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ON INFINITE-RANK SINGULAR PERTURBATIONS OF THE SCHRÖDINGER OPERATOR

ПРО СИНГУЛЯРНІ ЗБУРЕННЯ ОПЕРАТОРА ШРЕДІНГЕРА НЕСКІНЧЕННОГО РАНГУ

Schrödinger operators with infinite-rank singular potentials $V = \sum_{i,j=1}^{\infty} b_{ij}\langle\psi_j,\cdot\rangle\psi_i$ are studied under the condition that singular elements ψ_j are $\xi_j(t)$-invariant with respect to scaling transformations in $\mathbb{R}^3$.

Вивчається оператор Шредінгера з сингулярними потенціалами нескінченного рангу $V = \sum_{i,j=1}^{\infty} b_{ij}\langle\psi_j,\cdot\rangle\psi_i$ за умови, що сингулярні елементи ψ_j є $\xi_j(t)$-інваріантними відносно масштабних перетворень в $\mathbb{R}^3$.

1. Introduction. Let $-\Delta$, $\mathcal{D}(\Delta) = W_2^2(\mathbb{R}^3)$ be the Schrödinger operator in $L_2(\mathbb{R}^3)$ and let $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$ be the collection of unitary operators $U_t f(x) = t^{3/2} f(tx))$ in $L_2(\mathbb{R}^3)$ (so-called scaling transformations).

It is well known [1, 2] that $-\Delta$ is t^{-2}-homogeneous with respect to $\mathfrak{U}$ in the sense that

$$U_t\Delta u = t^{-2}\Delta U_t u \quad \forall t > 0, \quad u \in W_2^2(\mathbb{R}^3). \tag{1.1}$$

In other words, the set $\mathfrak{U}$ determines the structure of a symmetry and the property of $-\Delta$ to be t^{-2}-homogeneous with respect to $\mathfrak{U}$ means that $-\Delta$ possesses a symmetry with respect to $\mathfrak{U}$.

Consider the heuristic expression

$$-\Delta + \sum_{i,j=1}^{\infty} b_{ij}\langle\psi_j,\cdot\rangle\psi_i, \qquad \psi_j \in W_2^{-2}(\mathbb{R}^3), \quad b_{ij} = \overline{b_{ji}} \in \mathbb{C}. \tag{1.2}$$

We will say that $\psi \in W_2^{-2}(\mathbb{R}^3)$ is $\xi(t)$-*invariant with respect to* $\mathfrak{U}$ if there exists a real function $\xi(t)$ such that

$$\mathbb{U}_t\psi = \xi(t)\psi \quad \forall t > 0, \tag{1.3}$$

where $\mathbb{U}_t$ is the continuation of U_t onto $W_2^{-2}(\mathbb{R}^3)$ (see Section 2 for details).

The aim of the paper is to study self-adjoint operator realizations of (1.2) assuming that all ψ_j are $\xi_j(t)$-invariant with respect to the set of scaling transformations $\mathfrak{U}$.

It is well known, see e.g. [1 – 4] that the Schrödinger operators perturbed by potentials homogeneous with respect to a certain set of unitary operators play an important role in applications to quantum mechanics. To a certain extent this generates a steady interests to the study of self-adjoint extensions with various properties of symmetry [5 – 11]. In particular, an abstract framework to study finite rank singular perturbations with symmetries for an arbitrary nonnegative operator was developed in [6].

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In the present paper we generalize some results of [6] to the case of infinite rank perturbations of the Schrödinger operator in $L_2(\mathbb{R}^3)$. In particular, the description of all t^{-2}-homogeneous extensions of the symmetric operator $-\Delta_{\rm sym}$ is obtained. Another interesting property studied here is the possibility to get the Friedrichs and the Krein – von Neumann extension of $-\Delta_{\rm sym}$ as solutions of a system of equations involving the functions t^{-2} and $\xi(t)$.

Throughout the paper $\mathcal{D}(A)$, $\mathcal{R}(A)$, and $\ker A$ denote the domain, the range, and the null-space of a linear operator A, respectively, while $A \upharpoonright \mathcal{D}$ stands for the restriction of A to the set $\mathcal{D}$.

2. Auxiliary results. ***2.1. Preliminaries.*** Since the Sobolev space $W_2^{-2}(\mathbb{R}^3)$ coincides with the completion of $L_2(\mathbb{R}^3)$ with respect to the norm

$$\|f\|_{W_2^{-2}(\mathbb{R}^3)} = \left\|(-\Delta + I)^{-1} f\right\| \quad \forall f \in L_2(\mathbb{R}^3), \tag{2.1}$$

the resolvent operator $(-\Delta+I)^{-1}$ can be continuously extended to an isometric mapping $(-\Delta + I)^{-1}$ from $W_2^{-2}(\mathbb{R}^3)$ onto $L_2(\mathbb{R}^3)$ (we preserve the same notation for the extension). Hence, the relation

$$\langle \psi, u\rangle = \big((-\Delta + I)u, (-\Delta + I)^{-1}\psi\big), \quad u \in W_2^{-2}(\mathbb{R}^3), \tag{2.2}$$

enables one to identify the elements $\psi \in W_2^{-2}(\mathbb{R}^3)$ as linear functionals on $W_2^{2}(\mathbb{R}^3)$.

