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AN INFINITE-DIMENSIONAL BORSUK – ULAM TYPE GENERALIZATION OF THE LERAY – SCHAUDER FIXED POINT THEOREM AND SOME APPLICATIONS

НЕСКІНЧЕННОВИМІРНЕ УЗАГАЛЬНЕННЯ ТИПУ БОРСУКА – УЛАМА ДЛЯ ТЕОРЕМИ ЛЕРЕЯ – ШАУДЕРА ПРО НЕРУХОМУ ТОЧКУ ТА ДЕЯКІ ЗАСТОСУВАННЯ

A generalization of the classical Leray – Schauder fixed point theorem, based on the infinite-dimensional Borsuk – Ulam type antipode construction, is proposed. A new nonstandard proof of the classical Leray – Schauder fixed point theorem and a study of the solution manifold to a nonlinear Hamilton – Jacobi type equation are presented.

Запропоновано узагальнення класичної теореми Лерея – Шаудера про нерухому точку, що ґрунтується на нескінченновимірній конструкції антиподів типу Борсука – Улама. Наведено нестандартне доведення класичної теореми Лерея – Шаудера про нерухому точку та досліджено многовид розв'язків нелінійного рівняння типу Гамільтона – Якобі.

1. Introduction. The fixed point theorems are of very importance for many applications [1 – 3] in modern theories of differential equations and mathematical physics. Especially, the classical Leray – Schauder theorem and its diverse modifications [1, 4 – 9] in infinite-dimensional both Banach and Frechet spaces, being nontrivial generalizations of the well known finite-dimensional Brouwer fixed point theorem, are of special interest [4 – 7, 10, 11] in modern nonlinear mathematical analysis. In particular, there exist many problems in theories of differential and operator equations [1, 4, 9 – 12], which can be uniformly formulated as the following equation:

$$\hat{a}x = f(x), \tag{1}$$

where $x \in E_1$, $\hat{a}\colon E_1 \to E_2$ is a closed surgective linear operator from Banach space E_1 onto Banach space E_2, defined on a domain $D(\hat{a}) \subset E_1$ (which can be not dense) and $f\colon E_1 \to E_2$ is a nonlinear continuous mapping, whose domain $D(f) = D(a) \cap S_r(0)$. (Here $S_r(0) \subset E_1$ is the sphere in E_1 of radius $r > 0$, centered at zero.)

The following problem, important for many applications, is posed.

Problem. *Under what conditions on the linear operator $\hat{a}\colon E_1 \to E_2$ and the nonlinear continuous mapping $f\colon E_1 \to E_2$ does equation* (1) *possess a solution $x \in D(f)$, and what is the topological dimension $\dim \mathcal{N}(\hat{a}, f)$ of the solution set $\mathcal{N}(\hat{a}, f) \subset D(f)$?*

Recall also that the topological dimension of a closed compact set $A \subset X$ (X is a topological space) is defined as the number $\dim A := \inf\big\{k \in \mathbb{Z}_+\colon$ there holds the condition $\bigcap_{j=\overline{1,k+2}} U_{\alpha_j} = \varnothing$ for any subsets $U_{\alpha_j} \in \{U_{\alpha_\beta}\}$ of all specially chosen subcoverings $\{U_{\alpha_\beta}\}$ of any covering $\{U_\alpha\}$ of the set $A\big\}$.

a) In the case $:= id$ and $E_1 := E_2$ equation (1) reduces to the standard fixed point problem $f(x) = x$, $x \in S_r(0)$, studied before [1, 5, 8, 13, 14] by Banach, Leray, Schauder, Browder and many other mathematicians.

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b) In the odd case when $f(-x) = -f(x)$ for any $x \in D(f)$ equation (1) reduces to an infinite-dimensional generalization of the classical Borsuk – Ulam theorem on the sphere $S_r(0) \subset E_1$, which was recently stated by B. Gelman [11, 15].

Below we will prove a theorem, giving rise to a suitable solution to the Problem above, and give some its application to studying the solution set to a nonlinear Hamilton – Jacobi type equation.

