorem 4 implies Deng's condition (D_4). [Note: in [1], we used a stronger condition.] Now we know from Proposition 2 above that Deng's condition (D_4) is equivalent to the transversality of the subspaces $T_{p(0,\varepsilon)}\mathcal{W}^u$ and W^{cs}, where $\mathcal{W}^u$ is the unstable manifold of the equilibrium $q(\lambda(\varepsilon),\varepsilon)$ and W^{cs} is the *centre stable* subspace. We need to show that $W^{cs} = T_{p(0,\varepsilon)}\mathcal{M}^{cs}$ and $T_{p(0,\varepsilon)}\mathcal{W}^u = = T_{p(0,\varepsilon)}\mathcal{M}^{cu}$. For the first one, it suffices to show that $T_{p(0,\varepsilon)}\mathcal{M}^{cs} \subset W^{cs}$ since both subspaces have the same dimension. So we just need to show that the norm of a solution starting in $T_{p(0,\varepsilon)}\mathcal{M}^{cs}$ is bounded by a constant times $e^{\delta_0 t/2}$ for $t \geq 0$ since by taking ε small enough we can choose ν so that $\varepsilon(\mu + O(\varepsilon)) < \delta_0/2 < \nu < \delta_0$. Using the notation of Theorem 1 in [4], we see that, to prove this, it suffices to show that if $(x^+(t,\zeta_+,\xi,\lambda,\varepsilon), y^+(t,\zeta_+,\xi,\lambda,\varepsilon))$ is a solution in $\mathcal{M}^{cs}(\xi_0)$, then its derivatives with respect to ξ and ζ_+ are bounded by a constant times $e^{\delta_0 t/2}$ for $t \geq 0$. However, this follows from [4] (Theorem 1) provided ε is sufficiently small. In fact the result for the ζ_+-derivative directly follows from [4] (Theorem 1), since $x_c(\varepsilon t,\xi,\lambda,\varepsilon)$ does not depend on ζ_+. As for the ξ-derivative, from [4] (Theorem 1) it follows that this does not grow faster than $\dfrac{\partial x_c^+}{\partial \xi}(\varepsilon t,\xi,\lambda,\varepsilon)$, which is the solution $X(t)$ of

$$\dot{X} = \varepsilon F_x(x(t))X, \qquad X(0) = \mathbf{I},$$

where $F(x) = f(x, v(x,\lambda,\varepsilon),\lambda,\varepsilon)$, $x(t) = x_c^+(\varepsilon t,\xi,\lambda,\varepsilon)$. For the second one, it suffices to show that $T_{p(0,\varepsilon)}\mathcal{W}^u \subset T_{p(0,\varepsilon)}\mathcal{M}^{cu}$ since both subspaces have the same dimension. However it follows from [4] that $T_{p(0,\varepsilon)}\mathcal{M}^{cu}$ consists of the initial values of solutions of the variational system which do not grow at too high an exponential rate as $t \to -\infty$. However, all the solutions beginning in $T_{p(0,\varepsilon)}\mathcal{W}^u$ tend to zero as $t \to -\infty$. So the inclusion follows. Thus by (d) in the Remark after Theorem 3 holds and the proof of the theorem is complete.

Example. Consider the system

$$\dot{x} = \varepsilon[f(x) + y],$$

$$\dot{y}_1 = y_2 + [\lambda + \sqrt{-G'(0)}\sin(x_2)]y_1,$$

$$\dot{y}_2 = -G(y_1) + \lambda y_2,$$

where x and $y = (y_1, y_2)$ are in $\mathbf{R}^2$, and f, G are suitably smooth. f is chosen so that 0 is an unstable focus for

$$\dot{x} = f(x).$$

G satisfies $G(0) = 0$, $G'(0) < 0$ so that $(0,0)$ is a saddle for

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = -G(y_1).$$

Also we assume there is a solution $y_0(t) = (p(t), \dot{p}(t))$ of this last equation such that

$$y_0(t) \to (0,0) \qquad \text{as} \qquad t \to \pm\infty$$

and such that $p(t) > 0$ for all t. For example, if

$$G(y_1) = -y_1 + 2y_1^3,$$

then $p(t) = \operatorname{sech}(t) > 0$.

Conditions (i) – (v) of the Introduction are satisfied with $v^{\pm}(x, \lambda, \varepsilon) = 0$. Now

$$\int_{-\infty}^{\infty} \psi^*(t) g_\lambda(\xi_0, y_0(t), 0, 0) dt = -2 \int_{-\infty}^{\infty} (\dot{p}(t))^2 dt < 0.$$

So (vi) in Theorem 2 holds and Theorem 4 implies that there is a function $\lambda(\varepsilon)$ with $\lambda(0) = 0$ such that for ε sufficiently small, system

$$\dot{x} = \varepsilon\big[f(x) + y\big],$$

$$\dot{y}_1 = y_2 + \big[\lambda(\varepsilon) + \sqrt{-G'(0)}\sin(x_2)\big] y_1,$$

$$\dot{y}_2 = -G(y_1) + \lambda(\varepsilon) y_2,$$

has a solution $(x(t,\varepsilon), y(t,\varepsilon)) \neq (0,0)$ satisfying $(x(t,\varepsilon), y(t,\varepsilon)) \to (0,0)$ as $t \to \pm\infty$. Next we find that the last two conditions in Theorem 4 read:

$$\int_{-\infty}^{\infty} y_0(t) dt = \int_{-\infty}^{\infty} \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix} dt = \begin{bmatrix} \int_{-\infty}^{\infty} p(t) dt \\ 0 \end{bmatrix} \neq 0$$

and

$$\int_{-\infty}^{\infty} \psi^*(t) g_x(0, y_0(t), 0, 0) dt = \int_{-\infty}^{\infty} \begin{bmatrix} \ddot{p}(t) & -\dot{p}(t) \end{bmatrix} \begin{bmatrix} 0 & \sqrt{-G'(0)} p(t) \\ 0 & 0 \end{bmatrix} dt =$$

$$= \int_{-\infty}^{\infty} \begin{bmatrix} 0 & \sqrt{-G'(0)} p(t) \ddot{p}(t) \end{bmatrix} dt = -\sqrt{-G'(0)} \begin{bmatrix} 0 & \int_{-\infty}^{\infty} \dot{p}(t)^2 dt \end{bmatrix} \neq 0.$$

So Theorem 4 applies and we deduce that our homoclinic orbit is in fact a Sil'nikov saddle-focus homoclinic orbit.

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