It follows from (1.1), (2.1) that the operators $U_t \in \mathfrak{U}$ can be continuously extended to bounded operators $\mathbb{U}_t$ in $W_2^{-2}(\mathbb{R}^3)$ and for any $\psi \in W_2^{-2}(\mathbb{R}^3)$

$$\langle \mathbb{U}_t\psi, u\rangle = \langle \psi, U_t^* u\rangle = \langle \psi, U_{1/t} u\rangle. \tag{2.3}$$

Since the elements U_t of $\mathfrak{U}$ have the additional multiplicative property $U_{t_1}U_{t_2} = U_{t_2}U_{t_1} = U_{t_1t_2}$, relation (2.3) means that this relation holds for $\mathbb{U}_t$ also. But then, equality (1.3) gives $\xi(t_1)\xi(t_2) = \xi(t_1t_2)$ $(t_i > 0)$ that is possible only if $\xi(t) = 0$ or $\xi(t) = t^{-\alpha}$ $(\alpha \in \mathbb{R})$ [12] (Chap. IV). Hence, if an element $\psi \in W_2^{-2}(\mathbb{R}^3)$ is $\xi(t)$-invariant with respect to $\mathfrak{U}$, then $\xi(t) = t^{-\alpha}$ $(\alpha \in \mathbb{R})$ (the case $\xi(t) = 0$ is impossible because $\mathbb{U}_t$ has inverse).

***2.2. Operator realizations of* (2.1) *in* $L_2(\mathbb{R}^3)$.** Let us consider (1.2) assuming that all elements ψ_j are $t^{-\alpha}$-invariant with respect to $\mathfrak{U}$. This means that all elements of the linear span $\mathcal{X}$ of $\{\psi_j\}_{j=1}^{\infty}$ also satisfy (1.3) with $\xi(t) = t^{-\alpha}$. Obviously, the same is true for the closure $\overline{\mathcal{X}}$ of $\mathcal{X}$ in $W_2^{-2}(\mathbb{R}^3)$. Hence, if $\psi \in \overline{\mathcal{X}}$, then $\mathbb{U}_t\psi = t^{-\alpha}\psi$. This implies $\psi \in W_2^{-2}(\mathbb{R}^3) \setminus L_2(\mathbb{R}^3)$ (since the operator $U_t = \mathbb{U}_t \upharpoonright L_2(\mathbb{R}^3)$ is unitary in $L_2(\mathbb{R}^3)$. Thus $\overline{\mathcal{X}} \cap L_2(\mathbb{R}^3) = \{0\}$.

In that case, the perturbation $V = \sum_{i,j=1}^{n} b_{ij}\langle\psi_j, \cdot\rangle\psi_i$ turns out to be singular and the formula

$$\begin{gathered} -\Delta_{\rm sym} = -\Delta \upharpoonright \mathcal{D}(-\Delta_{\rm sym}), \\ \mathcal{D}(-\Delta_{\rm sym}) = \left\{u \in W_2^{-2}(\mathbb{R}^3)\colon\ \langle\psi_j, u\rangle = 0, \ \ j \in \mathbb{N}\right\} \end{gathered} \tag{2.4}$$

determines a closed densely defined symmetric operator in $L_2(\mathbb{R}^3)$.

Following [1] a self-adjoint operator realization $-\widetilde{\Delta}$ of (1.2) in $L_2(\mathbb{R}^3)$ are defined by

$$-\widetilde{\Delta} = -\Delta_R \restriction \mathcal{D}(-\widetilde{\Delta}), \qquad \mathcal{D}(-\widetilde{\Delta}) = \Big\{ f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}})\colon\ -\Delta_R f \in L_2(\mathbb{R}^3) \Big\}, \tag{2.5}$$

where

$$-\Delta_R = -\Delta + \sum_{i,j=1}^{\infty} b_{ij} \langle \psi_j^{\mathrm{ex}}, \cdot \rangle \psi_i \tag{2.6}$$

is seen as a regularization of (1.2) defined on $\mathcal{D}(-\Delta^*_{\mathrm{sym}})$. Here $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$ denote extensions of linear functionals $\langle \psi_j, \cdot \rangle$ onto $\mathcal{D}(-\Delta^*_{\mathrm{sym}})$.

In what follows, the elements $\{\psi_j\}_{j=1}^{\infty}$ in (1.2) are supposed to be a Riesz basis of the subspace $\overline{\mathcal{X}} \subset W_2^{-2}(\mathbb{R}^3)$. Then the vectors $h_j = (-\Delta + I)^{-1}\psi_j$, $j \in \mathbb{N}$, form a Riesz basis of the defect subspace $\mathcal{H} = \ker(-\Delta^*_{\mathrm{sym}} + I) \subset L_2(\mathbb{R}^3)$ of the symmetric operator $-\Delta_{\mathrm{sym}}$ (see (2.2) and (2.4)).