2. Main theorem. We will assume further that the following natural conditions are fulfilled:

i) domain $D(f) = D(a) \cap S_r(0)$;

ii) the mapping $f\colon E_1 \to E_2$ is $\hat{a}$-compact that is, it is continuous and for any bounded set $A_2 \subset E_2$, any bounded $A_1 \subset D(f)$ the set $f(A_1 \cap \hat{a}^{-1}(A_2))$ is relatively compact in E_2 (the empty set $\varnothing$ is considered, by definition, compact);

iii) there exists a bounded constant $k_f > 0$, such that

$$\sup_{x \in S_r(0)} \frac{1}{r} \|f(x)\|_2 := k_f^{-1};$$

iv) the inequality

$$k(\hat{a}) < k_f$$

holds, where, by definition,

$$k(\hat{a}) := \|\tilde{a}^{-1}\| = \sup_{y \in E_2} \frac{1}{\|y\|_2} \inf_{x \in D(\hat{a})} \big\{ \|x\|_1 \colon \hat{a}x = y \big\}, \tag{2}$$

and $\tilde{a} := \hat{a}|_{E_1/\ker \hat{a}}$ is an invertible susjective and continuous linear operator from the factor-space $E_1/\ker \hat{a}$ onto E_2.

Then the following main theorem [16 – 18] holds.

Theorem 1. *Let the dimension* $\dim \operatorname{Ker} \hat{a} \geq 1$ *and conditions* i) – iv) *hold. Then equation* (1) *possesses in* $D(f) \subset E_1$ *the nonempty solution set* $\mathcal{N}(\hat{a}, f)$, *whose topological dimension* $\dim \mathcal{N}(\hat{a}, f) \geq \dim \ker \hat{a} - 1$.

A proof of the theorem is based on the following lemmas.

Lemma 1. *For any constant* $k_s > k(\hat{a})$ *there exists a continuous odd selection* $s\colon E_2 \to E_1$ *for the mapping* $\tilde{a}^{-1}\colon E_2 \to E_1$, *satisfying the conditions*:

1) $\hat{a}s(y) = y$ *for any* $y \in E_2$;

2) $\|s(y)\|_1 \leq k_s \|y\|_2$, $y \in E_2$.

Proof. The lemma can be proved making use of the well known E. Michael theorem [19] on the selection for a linear surjective and continuous mapping, applied to the induced mapping $\tilde{a}\colon E_1/\ker \hat{a} \to E_2$. As the latter is invertible and continuous, there exists the bounded constant $k(\hat{a}) := \|\tilde{a}^{-1}\| < \infty$. The set-valued mapping $\tilde{a}^{-1}\colon E_2 \to \to E_1$ is lower semi-continuous with closed convex values. It is clear that $\tilde{a}^{-1}(-y) = = -\tilde{a}^{-1}(y)$ for any $y \in E_2$. Consider now, following [11, 15], another set-valaued mapping $\varphi\colon E_2 \to E_1$, such that $\varphi(y) = B_{r(y)}(0)$ for any $y \in E_2$, where $B_{r(y)}(0)$ is the closed ball of radius $r(y) = k(\hat{a})\|y\|_2 + 1$ in E_2. If to define a mapping $\varphi\colon E_2 \to E_1$ as $\tilde{\varphi}(y) := \tilde{a}^{-1}(y) \cap \varphi(y)$, one can see that $\tilde{\varphi}(-y) = -\tilde{\varphi}(y)$ for any $y \in E_2$. There exists a theorem proved by E. Michael [19], which says that any below semicontinuous set-valued mapping $\varphi\colon E_2 \to E_1$ of a paracompact space E_2 (in particular, of any metrized or Banach space E_2) into a Banach space E_1 with closed and convex values possesses a continuous selection. Moreover, by the theorem on equivariant selections [20] there exists an odd selection $s\colon E_2 \to E_1$, such that $s(y) \in \tilde{\varphi}(y)$ for each $y \in E_2$, whence $\hat{a}s(y) = y$. This mapping, in general, is nonlinear, if there does not exist the linear continuous projector from E_1 onto $\ker \hat{a} \subset E_1$. The selection $s\colon E_2 \to E_1$ allows also a more analytical construction. Really, since the set-valued mapping $\hat{a}^{-1}\colon E_2 \to E_1$ is defined on the whole Banach space E_2, one can write down that

$$\hat{a}^{-1}y = \bar{x}_y \oplus \operatorname{Ker} \hat{a} \tag{3}$$

for any $y \in E_2$ and some specified elements $\bar{x}_y \in E_1 \setminus \ker \hat{a}$, labelled by elements $y \in$ $\in E_2$. If the composition (3) is already specified, we can define a selection $s\colon E_2 \to E_1$ as follows:

$$s(y) := \frac{1}{2}(\bar{x}_y - \bar{x}_{-y}) \oplus \frac{1}{2}(\bar{c}_y - \bar{c}_{-y}), \tag{4}$$

where the elements $\bar{c}_y \in \ker \hat{a}$, $y \in E_2$, are chosen arbitrary, but fixed. It is now easy to check that

$$s(-y) = -s(y)$$

and

$$\hat{a}\ s(y) = \hat{a}\left(\frac{1}{2}(\bar{x}_y - \bar{x}_{-y}) \oplus \frac{1}{2}(\bar{c}_y - \bar{c}_{-y})\right) =$$

$$= \frac{1}{2}\hat{a}\bar{x}_y - \frac{1}{2}\hat{a}\bar{x}_{-y} = \frac{1}{2}y - \frac{1}{2}(-y) = y$$

for all $y \in E_2$, thereby the mapping (4) satisfies the main conditions i) and ii) above. To state the continuity of the mapping (4), we will consider below expression (2) for the norm $\|\tilde{a}^{-1}\| = k(\hat{a})$ of the inverse mapping $\tilde{a}^{-1}\colon E_2 \to E_1$. We can easily write down the following inequality:

$$\left\|s(y)\right\|_1 = \left\|\frac{1}{2}(\bar{x}_y - \bar{x}_{-y}) \oplus \frac{1}{2}(\bar{c}_y - \bar{c}_{-y})\right\|_1 =$$

$$= \frac{1}{2}\left\|(\bar{x}_y \oplus \bar{c}_y) - (\bar{x}_{-y} \oplus \bar{c}_{-y})\right\|_1 \leq$$

$$\leq \frac{1}{2}(\left\|(\bar{x}_y \oplus \bar{c}_y)\right\|_1 + \left\|(\bar{x}_{-y} \oplus \bar{c}_{-y})\right\|_1) \leq$$

$$\leq \frac{1}{2}k_s \left\|y\right\|_2 + \frac{1}{2}k_s \left\|y\right\|_2 = k_s \left\|y\right\|_2,$$

giving rise to the continuity of mapping (4), where we have assumed that there exists such a constant $k_s > 0$, that

$$\left\|(\bar{x}_y \oplus \bar{c}_y)\right\|_1 \leq k_s \left\|y\right\|_2,$$

for all $y \in E_2$. This constant $k_s > k(\hat{a})$ strongly depends on the choice of elements $\bar{c}_y \in \ker \hat{a}$, $y \in E_2$, what one can observe from definition (2). Really, owing to the definition of infimum, for any $\varepsilon > 0$ and all $y \in E_2$ there exist elements $\bar{x}_y^{(\varepsilon)} \oplus \bar{c}_y^{(\varepsilon)} \in E_1$, such that

$$k(\hat{a}) \leq \frac{\left\|\bar{x}_y^{(\varepsilon)} \oplus \bar{c}_y^{(\varepsilon)}\right\|_1}{\left\|y\right\|_2} < k(\hat{a}) + \varepsilon := k_s. \tag{5}$$

Now making now use of formula (4), we can construct a selection $s_\varepsilon \colon E_2 \to E_1$ as follows:

$$s_\varepsilon(y) := \frac{1}{2}\big(\bar{x}_y^{(\varepsilon)} - \bar{x}_{-y}^{(\varepsilon)}\big) \oplus \frac{1}{2}\big(\bar{c}_y^{(\varepsilon)} - \bar{c}_{-y}^{(\varepsilon)}\big),$$

satisfying, owing to inequalities (5), the searched for conditions i) and ii):

$$\hat{a}s_\varepsilon(y) = y, \qquad \|s_\varepsilon(y)\|_1 \le k_s\,\|y\|_2$$

for all $y \in E_2$ and $k_s := k(\hat{a}) + \varepsilon$, $\varepsilon > 0$. Moreover, the mapping $s_\varepsilon \colon E_2 \to E_1$ is, by construction, continuous [15, 19, 20] and odd that finishes the proof.

Lemma 2. *Let a mapping $f_r \colon E_1 \to E_2$ be defined as*

$$f_r(x) := \begin{cases} \dfrac{\|x\|_1}{r} f\left(\dfrac{rx}{\|x\|_1}\right), & \textit{if} \quad x \neq 0; \\ 0, & \textit{if} \quad x = 0. \end{cases}$$

Then the equation

$$t(t^2 + \varepsilon^2)^{-1} f_r(ts(y) + t^2\bar{c}) = y, \tag{6}$$

where $\bar{c} \in \ker \hat{a}$, is solvable for any $\varepsilon \neq 0$ with respect to $(t, y) \in [-1, 1] \times S_1(0)$, such that $\|y\|^2 + t^2 = 1$. Moreover, the corresponding solution $(t_\varepsilon, y_\varepsilon)$ satisfies the limiting condition: $\liminf_{\varepsilon \to 0} |t_\varepsilon| = \alpha_0 \in (0, 1)$.

Proof. Proof is based on a Borsuk – Ulam type theorem of [11, 15] and some standard functional-analytic resasonings.

As a consequence of Lemmas 1 and 2 one deduces the proof of the main Theorem 1. In particular, the solution set $\mathcal{N}(\hat{a}, f)$ depends on the kernel $\ker \hat{a}$, and whose topological dimension $\dim \mathcal{N}(\hat{a}, f) \ge \dim \ker \hat{a} - 1$, following from the form of equation (6).

3. Applications. ***3.1. The classical Leray – Schauder fixed point theorem.*** The following classical Leray – Schauder fixed point theorem holds.

Theorem 2. *Let a compact mapping $\bar{f} \colon B \to B$ in a Banach space B be such that there exists a cloesed convex and bounded set $M \subset M$, for which $\bar{f}(M) \subseteq M$. Then there exists a fixed point $\bar{x} \in M$, such that $\bar{f}(\bar{x}) = \bar{x}$.*

Proof. A proof of the theorem can be obtained from the main Theorem 1. Really, put, by definition, $E_1 := B \oplus \mathbb{R}$ and $E_2 := B$. For any point $x \in B$ one can define the set-valued projection mapping (metric projection)

$$B \ni x \to P_{\bar{f}}(x) \subset M_{\bar{f}} \subset B, \tag{7}$$

where $M_{\bar{f}} := \operatorname{conv} \bar{f}(M) \subseteq M$ and

$$\inf_{y \in M_{\bar{f}}} \|x - y\| := \big\|x - P_{\bar{f}}(x)\big\|. \tag{8}$$

The constructed mapping (7) is well-defined [1, 21, 22] and below semi-continuous, owing to the compactness, closedness and convexity of the set $M_{\bar{f}} \subset B$. Take now the unite sphere $S_1(0) \subset E_1$, a compact surjective linear operator $\hat{b} \colon B \to B$, whose $\dim \ker \hat{b} \ge 1$, a continuous selection $\bar{P}_{\bar{f}} \colon B \to M_{\bar{f}}$ for the set-valued mapping (7), existing owing to the aboave mentioned E. Michael theorem [19], and construct a mapping $f \colon S_1(0) \to E_2$, where, by definition, for any $(x, \tau) \in S_1(0)$ and $\lambda \in \mathbb{R}$

$$f(x, \tau) := \bar{f}(\bar{P}_{\bar{f}}(x)) - \bar{P}_{\bar{f}}(x) + \lambda \hat{b} x. \tag{9}$$

If to define now a related with (8) mapping $\hat{a}\colon E_1 \to E_2$ as

$$\hat{a}(x,\tau) := \lambda\hat{b}x$$

for any $(x,\tau) \in E_1$, the fixed point problem for the mapping $\bar{f}\colon B \to B$ becomes equivalent to the following equation:

$$\hat{a}\ (x,\tau) = f(x,\tau) \iff \bar{f}(\bar{P}_{\bar{f}}(x)) = \bar{P}_{\bar{f}}(x).$$

The following simple lemma holds.

Lemma 3. *The mapping* (9) *is continuoes, $\hat{a}$-compact and satisfying for some nonzero value $\lambda \in \mathbb{R}$ the condition $k_f > k(\hat{a})$.*

Thereby, based on main Theorem 1 *there exists a point $(x_\tau,\tau) \in S_1(0) \subset E_1$, such that*

$$\bar{f}(\bar{P}_{\bar{f}}(x_\tau)) = \bar{P}_{\bar{f}}(x_\tau) \iff \bar{f}(\bar{x}) = \bar{x},$$

where $x = \bar{P}_{\bar{f}}(x_\tau) \in M_{\bar{f}}$, prooving the theorem.

***Remark* 1.** There exists [16 – 18] another nonstandard proof of the classical Leray – Schauder fixed point theorem, based on the measure theory and a Krein – Milman type theorem about a representation of convex compact sets by means of their extreme points.