Let $\{e_j\}_1^{\infty}$ be the canonical basis of the Hilbert space l^2 (i.e., $e_j = (\ldots, 0, 1, 0, \ldots)$, where 1 occurs on the j th place only). Putting $\Psi e_j := \psi_j$, $j \in \mathbb{N}$, we define an injective linear mapping $\Psi\colon\ l^2 \to W_2^{-2}(\mathbb{R}^3)$ such that $\mathcal{R}(\Psi) = \mathcal{X}$.

Let $\Psi^*\colon\ W_2^2(\mathbb{R}^3) \to \mathbb{C}^n$ be the adjoint operator of Ψ (i.e., $\langle u, \Psi d \rangle = (\Psi^* u, d)_{l^2}$ $\forall u \in W_2^2(\mathbb{R}^3)$ $\forall d \in l^2$). It is easy to see that

$$\Psi^* u = \big(\langle \psi_1, u \rangle, \ldots, \langle \psi_j, u \rangle, \ldots \big) \quad \forall u \in W_2^2(\mathbb{R}^3). \tag{2.7}$$

It follows from (2.7) that the extended functionals $\langle \psi_j^{\mathrm{ex}}, \cdot \rangle$ in (2.6) are completely defined by an extension Ψ_R^* of Ψ^* onto $\mathcal{D}(-\Delta^*_{\mathrm{sym}})$, i.e.,

$$\Psi_R^* f = \big(\langle \psi_1^{\mathrm{ex}}, f \rangle, \ldots, \langle \psi_j^{\mathrm{ex}}, f \rangle, \ldots \big) \quad \forall f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}}). \tag{2.8}$$

Since $\mathcal{D}(-\Delta^*_{\mathrm{sym}}) = W_2^2(\mathbb{R}^3) \dot{+} \mathcal{H}$, where $\mathcal{H} = \ker(-\Delta^*_{\mathrm{sym}} + I)$ the formula (2.8) can be rewritten as

$$\Psi_R^* f = \Psi_R^* \left(u + \sum_{k=1}^{\infty} d_k h_k \right) = \Psi^* u + R d \quad \forall f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}}), \tag{2.9}$$

where $u \in W_2^2(\mathbb{R}^3)$, $d = (d_1, d_2, \ldots) \in l_2$, and R is an arbitrary bounded operator acting in l^2.

Using the definition of Ψ and Ψ_R^*, the regularization (2.6) takes the form

$$-\Delta_R = -\Delta + \Psi B \Psi_R^*, \tag{2.10}$$

where the self-adjoint operator B is defined in l^2 by the infinite-dimensional Hermitian matrix $\mathbf{B} = \|b_{ij}\|_{i,j=1}^{\infty}$.

2.3. Description in terms of boundary triplets. The formulas (2.5) and (2.10) do not provide an explicit description of operator realizations $-\widetilde{\Delta}$ of (1.2) through the parameters b_{ij} of the singular perturbation V. To get the required description the method of boundary triplets is now incorporated.

Definition 2.1 [13]. *Let A_{sym} be a closed densely defined symmetric operator in a Hilbert space $\mathfrak{H}$. A triplet (N, Γ_0, Γ_1), where N is an auxiliary Hilbert space and Γ_0, Γ_1 are linear mappings of $\mathcal{D}(A^*_{\mathrm{sym}})$ into N, is called a boundary triplet of A^*_{sym} if $(A^*_{\mathrm{sym}} f, g) - (f, A^*_{\mathrm{sym}} g) = (\Gamma_1 f, \Gamma_0 g)_N - (\Gamma_0 f, \Gamma_1 g)_N$ for all $f, g \in \mathcal{D}(A^*_{\mathrm{sym}})$ and the mapping $(\Gamma_0, \Gamma_1)\colon\ \mathcal{D}(A^*_{\mathrm{sym}}) \to N \oplus N$ is surjective.*

The next two results (Lemma 2.1 and Theorem 2.3) are some 'folk-lore' of the extension theory (see, e.g., [14–16]). Basically their proofs are the same as in [14], where the case of finite defect numbers has been considered.

Lemma 2.1. *Let R in* (2.9) *be a bounded self-adjoint operator in l^2. Then the triplet $(l^2, \Gamma_0, \Gamma_1)$, where the linear operators $\Gamma_i\colon\ \mathcal{D}(-\Delta^*_{\rm sym}) \to l^2$ are defined by the formulas*

$$\Gamma_0 f = \Psi^*_R f, \qquad \Gamma_1 f = -\Psi^{-1}(-\Delta + I)h, \tag{2.11}$$

(*where $f = u + h,\ u \in W^2_2(\mathbb{R}^3),\ h \in \mathcal{H}$*) *is a boundary triplet of $-\Delta^*_{\rm sym}$.*

Theorem 2.1. *The operator realization $-\widetilde{\Delta}$ of* (1.2) *defined by* (2.5) *and* (2.10) *is a self-adjoint extension of $-\Delta_{\rm sym}$ which coincides with the operator*

$$-\Delta_B = -\Delta^*_{\rm sym} \upharpoonright \mathcal{D}(\Delta_B), \qquad \mathcal{D}(\Delta_B) = \big\{ f \in \mathcal{D}(\Delta^*_{\rm sym})\colon\ B\Gamma_0 f = \Gamma_1 f \big\}, \tag{2.12}$$

where Γ_i are defined by (2.11) *and a self-adjoint operator B is defined in l^2 by the Hermitian matrix $\mathbf{B} = \|b_{ij}\|^{\infty}_{i,j=1}$.*

3. t^{α}-Invariant singular perturbations of $-\Delta$. ***3.1. Description of all t^{α}-invariant elements.*** An additional study of $\mathbb{U}_t$ allows one to restrict the variation of the parameter α for $t^{-\alpha}$-invariant elements.