3.2. A Hamilton – Jacobi type nonlinear equation in $\mathbb{R}^n$. There is considered the Cauchy problem to the following nonlinear Hamilton – Jacobi type equation in $\mathbb{R}^n$:

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(|u_x|^2 + \beta u|x|^2\right) = 0, \tag{10}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, $\beta \in \mathbb{R}$ is a constant parameter and

$$u|_{t=+0} = v$$

for $v\colon \mathbb{R}^n \to \mathbb{R}$ being a given mapping. The corresponding classical and generalized solutions to equation (10), when $v \in BSC(\mathbb{R}^n)$ is a below semi-continuous function, can be represented [2, 23 – 27] for $t \in \mathbb{R}_+$ as

$$u(x,t) = \inf_{y\in\mathbb{R}^n}\left\{v(y) - \frac{1}{2}\langle y,\dot{\alpha}\rangle|_{\tau=0} - \frac{\beta}{16}\left(|x|^4 - |y|^4\right) + \frac{1}{2}\langle x,\dot{\alpha}\rangle|_{\tau=t}\right\},$$

where we denoted "$\cdot$" $:= \dfrac{d}{d\tau}$, "$\cdot\cdot$" $:= \dfrac{d^2}{d\tau^2}$ and $\alpha\colon \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ is the vector-valued solution to the following set of nonlinear ordinary differential equations:

$$-\ddot{\alpha} = \beta\left(u\alpha + \frac{1}{2}|\alpha|^2\dot{\alpha}\right), \tag{11}$$

$$\dot{u} = \frac{1}{2}\left(|\dot{\alpha}|^2 - \beta u|\alpha|^2\right)$$

under the boundary conditions

$$\alpha|_{\tau=+0} = y, \qquad \alpha|_{\tau=t} = x, \tag{12}$$

$$u|_{\tau=+0} = v(y)$$

for any $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. The problems like (11) are of very importance in the mathematical theory of nonlinear oscillations [3] and were before extensively studied in [2, 3, 28] by A. M. Samoilenko and his co-workers.

To show that problem (11) and (12) is solvable, we rewrite it in the following canonical form:

$$\hat{a}(\alpha, u) = f_\beta(\alpha, u), \tag{13}$$

where $(\alpha, u) \in H(0, t; \mathbb{R}^n) \oplus H(0, t; \mathbb{R}) := E_1$, $D(\hat{a}) = H^2(0, t; \mathbb{R}^n) \oplus H^1(0, t; \mathbb{R})$, $E_2 := H(0, t; \mathbb{R}^n) \oplus H(0, t; \mathbb{R})$ and

$$\hat{a}(\alpha, u) := (-\ddot{\alpha}, \dot{u}),$$
$$f_\beta(\alpha, u) := \left(\beta\left(u\alpha + \frac{1}{2}|\alpha|^2\dot{\alpha}\right), \frac{1}{2}\left(|\dot{\alpha}|^2 - \beta u|\alpha|^2\right)\right). \tag{14}$$

The corresponding solution set $\mathcal{N}(\hat{a}, f_\beta) \in D(\hat{a})$ to problem (13) can be studied making use of the main Theorem 1. Namely, the following theorem holds.

Theorem 3. *Let a parameter $\beta \in \mathbb{R}$ be chosen in such a way that $k_{f_\beta} > k(\hat{a})$, where*

$$k_{f_\beta}^{-1} := \sup_{\|(\alpha,u)\|_1 = r} \frac{1}{r}\|f_\beta(\alpha, u)\|_2,$$

$$k(\hat{a}) := \|\tilde{a}^{-1}\| = \sup_{\|w\|_2 = 1} \inf_{(\alpha,u)\in D(\hat{a})} \Big\{\|(\alpha, u)\|_1 \colon (-\ddot{\alpha}, \dot{u}) = w\Big\},$$

for some $r > 0$. Then there exists a nonempty solution set $\mathcal{N}(\hat{a}, f_\beta) \in D(\hat{a})$ to equation (14), *whose topological dimension* $\dim \mathcal{N}(\hat{a}, f_\beta) \geq 2$.

Thereby, the Cauchy problem for problem (11) and (12) is solvable and the space of the corresponding solutions is not trivial (in general, it is nonunique!). Based now on Theorem 3 the searched for solvability of the Cauchy problem to equation (10) is completely stated.

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