Theorem 3.1 [6]. *$t^{-\alpha}$-Invariant elements $\psi \in W^{-2}_2(\mathbb{R}^3)$ with respect to scaling transformations exist if and only if $0 < \alpha < 2$.*

Proof. For the convenience of the reader we briefly outline the principal stages of the proof. Consider a family of self-adjoint operators on $L_2(\mathbb{R}^3)$

$$G_t = (-t^{-2}\Delta + I)(-\Delta + I)^{-1}, \quad t > 0. \tag{3.1}$$

It follows from (1.1), (2.2), and (2.3) that for all $u \in W^2_2(\mathbb{R}^3)$

$$\langle \mathbb{U}_t\psi, u\rangle = \big((-\Delta + I)U_{1/t}u, h\big) = \big(U_{1/t}(-t^{-2}\Delta + I)u, h\big) =$$

$$= \big((-t^{-2}\Delta + I)u, U_t h\big) = \big(G_t(-\Delta + I)u, U_t h\big) = \big((-\Delta + I)u, G_t U_t h\big), \tag{3.2}$$

where $h = (-\Delta + I)^{-1}\psi$. On the other hand, if ψ is $t^{-\alpha}$-invariant, then

$$\langle \mathbb{U}_t\psi, u\rangle = t^{-\alpha}\langle \psi, u\rangle = \big((-\Delta + I)u, t^{-\alpha}h\big).$$

Combining the obtained relation with (2.3) one gets that an element ψ is $t^{-\alpha}$-invariant with respect to scaling transformations if and only if

$$G_t U_t h = t^{-\alpha}h, \qquad t > 0, \quad h = (\mathbb{A}_0 + I)^{-1}\psi. \tag{3.3}$$

The formula for G_t in (3.1) with an evident reasoning leads to the estimates

$$\alpha(t)\|h\| = \alpha(t)\|U_t h\| < \|G_t U_t h\| < \beta(t)\|U_t h\| = \beta(t)\|h\|,$$

where $\alpha(t) = \min\{1, t^{-2}\}$ and $\beta(t) = \max\{1, t^{-2}\}$. Therefore $\alpha(t) < t^{-\alpha} < \beta(t)$ for all $t > 0$. This estimation can be satisfied for $0 < \alpha < 2$ only.

To complete the proof it suffices to construct $t^{-\alpha}$-invariant elements ψ for $0 < \alpha < 2$.

Fix $m(w) \in L_2(S^2)$, where $L_2(S^2)$ is the Hilbert space of square-integrable functions on the unit sphere S^2 in $\mathbb{R}^3$, and determine the functional $\psi(m,\alpha) \in W_2^{-2}(\mathbb{R}^3)$ by the formula

$$\langle \psi(m,\alpha), u\rangle = \int\limits_{\mathbb{R}^3} \frac{\overline{m(w)}}{|y|^{3/2-\alpha}(|y|^2+1)}\big(|y|^2+1\big)\widehat{u}(y)dy \quad \big(y = |y|w \in \mathbb{R}^3\big), \tag{3.4}$$

where $\widehat{u}(y) = \dfrac{1}{(2\pi)^{3/2}} \displaystyle\int_{\mathbb{R}^3} e^{ix\cdot y}u(x)dx$ is the Fourier transformation of $u(\cdot) \in W_2^2(\mathbb{R}^3)$.

It is easy to verify that

$$(\widehat{U_{1/t}u})(y) = \frac{1}{(2\pi t)^{3/2}}\int\limits_{\mathbb{R}^3} e^{iy\cdot x}u(x/t)dx = U_t\widehat{u}(y) = t^{3/2}\widehat{u}(ty). \tag{3.5}$$

Using (3.4) and (3.5), one obtains $\langle \psi(m,\alpha), U_{1/t}u\rangle = t^{-\alpha}\langle \psi(m,\alpha), u\rangle$ for all $u \in \in W_2^2(\mathbb{R}^3)$. By (1.3) and (2.3) this means that $\psi(m,\alpha)$ is $t^{-\alpha}$-invariant with respect to $\mathfrak{U}$.

Theorem 3.1 is proved.

The next statement describes all $t^{-\alpha}$-invariant elements for a fixed $\alpha \in (0,2)$.

Proposition 3.1. *An element $\psi \in W_2^{-2}(\mathbb{R}^3)$ is $t^{-\alpha}$-invariant with respect to scaling transformations if and only if $\psi = \psi(m,\alpha)$ where $\psi(m,\alpha)$ is defined by* (3.4).

Proof. Let $\psi \in W_2^{-2}(\mathbb{R}^3)$ be $t^{-\alpha}$-invariant with respect to $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$. This means that (3.3) holds for $h = (\mathbb{A}_0 + I)^{-1}\psi$. Using (3.5) one can rewrite (3.3) as

$$\frac{t^{-2}|y|^2+1}{|y|^2+1}t^{-3/2}\widehat{h}\left(\frac{y}{t}\right) = t^{-\alpha}\widehat{h}(y), \quad t > 0, \tag{3.6}$$

where the equality is understood in the sense of $L_2(\mathbb{R}^3)$. Setting $t = |y|$, $(w = y/|y|)$ one derives that (3.6) holds if and only if

$$\widehat{h}(y) = \frac{m(w)}{|y|^{3/2-\alpha}(|y|^2+1)}, \qquad m(w) = 2\widehat{h}(w), \tag{3.7}$$

where $m(w) \in L_2(S^2)$ (because $\widehat{h}(w) \in L_2(\mathbb{R}^3)$). Combining (3.7) with (2.2) and (3.4) one concludes that $\psi = \psi(m,\alpha)$.

Proposition 3.1 is proved.

***Remark* 3.1.** Proposition 3.1 generalizes Proposition 3.1 in [9] where the case $\alpha = 3/2$ was considered.

3.2. t^{-2}-Homogeneous extensions of $-\Delta_{\mathrm{sym}}$ transversal to $-\Delta$. Denote $-\Delta_R = -\Delta^*_{\mathrm{sym}} \upharpoonright \ker\Gamma_0$, where Γ_0 is defined by (2.11). Since $(l^2, \Gamma_0, \Gamma_1)$ is a boundary triplet of $-\Delta^*_{\mathrm{sym}}$ and the initial operator $-\Delta$ coincides with $-\Delta^*_{\mathrm{sym}} \upharpoonright \ker\Gamma_1$, one concludes that $-\Delta_R$ and $-\Delta$ are transversal self-adjoint extensions of $-\Delta_{\mathrm{sym}}$, i.e., $\mathcal{D}(-\Delta_R) \cap \mathcal{D}(-\Delta) = \mathcal{D}(-\Delta_{\mathrm{sym}})$ and $\mathcal{D}(-\Delta_R) + \mathcal{D}(-\Delta) = \mathcal{D}(-\Delta^*_{\mathrm{sym}})$ [13].

In view of (1.3) and (2.3) the $t^{-\alpha_j}$-invariance of an element ψ_j in (1.2) is equivalent to the relation

$$t^{-\alpha_j}\langle\psi_j, u\rangle = \langle\psi_j, U_{1/t}u\rangle \quad \forall u \in W_2^2(\mathbb{R}^3), \quad t > 0. \tag{3.8}$$

It turns out that the preservation of (3.8) for the extended functionals $\langle\psi_j^{\rm ex}, \cdot\rangle$ is equivalent to the t^{-2}-homogeneity of $-\Delta_R$.

Proposition 3.2. *Let $\psi_j^{\rm ex}$ be defined by* (2.8). *Then the relations*

$$t^{-\alpha_j}\langle\psi_j^{\rm ex}, f\rangle = \langle\psi_j^{\rm ex}, U_{1/t}f\rangle \quad \forall j \in \mathbb{N} \quad \forall t > 0 \tag{3.9}$$

hold for all $f \in \mathcal{D}(-\Delta^*_{\rm sym})$ if and only if the operator $-\Delta_R$ is t^{-2}-homogeneous with respect to $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$.

Proof. It follows from (2.2) and (2.3) that

$$\langle\psi_j, U_t u\rangle = \langle\mathbb{U}_{1/t}\psi_j, u\rangle = t^{\alpha_j}\langle\psi_j, u\rangle = 0$$

for every $u \in \mathcal{D}(-\Delta_{\rm sym})$. Thus $U_t\colon \mathcal{D}(-\Delta_{\rm sym}) \to \mathcal{D}(-\Delta_{\rm sym})$ and, by (1.1) and (2.4), the symmetric operator $-\Delta_{\rm sym}$ is t^{-2}-homogeneous: $U_t\Delta_{\rm sym} = t^{-2}\Delta_{\rm sym}U_t$. But then the adjoint $-\Delta^*_{\rm sym}$ of $-\Delta_{\rm sym}$ is also t^{-2}-homogeneous. This means that a self-adjoint extension $-\widetilde{\Delta}$ of $-\Delta_{\rm sym}$ is t^{-2}-homogeneous with respect to $\mathfrak{U} = \{U_t\}_{t\in(0,\infty)}$ if and only if $U_t\mathcal{D}(-\widetilde{\Delta}) = \mathcal{D}(-\widetilde{\Delta})$ for all $t > 0$. Since $U_tU_{1/t} = I$ the last equality is equivalent to the inclusion

$$U_t\mathcal{D}(-\widetilde{\Delta}) \subset \mathcal{D}(-\widetilde{\Delta}) \quad \forall t > 0. \tag{3.10}$$

Using (2.8) one can rewrite relations (3.9) as follows:

$$\Xi(t)\Psi_R^* f = \Psi_R^* U_{1/t} f \quad \forall f \in \mathcal{D}(-\Delta^*_{\rm sym}) \quad \forall t > 0, \tag{3.11}$$

where a bounded invertible operator $\Xi(t)$ in l^2 is defined by the formulas

$$\Xi(t)e_j = t^{-\alpha_j}e_j, \quad j \in \mathbb{N}. \tag{3.12}$$

Since $\mathcal{D}(-\Delta_0) = \ker\Gamma_0 = \ker\Psi_R^*$, (3.11) implies that $\mathcal{D}(-\Delta_R)$ satisfies (3.10). Thus $-\Delta_R$ is t^{-2}-homogeneous with respect to $\mathfrak{U}$.

Conversely, assume that $-\Delta_R$ is t^{-2}-homogeneous. According to (2.9) and (3.10) this is equivalent to the relation

$$\Psi_R^* U_{1/t} f = 0 \quad \forall f = u + \sum_{j=1}^{\infty} d_j h_j \in \mathcal{D}(-\Delta_R) \quad \forall t > 0. \tag{3.13}$$

Let us study (3.13) more detail. Using (3.1) and (3.3) it is seen that

$$U_{1/t}h_j = t^{-2}G_{1/t}U_{1/t}h_j + (I - t^{-2}G_{1/t})U_{1/t}h_j =$$

$$= \frac{t^{-2}}{t^{-\alpha_j}}h_j + (1 - t^{-2})(-\Delta + I)^{-1}U_{1/t}h_j,$$

where $h_j = (-\Delta + I)^{-1}\psi_j$. Therefore,

$$U_{1/t}f = v + \sum_{j=1}^{\infty} t^{\alpha_j - 2} d_j h_j, \tag{3.14}$$

where the element $v = U_{1/t}u + (1 - t^{-2})(-\Delta + I)^{-1}U_{1/t}\sum_{i=1}^{\infty} d_j h_j$ belongs to $\mathcal{D}(-\Delta)$. Substituting the obtained expression for $U_{1/t}f$ into (3.13) and using (2.9) one gets

$$\Psi^* U_{1/t}u + (1 - t^{-2})\Psi^*(-\Delta + I)^{-1}U_{1/t}\sum_{j=1}^{\infty} d_j h_j + t^{-2}R\Xi^{-1}(t)d = 0. \tag{3.15}$$

Here $\Psi^* U_{1/t}u = \Xi(t)\Psi^* u$ by (2.3) and (2.7). Moreover $\Psi^* u = -Rd$ since the vector $f = u + \sum_{j=1}^{\infty} d_j h_j$ belongs to $\mathcal{D}(-\Delta_R) = \ker \Psi_R^*$. Thus $\Psi^* U_{1/t}u = -\Xi(t)Rd$.

On the other hand, employing (2.2) and (2.7), one gets

$$\Psi^*(-\Delta + I)^{-1}U_{1/t}\sum_{j=1}^{\infty} d_j h_j = K_t d,$$

where K_t is a bounded operator in l^2 that is defined by the infinite-dimensional matrix $\mathbf{K} = \|k_{ij}\|_{i,j=1}^{\infty}$, $k_{ij} = (h_j, U_t h_i)$ with respect to the canonical basis $\{e_j\}_1^{\infty}$ (see Subsection 2.2). The obtained relations allow one to rewrite (3.15) as follows:

$$\big[-\Xi(t)R + t^{-2}R\Xi^{-1}(t) + (1 - t^{-2})K_t \big] d = 0 \quad \forall t > 0,$$

where d is an arbitrary element from l^2 (it follows from the presentation $f \in \mathcal{D}(-\Delta_R)$ in (3.13) and the transversality $-\Delta$ and $-\Delta_R$ with respect to $-\Delta_{\text{sym}}$). Therefore, the t^{-2}-homogeneity of $-\Delta_R$ is equivalent to the operator equality in l^2:

$$\Xi(t)R - t^{-2}R\Xi^{-1}(t) = (1 - t^{-2})K_t \quad \forall t > 0. \tag{3.16}$$

Finally, employing (2.9) and (3.15) it is easy to see that equality (3.16) is equivalent to (3.11). Therefore, the extended functionals $\langle \psi_j^{\text{ex}}, \cdot \rangle$ satisfy (3.9).

Proposition 3.2 is proved.

***Remark* 3.2.** The result similar to Proposition 3.2 was proved in [6] for the case of finite rank perturbations of a self-adjoint operator acting in an abstract Hilbert space $\mathfrak{H}$.

Theorem 3.2. *Let $\alpha_j \in (1, 2)$ for any $t^{-\alpha_j}$-invariant element ψ_j in the definition* (2.4) *of $-\Delta_{\text{sym}}$. Then there exists a unique t^{-2}-homogeneous self-adjoint extension of $-\Delta_{\text{sym}}$ transversal to $-\Delta$.*

Proof. It follows from the general theory of boundary triplets [13, 17] that an arbitrary self-adjoint extension $-\widetilde{\Delta}$ of $-\Delta_{\text{sym}}$ transversal to $-\Delta$ coincides with $-\Delta_R$ for a certain choice of a bounded self-adjoint operator R in l^2. As was shown in the proof of Proposition 3.2, $-\Delta_R$ is t^{-2}-homogeneous with respect to scaling transformations if and only if the operator R is a solution of (3.16) that does not depend on $t > 0$. Using (3.12) and the definition of K_t one can rewrite (3.16) componentwise as follows:

$$(t^{-\alpha_i} - t^{\alpha_j - 2})r_{ij} = (1 - t^{-2})(h_j, U_t h_i), \quad \mathbf{R} = \|r_{ij}\|_{i,j=1}^{\infty} \tag{3.17}$$

where the infinite-dimensional matrix $\mathbf{R}$ is the matrix presentation of R with respect to the canonical basis $\{e_j\}_1^{\infty}$.

Let us calculate $(h_j, U_t h_i)$ in (3.17). According to Proposition 3.1, $t^{-\alpha_j}$-invariant elements ψ_j in (1.2) have the form $\psi_j = \psi(m_j, \alpha_j)$, where $m_j(\cdot) \in L_2(S^2)$ and elements $h_j = (-\Delta + I)^{-1}\psi(m_j, \alpha_j)$ are defined by (3.7).

It follows from (3.5) that

$$\widehat{U_t h_i}(y) = t^{-3/2}\widehat{h}\left(\frac{y}{t}\right) = t^{2-\alpha_i}\frac{m_i(w)}{|y|^{3/2-\alpha_i}(|y|^2+t^2)}.$$

Hence,

$$(h_j, U_t h_i) = t^{2-\alpha_i}\int\limits_{\mathbb{R}^3}\frac{m_j(w)\overline{m_i(w)}}{|y|^{3-(\alpha_j+\alpha_i)}(|y|^2+t^2)(|y|^2+1)}\,dy =$$

$$= (m_j, m_i)_{L_2}\int\limits_0^\infty \frac{t^{2-\alpha_i}}{|y|^{1-(\alpha_i+\alpha_j)}(|y|^2+t^2)(|y|^2+1)}d|y| =$$

$$= c_{ij}\frac{t^{\alpha_j} - t^{2-\alpha_i}}{t^2-1}(m_j, m_i)_{L_2},$$

where $c_{ij} = \displaystyle\int_0^\infty \frac{|y|^{3-(\alpha_i+\alpha_j)}}{|y|^2+1}d|y|$ and $(m_i, m_j)_{L_2} = \displaystyle\int_{S^2} m_i(w)\overline{m_j(w)}dw$ is the scalar product in $L_2(S^2)$. Substituting the obtained expression for $(h_j, U_t h_i)$ into (3.17) one finds $r_{ij} = -c_{ij}(m_j, m_i)_{L_2}$. The matrix $\mathbf{R} = \|r_{ij}\|_{i,j=1}^{\infty}$ determined in such a way is the matrix representation of a unique solution R of (3.16) that does not depend on $t > 0$.

Theorem 3.2 is proved.

3.3. The Friedrichs and Krein – von Neumann extensions. As was shown in the proof of Proposition 3.2, the symmetric operator $-\Delta_{\mathrm{sym}}$ is t^{-2}-homogeneous with respect to scaling transformations. According to general results obtained in [6, 10], the Friedrichs $-\Delta_F$ and the Krein – von Neumann $-\Delta_N$ extensions of $-\Delta_{\mathrm{sym}}$ are also t^{-2}-homogeneous.

Theorem 3.3. *Let $\alpha_j \in (1,2)$ for any $t^{-\alpha_j}$-invariant element ψ_j in the definition* (2.4) *of $-\Delta_{\mathrm{sym}}$ and let the spectrum of $-\Delta_R$, where R is a unique solution of* (3.16) *does not cover real line $\mathbb{R}$. Then the Krein – von Neumann extension $-\Delta_N$ coincides with $-\Delta_R$ and the Friedrichs extension $-\Delta_F$ coincides with the initial operator $-\Delta$.*

Proof. A simple analysis of (3.7) shows that $h_j \in L_2(\mathbb{R}^3) \setminus W_2^1(\mathbb{R}^3)$ for $1 \le \alpha < 2$, i.e., singular elements ψ_j in (2.4) form a $W_2^{-1}(\mathbb{R}^3)$-independent system. This means that the initial operator $-\Delta$ coincides with the Friedrichs extension $-\Delta_F$.

Since $-\Delta_R$ is t^{-2}-homogeneous and $\sigma(-\Delta_R) \neq \mathbb{R}$, the equality

$$U_t(-\Delta_R - \lambda I) = t^{-2}(-\Delta_R - t^2\lambda I)U_t, \quad t > 0,$$

means that the spectrum of $-\Delta_R$ is nonnegative. Therefore, $-\Delta_R$ is a nonnegative extension of $-\Delta_{\mathrm{sym}}$ transversal to the Friedrichs extension $-\Delta$. But then the Krein – von Neumann extension $-\Delta_N$ is also transversal to $-\Delta$. Since $-\Delta_N$ is t^{-2}-homogeneous, Theorem 3.2 gives $-\Delta_N = -\Delta_R$ that completes the proof.

3.4. t^{-2}-Homogeneous extensions of $-\Delta_{\mathrm{sym}}$. Let us consider the heuristic expression (1.2), where all elements ψ_j are assumed to be $t^{-\alpha}$-invariant with respect to scaling transformations, i.e., $\psi_j = \psi(m_j, \alpha)$, where $\alpha \in (1,2)$ is fixed.

It follows from (1.3) and (2.3) that the singular potential $V = \sum_{i,j=1}^{\infty} b_{ij}\langle\psi_j,\cdot\rangle\psi_i$ in (1.2) is $t^{-2\alpha}$-homogeneous in the sense that

$$\mathbb{U}_t V u = t^{-2\alpha} V U_t u \quad \forall u \in W_2^2(\mathbb{R}^3).$$

Hence, the initial operator $-\Delta$ and its singular perturbation V possess the homogeneity property with different index of homogeneity: t^{-2} and $t^{-2\alpha}$, respectively. In view of this, it is natural to expect that any self-adjoint extension $-\widetilde{\Delta}$ of $-\Delta_{\mathrm{sym}}$ having the t^{-2}-homogeneity property (as well as $-\Delta$ and $-\Delta_R$) is closely related to $-\Delta$ and $-\Delta_R$.

Let $(l^2, \Gamma_0, \Gamma_1)$ be a boundary triplet of $-\Delta^*_{\mathrm{sym}}$ defined by (2.11), where R is a unique solution of (3.16).

Theorem 3.4. *Let all elements ψ_j be $t^{-\alpha}$-invariant with respect to scaling transformations, where $\alpha \in (1,2)$ is fixed. Then an arbitrary t^{-2}-homogeneous self-adjoint extension $-\widetilde{\Delta}$ of $-\Delta_{\mathrm{sym}}$ coincides with the restriction of $-\Delta^*_{\mathrm{sym}}$ onto the domain*

$$\mathcal{D}(-\widetilde{\Delta}) = \{f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}}) \colon\ (I - V)\Gamma_0 f = i(I + V)\Gamma_1 f\}, \tag{3.18}$$

where V is taken from the set of unitary and self-adjoint operators in l^2.

Proof. If Γ_0 is a boundary operator defined by (2.11), where R is a unique solution of (3.16), then formulas (3.11) and (3.12) give

$$\Gamma_0 U_{1/t} f = t^{-\alpha}\Gamma_0 f \quad \forall f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}}) \quad \forall t > 0. \tag{3.19}$$

On the other hand, using (3.14), one derives

$$\Gamma_1 U_{1/t} f = t^{\alpha-2}\Gamma_1 f \quad \forall f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}}) \quad \forall t > 0. \tag{3.20}$$

It is known [13] that an arbitrary self-adjoint extension $-\widetilde{\Delta}$ of $-\Delta_{\mathrm{sym}}$ is the restriction of $-\Delta^*_{\mathrm{sym}}$ onto the domain (3.18) where V is a unitary operator in l^2. By (3.19), (3.20),

$$U_{1/t}\mathcal{D}(-\widetilde{\Delta}) = \{f \in \mathcal{D}(-\Delta^*_{\mathrm{sym}}) \colon\ t^{\alpha}(I - V)\Gamma_0 f = it^{2-\alpha}(I + V)\Gamma_1 f\}. \tag{3.21}$$

The operator $-\widetilde{\Delta}$ is t^{-2}-homogeneous if and only if its domain $\mathcal{D}(-\widetilde{\Delta})$ satisfies (3.10). Comparing (3.18) and (3.21) and taking into account that $\alpha > 1$, one concludes that (3.10) holds if and only if $\Gamma_0\mathcal{D}(-\widetilde{\Delta}) = \ker(I - V)$ and $\Gamma_1\mathcal{D}(-\widetilde{\Delta}) = \ker(I + V)$. These relations give

$$\ker(I - V) \oplus \ker(I + V) = l^2 \tag{3.22}$$

since $-\widetilde{\Delta}$ is a self-adjoint operator and $(l^2, \Gamma_0, \Gamma_1)$ is a boundary triplet of $-\Delta^*_{\mathrm{sym}}$. The obtained identity implies that the unitary operator V also is self-adjoint.

Conversely, if V is unitary and self-adjoint, then (3.22) is satisfied. Hence, (3.10) holds and $-\widetilde{\Delta}$ is t^{-2}-homogeneous.

Theorem 3.4 is proved.

***Corollary* 3.1.** *There are no t^{-2}-homogeneous operators among nontrivial ($\neq -\Delta$) self-adjoint operator realizations of* (1.2).

Proof. According to Theorem 2.1 an operator realization $-\Delta_B$ of (1.2) is defined by (2.12). It follows from (2.12) and (3.18) that $B = -i(I - V)(I + V)^{-1}$. If the operator V has the additional property (3.22) (the condition of t^{-2}-homogeneity of $-\Delta_B$), then $B = 0$. Hence $-\Delta_B$ is t^{-2}-homogeneous if and only if $-\Delta_B = -\Delta$.

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